From Parametricity to Conservation Laws, via Noether's Theorem

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Invariance in Programming: (Reynolds, 1983)

$$\vdash M : \forall \alpha.\ \mathsf{List}\ \alpha \to \mathsf{Int}$$

implies the property

$$\forall f : A \to B, l : \mathsf{List}\ A.\ M\, l = M\, (\mathrm{map}\ f\ \ l)$$

... because M does not “know” what α is.

Invariance in Classical Mechanics:

A Lagrangian:

$$L(t, q, \dot{q})$$

If

$$\forall x.\ L(t, q, \dot{q}) = L(t, q + x, \dot{q})$$

then, for all "physically realisable" paths q:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

... a conservation law via Noether's theorem.

A Concrete Lagrangian: (Two particles connected by a spring)

$$L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1{}^2 + \dot{x}_2{}^2) - \frac{1}{2}k(x_1 - x_2)^2$$

This satisfies:

$$\forall y.\ L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = L(t, x_1 + y, x_2 + y, \dot{x}_1, \dot{x}_2)$$

so, for all “physically realisable” paths for x_1 and x_2:

$$\frac{d}{dt}m(\dot{x}_1 + \dot{x}_2) = 0$$

Compare:

$$\forall f : A \to B, l : \mathsf{List}\ A.\ M\, l = M\, (\mathrm{map}\ f\ \ l)$$

with

$$\forall x.\ L(t, q, \dot{q}) = L(t, q + x, \dot{q})$$

The Plan:

- Types yield invariance properties

  (*via Parametricity*)

- Invariance properties yield conservation laws

  (*via Noether's theorem*)

Lagrangian Mechanics and Noether's Theorem

Lagrangian Mechanics and Noether's Theorem

From Invariance to Conservation Laws

Lagrangians:

$$L(t, q, \dot{q}) = T - V$$

where:

T is the total *kinetic energy* of the system

V is the total *potential energy* of the system

The Action:

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q(t), \dot{q}(t))\, dt$$

Principle of Stationary Action: *(Euler-Lagrange Equations)*

$$\delta \mathcal{S} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

The "physically realisable" paths q satisfy these ODEs

An Example Lagrangian: *(Particle under gravity)*

$$L(t, x, y, \dot{x}, \dot{y}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

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Euler-Lagrange Equations:

$$m\ddot{x} = 0 \qquad m\ddot{y} = -mg$$

Both equations are of the form $F = m\ddot{x}$ ...
... Newton's second law is derived in Lagrangian mechanics

Given an Action:

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q, \dot{q})\, dt$$

assume transformations of time $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$
assume transformations of space $\Psi_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$
where Φ_0 and Ψ_0 are the identity, and $\Phi_{-\epsilon} = \Phi_\epsilon^{-1}$

Invariance of the Action:

The action is invariant if (for all q, a, b, ϵ):

$$\int_a^b L(t, q(t), \dot{q}(t))\, dt = \int_{\Phi_\epsilon(a)}^{\Phi_\epsilon(b)} L(s, q^*(s), \dot{q}^*(s))\, ds$$

where $q^* = \Psi_\epsilon \circ q \circ \Phi_\epsilon^{-1}$

Noether's Theorem:

If the action

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q, \dot{q})\, dt$$

is invariant under Φ_ϵ and Ψ_ϵ, then

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\psi_i + \left(L - \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)\phi\right) = 0$$

where $\phi = \frac{\partial \Phi}{\partial \epsilon}\Big|_{\epsilon=0}$ and $\psi = \frac{\partial \Psi}{\partial \epsilon}\Big|_{\epsilon=0}$

Noether's Theorem:

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where $\phi = \left.\frac{\partial \Phi}{\partial \epsilon}\right|_{\epsilon=0}$ and $\psi = \left.\frac{\partial \Psi}{\partial \epsilon}\right|_{\epsilon=0}$

Special case of Noether's "first theorem", (Noether, 1918).

The Spring Lagrangian:

$$L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1{}^2 + \dot{x}_2{}^2) - \frac{1}{2}k(x_1 - x_2)^2$$

satisfies:

$$L(t + \epsilon t', x_1, x_2, \dot{x}_1, \dot{x}_2) = L(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$$

and so, by Noether's theorem:

$$\frac{d}{dt}\left(\frac{1}{2}m(\dot{x}_1{}^2 + \dot{x}_2{}^2) + \frac{1}{2}k(x_1 - x_2)^2\right) = 0$$

i.e., *energy* is conserved.

Simplified Invariance:

when $\Phi_\epsilon(t) = t + \epsilon t'$, and $\Psi_\epsilon(x) = G_\epsilon x + \epsilon x'$,

then invariance reduces:

$$L(t, q, \dot{q}) = L(t + t', Gq + x', G\dot{q})$$

Deduce invariance properties like this from types:

$$\forall t' : \mathsf{T}(1), x' : \mathsf{T}(3), o : \mathsf{O}(3).$$
$$C^\infty(\mathbb{R}\langle 1, t'\rangle \times \mathbb{R}^3\langle o, x'\rangle \times \mathbb{R}^3\langle o, 0\rangle, \mathbb{R}\langle 1, 0\rangle)$$

Relational Parametricity

Relational Parametricity

From Types to Invariance (Reynolds, 1983)

Type Abstraction

$$M : \forall\alpha{:}{*}.\ \alpha \to (\alpha \to \alpha) \to \alpha$$

The implementation M only “knows” two things about α:

- at least one $z : \alpha$ exists;
- and, given one, there is another, by $s : \alpha \to \alpha$.

The program M is uniform under changes of representation of α.

Reynolds’ Idea

Formalise M’s symmetry by preservation of relations

For example,

$$M : \forall\alpha{:}*.\ \alpha \to (\alpha \to \alpha) \to \alpha$$

let X and Y be sets, and let $R \subseteq X \times Y$

if we have $z_1 \in X$, $z_2 \in Y$ such that:

$$(z_1, z_2) \in R$$

and $s_1 : X \to X$, $s_2 : Y \to Y$ such that:

$$\forall (a, b) \in R.\ (s_1\ a, s_2\ b) \in R$$

then

$$(M\ [X]\ z_1\ s_1, M\ [Y]\ z_2\ s_2) \in R$$

Preservation of Relations

implies $(\forall\alpha.\ \alpha \to (\alpha \to \alpha) \to \alpha) \cong \mathbb{N}$

Higher Kinds
and
Reflexive Graphs

System F: Quantification over *types*:

$$\forall \alpha{:}*.\ \text{List}\ \alpha \to \text{List}\ \alpha$$

System Fω: Quantification over *type operators*:

$$\forall f : * \to *.\ \forall \alpha{:}*.\ f\,\alpha \to f\,\alpha$$

and type-level λ-abstraction:

$$\begin{aligned}
\text{List} &= \lambda\alpha : *.\ \forall_{*}\beta.\ \beta \to (\alpha \to \beta \to \beta) \to \beta \\
\text{Monad} &= \lambda m : * \to *.\ (\forall \alpha{:}*.\ \alpha \to m\ \alpha)\times \\
&\qquad (\forall \alpha, \beta{:}*.\ m\ \alpha \to (\alpha \to m\ \beta) \to m\ \beta)
\end{aligned}$$

Present in Haskell, Scala, and ML (via the module system)

Model invariance using Reflexive Graphs:

(Hasegawa, 1994) (Robinson and Rosolini, 1994) (Dunphy and Reddy, 2004)

Kinds are reflexive graphs:

$A : \mathrm{Set}$	a large set of objects
$R : A \times A \to \mathrm{Set}$	sets of edges/relations between objects
$\mathrm{refl} : \forall a.\ R(a, a)$	distinguished identity edges

For example, $[\![*]\!] = (\mathrm{set}, \mathrm{rel}, \equiv)$

$* \to *$, $(* \to *) \to *$ etc. are also reflexive graphs

Types are morphisms between reflexive graphs:

$$(f, r) : (A, R_A, \mathrm{refl}_A) \to (B, R_B, \mathrm{refl}_B)$$

where:

$f : A \to B$

$r : \forall a, a' \in A.\ R_A(a, a') \to R_B(f\,a, f\,a')$

such that $r\ a\ a\ (\mathrm{refl}_A\ a) = \mathrm{refl}_B\ (f\,a)$

For example $\mathsf{List} : * \to *$,

$$\begin{array}{lcl} [\![\mathsf{List}]\!]^f X & = & X^* \\ [\![\mathsf{List}]\!]^r X\,Y\,R & = & \{(l, l') \mid |l| = |l'| \wedge \forall i.\ R(l[i], l'[i])\} \end{array}$$

Terms are transformations between morphisms:

Given two morphisms $X, Y : (A, R_A, \mathrm{refl}_A) \to [\![*]\!]$,

A *term* is a function:

$$M : \forall a \in A.\ X^f a \to Y^f a$$

such that for all $a, a' \in A$, $r \in R_A(a, a')$,

$$(x, x') \in X^r aa' r \text{ implies } (M\ a\ x, M\ a'\ x') \in Y^r aa' r$$

Unpacking this definition:

For concrete X and Y we get “free theorems” about M;

or prove two programs equivalent;

or prove that some types are uninhabited.

Higher-kinded Types
for
Classical Mechanics

Higher kinded types for Classical Mechanics:

$$\begin{array}{lcl} \mathbb{R}^n & : & \mathsf{GL}(n) \longrightarrow \mathsf{T}(n) \longrightarrow \mathsf{CartSp} \\ C^\infty & : & \mathsf{CartSp} \longrightarrow \mathsf{CartSp} \longrightarrow * \end{array}$$

where

$\mathsf{GL}(n)$	–	the kind of invertible linear transformations
$\mathsf{T}(n)$	–	the kind of translations
CartSp	–	the kind of cartesian spaces

Classical Mechanics' kinds are groupoids:

$$[\![\mathsf{GL}(n)]\!] = (\{*\}, \mathrm{GL}(n), I)$$

GL(n) is the group of invertible linear transformations on $\mathbb{R}^n$

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$[\![\mathsf{O}(n)]\!] = (\{*\}, \mathrm{O}(n), I)$

$\mathrm{O}(n)$ is the group of orthogonal transformations on $\mathbb{R}^n$

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$\mathrm{GL}(n)$ is the group of invertible linear transformations on $\mathbb{R}^n$

$[\![\mathsf{O}(n)]\!] = (\{*\}, \mathrm{O}(n), I)$
$\mathrm{O}(n)$ is the group of orthogonal transformations on $\mathbb{R}^n$

$[\![\mathsf{T}(n)]\!] = (\{*\}, \mathrm{T}(n), 0)$
$\mathrm{T}(n)$ is the group of translations on $\mathbb{R}^n$

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$[\![\mathsf{T}(n)]\!] = (\{*\}, \mathrm{T}(n), 0)$
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$[\![\mathsf{Z}]\!] = (\{*\}, \mathbb{Z}, 0)$
$\mathbb{Z}$ is the additive group of integers

Classical Mechanics' kinds are groupoids:

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GL(n) is the group of invertible linear transformations on $\mathbb{R}^n$

$[\![\mathsf{O}(n)]\!] = (\{*\}, \mathrm{O}(n), I)$
O(n) is the group of orthogonal transformations on $\mathbb{R}^n$

$[\![\mathsf{T}(n)]\!] = (\{*\}, \mathrm{T}(n), 0)$
T(n) is the group of translations on $\mathbb{R}^n$

$[\![\mathsf{Z}]\!] = (\{*\}, \mathbb{Z}, 0)$
$\mathbb{Z}$ is the additive group of integers

$[\![\mathsf{CartSp}]\!] = (\mathbb{N}, \text{diffeomorphisms on } \mathbb{R}^n, \mathrm{id})$
Diffeomorphisms are smooth functions with smooth inverses

Higher-kinded types for Classical Mechanics:

$$\begin{array}{lcl} \mathbb{R}^n & : & \mathsf{GL}(n) \longrightarrow \mathsf{T}(n) \longrightarrow \mathsf{CartSp} \\ C^\infty & : & \mathsf{CartSp} \longrightarrow \mathsf{CartSp} \longrightarrow * \end{array}$$

... with interpretations:

$$\begin{array}{lcl} [\![\mathbb{R}^n]\!]^f \; * \; * & = & n \\ [\![\mathbb{R}^n]\!]^r \; * \; * \; G \; * \; * \; t & = & \lambda \vec{x}.\, G\vec{x} + t \end{array}$$

$$\begin{array}{lcl} [\![C^\infty]\!]^f \; m \; n & = & \text{smooth functions } \mathbb{R}^m \longrightarrow \mathbb{R}^n \\ [\![C^\infty]\!]^r \; m \; m \; d_1 \; n \; n \; d_2 & = & \{(f, f') \mid d_2 \circ f = f' \circ d_1\} \end{array}$$

Term constants for Classical Mechanics:

$$\underline{\vec{c}} \quad : \wr \mathbb{R}^n\langle 1, 0\rangle \int$$

$$0 \quad : \forall g{:}\mathsf{GL}(n).\ \wr\, \mathbb{R}^n\langle g, 0\rangle \int$$

$$(+) : \forall g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 + t_2\rangle)$$

$$(-) : \forall g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 - t_2\rangle)$$

$$(*) \ : \forall g_1{:}\mathsf{GL}(1), g_2{:}\mathsf{GL}(n).$$
$$C^\infty(\mathbb{R}\langle g_1, 0\rangle \times \mathbb{R}^n\langle g_2, 0\rangle, \mathbb{R}\langle \text{scale}_n(g_1)g_2, 0\rangle)$$

$$(\cdot) \ : \forall g{:}\mathsf{GL}(1), o{:}\mathsf{O}(n).$$
$$C^\infty(\mathbb{R}^n\langle(\text{scale}_n\ g)(\text{ortho}_n\ o), 0\rangle \times$$
$$\mathbb{R}^n\langle(\text{scale}_n\ g)(\text{ortho}_n\ o), 0\rangle, \mathbb{R}\langle(\text{scale}_n\ g)^2, 0\rangle)$$

More term constants for Classical Mechanics:

$$\sin\ : \forall z{:}\mathsf{Z}.\ C^\infty(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle)$$

$$\cos\ : \forall z{:}\mathsf{Z}.\ C^\infty(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle)$$

$$\exp\ : \forall t{:}\mathsf{T}(1).\ C^\infty(\mathbb{R}\langle 1, t\rangle, \mathbb{R}\langle \exp\ t, 0\rangle)$$

$$(/)\ : \forall g_1, g_2{:}\mathsf{GL}(1).\ C^\infty(\mathbb{R}\langle g_1, 0\rangle \times \mathbb{R}\langle g_2, 0\rangle, \mathbb{R}\langle g_1 g_2^{-1}, 0\rangle)$$

$$\mathrm{sqrt}\ : \forall g{:}\mathsf{GL}(1).\ C^\infty(\mathbb{R}\langle g^2, 0\rangle, \mathbb{R}\langle g, 0\rangle)$$

A syntax for smooth invariant functions:

$$\Theta|\Gamma;\Delta \vdash M : X$$

where

$\Theta = \alpha_1 : \kappa_1, ..., \alpha_n : \kappa_n$ - kinding context
$\Gamma = z_1 : A_1, ..., z_m : A_m$ - typing context
$\Delta = x_1 : X_1, ..., x_k : X_k$ - spatial context

- Semantics is given by translation into Fω
- $\Theta|\Gamma;\Delta \vdash M : X \quad \Rightarrow \quad \Theta \vdash \Gamma \vdash \lfloor M \rfloor : C^\infty(\Delta, X)$

Examples

Free Particle

$$\begin{aligned}
\Theta &= t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3) \\
\Gamma &= m : \lbag \mathbb{R}\langle 1, 0\rangle \rbag \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), t_x\rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle
\end{aligned}$$

$$L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) : \mathbb{R}\langle 1, 0\rangle$$

Free Particle

$$
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\Theta &= t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3) \\
\Gamma &= \ m : \wr \mathbb{R}\langle 1, 0\rangle \int \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), t_x\rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle
\end{aligned}
$$

$$
L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) : \mathbb{R}\langle 1, 0\rangle
$$

Free theorems

$$
\begin{aligned}
&\forall t_t \in \mathbb{R}.\ [\![L]\!](t + t_t, \vec{x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}}) && \Rightarrow \quad \text{energy} \\
&\forall \vec{t_x} \in \mathbb{R}^3.\ [\![L]\!](t, \vec{x} + \vec{t_x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}}) && \Rightarrow \quad \text{linear momentum} \\
&\forall O \in \mathrm{O}(3).\ [\![L]\!](t, O\vec{x}, O\dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}}) && \Rightarrow \quad \text{angular momentum}
\end{aligned}
$$

In detail:

$$\forall O \in \mathrm{O}(3).\ [\![L]\!](t, O\vec{x}, O\dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$$

In particular (rotation around the z-axis):

$$O_\epsilon = \begin{pmatrix} \cos\epsilon & \sin\epsilon & 0 \\ -\sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Apply Noether's theorem with:

$$\Psi_\epsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = O_\epsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \cos\epsilon + x_2 \sin\epsilon \\ -x_1 \sin\epsilon + x_2 \cos\epsilon \\ x_3 \end{pmatrix}$$

Derive the conservation law:

$$\frac{d}{dt}(m\dot{x}x_2 - m\dot{x}x_1) = 0$$

Particle in an arbitrary potential field:

$$\begin{aligned}
\Theta &= t_t : \mathsf{T}(1), o : \mathsf{O}(3) \\
\Gamma &= m : \wr\mathbb{R}\langle 1, 0\rangle\int, \\
&\quad\; V : \forall o{:}\mathsf{O}(3).\ C^\infty(\mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle, \dot{x} : \mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle
\end{aligned}$$

$$L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) - V(x) : \mathbb{R}\langle 1, 0\rangle$$

Conserved Quantities:

- Energy
- Angular momentum

    Even though V is unknown

n-body problem

$$\begin{aligned}
\Theta &= n : \mathsf{Nat}, t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3) \\
\Gamma &= \; m : \wr \mathbb{R}\langle 1, 0\rangle \int \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, \\
&\quad\; x : \text{vec } n \; (\mathbb{R}^3\langle \text{ortho}_3(o), t_x\rangle), \\
&\quad\; \dot{x} : \text{vec } n \; (\mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle)
\end{aligned}$$

$$\begin{aligned}
L = \tfrac{1}{2} m(&\text{sum } (\text{map } (\dot{x}_i.\ \dot{x}_i \cdot \dot{x}_i))\ \dot{x}) - \\
&\text{sum } (\text{map } ((x_i, x_j).\ Gm^2/|x_i - x_j|)\ (\text{cross } x\ x)) : \mathbb{R}\langle 1, 0\rangle
\end{aligned}$$

n-body problem

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\Theta &= n : \mathsf{Nat}, t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3) \\
\Gamma &= \; m : \wr \mathbb{R}\langle 1, 0\rangle \int \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, \\
&\quad\; x : \text{vec } n \; (\mathbb{R}^3\langle \text{ortho}_3(o), t_x\rangle), \\
&\quad\; \dot{x} : \text{vec } n \; (\mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle)
\end{aligned}$$

$$\begin{aligned}
L = \tfrac{1}{2} m (&\text{sum } (\text{map } (\dot{x}_i.\ \dot{x}_i \cdot \dot{x}_i))\ \dot{x}) - \\
&\text{sum } (\text{map } ((x_i, x_j).\ Gm^2 / |x_i - x_j|)\ (\text{cross } x\ x)) : \mathbb{R}\langle 1, 0\rangle
\end{aligned}$$

Conserved quantities:

- Energy
- Linear momentum
- Angular momentum

Pendulum:

$$\Theta = t_t : \mathsf{T}(1), z : \mathsf{Z}$$
$$\Gamma = m : \wr \mathbb{R}\langle 1, 0\rangle \int, l : \wr \mathbb{R}\langle 1, 0\rangle \int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle$$

Pendulum:

$$\begin{aligned}\Theta &= t_t : \mathsf{T}(1), z : \mathsf{Z}\\ \Gamma &= \ m : \wr\mathbb{R}\langle 1, 0\rangle\int, l : \wr\mathbb{R}\langle 1, 0\rangle\int\\ \Delta &= t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle\end{aligned}$$

$$\begin{aligned}L = &\ \text{let } y = l \sin\theta \text{ in}\\ &\ \text{let } \dot{x} = l\dot{\theta}\cos\theta \text{ in}\\ &\ \text{let } \dot{y} = -l\dot{\theta}\sin\theta \text{ in}\\ &\quad \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy : \mathbb{R}\langle 1, 0\rangle\end{aligned}$$

Pendulum:

$$\begin{aligned}
\Theta &= t_t : \mathsf{T}(1), z : \mathsf{Z} \\
\Gamma &= \; m : \wr \mathbb{R}\langle 1, 0\rangle \int, l : \wr \mathbb{R}\langle 1, 0\rangle \int \\
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\end{aligned}$$

$$\begin{aligned}
L = {} & \text{let } y = l \sin\theta \text{ in} \\
& \text{let } \dot{x} = l\dot{\theta}\cos\theta \text{ in} \\
& \text{let } \dot{y} = -l\dot{\theta}\sin\theta \text{ in} \\
& \quad \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy : \mathbb{R}\langle 1, 0\rangle
\end{aligned}$$

Free theorems:

Energy conservation

Invariance under $z : \mathsf{Z}$ not smooth $\Rightarrow$ no conserved property

Damped spring

$$\Theta = t_t : \mathsf{T}(1)$$
$$\Gamma = k : \wr \mathbb{R}\langle 1, 0\rangle \int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t + t_t\rangle, x : \mathbb{R}\langle \exp(-t_t), 0\rangle, \dot{x} : \mathbb{R}\langle \exp(-t_t), 0\rangle$$

$$L = \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\right)\exp(t) : \mathbb{R}\langle 1, 0\rangle$$

Damped spring

$$\begin{aligned}\Theta &= t_t : \mathsf{T}(1) \\ \Gamma &= \ k : \wr\mathbb{R}\langle 1, 0\rangle\int \\ \Delta &= t : \mathbb{R}\langle 1, t_t + t_t\rangle, x : \mathbb{R}\langle \exp(-t_t), 0\rangle, \dot{x} : \mathbb{R}\langle \exp(-t_t), 0\rangle\end{aligned}$$

$$L = \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\right)\exp(t) : \mathbb{R}\langle 1, 0\rangle$$

Conservation Law:

$$\frac{d}{dt}\left[\left(\frac{1}{2}x\dot{x} + \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right)e^t\right] = 0$$

Conclusions…

What's been done:

A type system for Lagrangians

Types ⇒ Free theorems ⇒ Noether's Thm. ⇒ Conservation Laws