Diophantine properties of definable sets: applications of model theory to number theory

Alex Wilkie

August 26, 2016

First of all, a challenge. Suppose that α is a real number having the property that for every positive integer n, the real number n^α happens to be a positive integer. Prove that α itself is an integer.

I have often posed this problem at coffee time in the various mathematics departments in which I have worked, and despite the fact that the proof only takes a few lines, it never fails to frustrate many colleagues. Actually, it is an example of a particular class of problems in *transcendental number theory*. Namely, to show that those functions f that crop up naturally in analysis and calculus (as, say, solutions to differential equations) have the property that they very rarely take integer (or algebraic) values for integer (or algebraic) arguments unless there is a good algebraic reason for them doing so, e.g. the function is a polynomial with rational coefficients (or an algebraic function). Thus, in the problem above the function under consideration is $f(x) = x^\alpha$ which is clearly not a polynomial unless α is a positive integer.

Now there are obvious restrictions that have to be made here before formulating a precise general conjecture. Firstly, there are functions like $f(x) = 2^x$ or $f(x) = x^x$ and it is not clear what we might mean by there being "algebraic reasons" to explain the fact that $f(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$ in these cases. So let us restrict to functions of at most polynomial growth (or, possibly, sub-exponential growth). But then there are functions like $f(x) = \sin(2\pi x)$ (or even $f(x) = x^{\sin(2\pi x^x)}$). But these are only a counterexamples because of the periodicity of the sin function.

So here's the "conjecture". Let f be a real function of a real variable that crops up naturally in calculus, has at most polynomial growth, and does not entail (via simple logical and algebraic reasoning) the existence of any periodic function. Suppose that $f(n)$ is an integer for all natural numbers n. Then f is a polynomial with rational coefficients (at least, for sufficiently large values of its argument).

The purpose of my talk is to show how model theory can help to both formulate this conjecture precisely, and then prove it.

By the way, if you did manage to solve the challenge problem (which certainly can be deduced from the result I discuss here, but has a much more direct proof) then you might like to try to get the same conclusion but under the much weaker assumption that 2^α, 3^α and 5^α are positive integers (Siegel's theorem). And what about only assuming that 2^α and 3^α are positive integers? Does this alone force α to be an integer? That is completely open!

Mathematical Institute
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford
OX2 6GG
wilkie@maths.ox.ac.uk