Trace semantics of well-founded processes via commutativity

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Let $\mathcal{S} = (A_k)_{k \in K}$ be a *signature*, i.e. family of sets. We call $k \in K$ an *operation* and the set A_k its *arity*, which may be empty. Consider the language of I/O and nondeterminism inductively defined as follows:

$$M ::= \quad \texttt{input}_k(M_i)_{i \in A_k} \mid M \texttt{ or } M$$

Informally: the command $\texttt{input}_k(M_i)_{i \in A_k}$ first prints k. Then, if the user inputs $i \in A_k$, it proceeds to execute M_i. The command M $\texttt{or}$ M' nondeterministically chooses to execute M or M'.

Write $\mathcal{P}_\mathsf{f}^+$ for the finite nonempty powerset monad on **Set**, whose Eilenberg-Moore algebras are semilattices. Write $H^\mathcal{S}$ for the endofunctor $X \mapsto \sum_{k \in K} X^{A_k}$, whose algebras are $\mathcal{S}$-algebras. Let Q be the set of commands; the "medium-step" operational semantics [4] is the evident map $\zeta : Q \to \mathcal{P}_\mathsf{f}^+ H^\mathcal{S} Q$. When $(k, (N_i)_{i \in A_k}) \in \zeta M$, we write $M \Downarrow_k$ and also $M \Downarrow_k \overset{i}{\rightsquigarrow} N_i$ for each $i \in A_k$.

Clearly, bisimilarity is the least congruence generated by the basic laws of $\texttt{or}$, viz. commutativity, associativity and idempotency. (This gives a *sum of monads* [3].) We now consider trace equivalence. A *trace* s is a finite or infinite sequence $k_0, i_0, k_1, i_1, \ldots$, where $k_n \in K$ and $i_n \in A_{k_n}$ for each n. A command M *has* this trace s when $M_0 \Downarrow_{k_0} \overset{i_0}{\rightsquigarrow} M_1 \Downarrow_{k_1} \overset{i_1}{\rightsquigarrow} \cdots$, for some sequence of commands $M = M_0, M_1, \ldots$. Commands in our language have no infinite traces, because (R, ζ) is a *well-founded* $\mathcal{P}_\mathsf{f}^+ H^\mathcal{S}$-coalgebra [6]. Two commands with the same traces are *trace equivalent*. Plotkin [private communication] showed this to be the congruence generated by the basic laws of $\texttt{or}$ and commutativity, for all $k \in K$, of $\texttt{or}$ with $\texttt{input}_k$:

$$\forall M, M' \in R^{A_k}. \quad \texttt{input}_k(M_i \texttt{ or } M'_i)_{i \in A_k} \quad \equiv \quad \texttt{input}_k(M_i)_{i \in A_k} \texttt{ or } \texttt{input}_k(M'_i)_{i \in A_k}$$

A *trace process* D is a prefix-closed set of odd-length traces. The even-length traces *enabled* by D are given by $\mathsf{en}(D) = \{\varepsilon\} \cup \{ski \mid sk \in D,\, i \in A_k\}$, and for $t \in \mathsf{en}(D)$ its *response set* is $t^D = \{k \in K \mid tk \in D\}$. A *tree* is a trace process D such that t^D is singleton for all $t \in \mathsf{en}(D)$. A tree is *well-founded* when no infinite trace has all its odd-length prefixes in D. As is well-known, the set of well-founded trees gives an initial $H^\mathcal{S}$-algebra, whilst the set of all trees gives a final $H^\mathcal{S}$-coalgebra.

A trace process is *finitely nondeterministic, total and König* when t^D is finite and nonempty for all $t \in \mathsf{en}(D)$, and no infinite trace has all its odd-length prefixes in D. Let FNTK be the set of all such trace processes. The trace set of each command in our language—indeed, of any state of a well-founded $\mathcal{P}_\mathsf{f}^+ H^\mathcal{S}$-coalgebra—has these properties. Plotkin's argument shows the converse: each $D \in \mathsf{FNTK}$ is the trace set of a command. Thus FNTK is initial among semilattice $\mathcal{S}$-algebras in which $\mathcal{S}$-operations commute with $\vee$, and hence $\mathcal{S}$-operations are monotone. (This gives a *tensor of monads* [1, 2, 3].)

We have developed Plotkin's result in two directions.

- Replacing nondeterministic choice by probabilistic choice M $\texttt{or}_p$ M', where $p \in [0, 1]$. Then the corresponding result holds, with essentially the same proof.

- Replacing finite nondeterminism by countable nondeterminism (or greater). A similar result holds, with a quite different proof. A trace process is *countably nondeterministic and well-foundedly total* when, for all $t \in \mathsf{en}(D)$, the set t^D is countable and there is a well-founded tree E such that $\{ts \mid s \in E\} \subseteq D$, cf. [5]. The trace set of every command has these properties; conversely, every such trace process is the trace set of some command. The set of such trace processes is initial among semilattice $\mathcal{S}$-algebras with countable suprema, in which $\mathcal{S}$-operations commute with countable supremum.

References

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