Why nonassociativity?
Actually associative
Toolbox
Existence?
Reflections

A category theoretic framework for noncommutative and nonassociative geometry

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Overview

1. Why nonassociativity?
2. Actually associative
3. Toolbox
4. Existence?
5. Reflections

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A category theoretic framework for noncommutative and nonassociative geometries
Why nonassociativity?

- Flux compactifications of closed string theory
- Coordinate algebra probed by closed strings winding and propagating in $R$-flux compactification is noncommutative and nonassociative
  

\[
[x^i, x^j] = \frac{i \ell_s^4}{3 \hbar} R^{ijk} \partial_k \\
[x^i, x^j, x^k] = \ell_s^4 R^{ijk}
\]
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Actually associative

[Mylonas, Schupp, Szabo, 2012, 2014] found $g$ with $F \in Ug \otimes Ug$ (invertible, normalised) such that

- $\ast = \mu \circ F^{-1}$ gives
  
  $$[x^i, x^j]_\ast = x^i \ast x^j - x^j \ast x^i = \frac{i \ell^4} {3 \hbar} R^{ijk} \partial_k,$$

  $$[x^i, x^j, x^k]_\ast = [[x^i, x^j]_\ast, x^k]_\ast + \text{cycl.} = \ell^4 R^{ijk}.$$

- $R_F = F_2 F^{-1} = \sum R^{(1)}_F \otimes R^{(2)}_F$ gives
  
  $$[x^i, x^j]_{R_F} = x^i \ast x^j - \sum R^{(2)}_F(x^i) \ast R^{(1)}_F(x^j) = 0.$$

- $\phi_F = (1 \otimes F) \cdot (\text{id}_H \otimes \Delta)(F) \cdot (\Delta \otimes \text{id}_H)(F^{-1}) \cdot (F^{-1} \otimes 1)$ gives
  
  $$[x^i, x^j, x^k]_{R_F, \phi_F} = 0.$$

Thm $(Ug, R_F, \phi_F)$ is a triangular quasi-Hopf algebra.
Abstract this: consider arbitrary algebra $A$, arbitrary triangular quasi-Hopf algebra $(H, R,\phi)$ such that, for $a, a', a'' \in A$

$$[a, a']_R = 0$$
$$[a, a', a'']_{R,\phi} = 0$$

Find that

- $A$ is a commutative and associative algebra object in $^H\mathcal{M}$, the representation category of $H$

Strategy

- Consider the representation category of an arbitrary triangular quasi-Hopf algebra $H$
- Develop notions of geometry on one algebra object and its bimodule objects internal to such a representation category
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Representation category of triangular quasi-Hopf algebra, $H\mathcal{M}$

- Objects: All bounded $\mathbb{Z}$-graded left $H$-modules with $\triangleright : H \otimes V \to V$
- Morphisms: All $H$-equivariant degree preserving graded $k$-module maps

$$R = \sum R^{(1)} \otimes R^{(2)}$$
$$\phi = \sum \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$$

$H\mathcal{M}$ is Closed Symmetric Monoidal category:

- Monoidal functor $\otimes$
  - associator: $\Phi : (v \otimes w) \otimes x \mapsto \sum (\phi^{(1)} \triangleright v) \otimes ((\phi^{(2)} \triangleright w) \otimes (\phi^{(3)} \triangleright x))$
  - symmetric braiding: $\tau : v \otimes w \mapsto \sum (-1)^{|v||w|} (R^{(2)} \triangleright w) \otimes (R^{(1)} \triangleright v)$
- Internal hom functor $\text{hom}$

$$\cdot \otimes V \dashv \text{hom}(V, \cdot) : H\mathcal{M} \to H\mathcal{M} , \forall \ V \in \text{Obj}(H\mathcal{M})$$

with currying natural bijection $\zeta$

$$\zeta_{V,W,X} : \text{Hom}_{H\mathcal{M}}(V \otimes W, X) \to \text{Hom}_{H\mathcal{M}}(V, \text{hom}(W, X))$$

NB Internal homomorphisms are objects in $H\mathcal{M}$
Commutative algebra object and its bimodule objects in $^H\mathcal{M}$

**NCA space:** A commutative algebra object in $^H\mathcal{M}$ is a triple $(A, \mu_A, \eta_A)$ consisting of an $^H\mathcal{M}$-object $A$ together with two $^H\mathcal{M}$-morphisms $\mu_A : A \otimes A \to A$ (product) and $\eta_A : I \to A$ (unit) such that

$$
\Phi_A, A, A \quad \Phi_A, A, A
\downarrow \quad \downarrow
A \otimes (A \otimes A) \quad A \otimes (A \otimes A)
\mu_A \quad \mu_A
\downarrow \quad \downarrow
A \otimes A \quad A \otimes A
\mu_A \quad \mu_A
\downarrow \quad \downarrow
A \quad A
$$

and $\mu_A = \mu_A \circ \tau$.

**NCA vector bundles:** A symmetric $A$-bimodule object in $^H\mathcal{M}$ is is a triple $(V, l_V, r_V)$ consisting of an $^H\mathcal{M}$-object $V$ with two $^H\mathcal{M}$-morphisms $l_V : A \otimes V \to V$ (left $A$-action) and $r_V : V \otimes A \to V$ (right $A$-action) satisfying the usual bimodule relations and also $l_V = r_V \circ \tau$ and $r_V = l_V \circ \tau$.

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A category theoretic framework for noncommutative and nonassociative geometries.
Internal hom-objects in $H\mathcal{M}$

Regard all geometric quantities as internal hom-objects.

- Indispensable when geometric quantities are dynamical (e.g. metric field in gravity or curvature field of connection in Yang-Mills theory)
- Richer framework for nonassociative geometry than [Beggs, Majid, 2010], configuration space of nca space as large as corresponding classical space

For any braided closed monoidal category there are canonical
- $\text{ev} : \text{hom}(V, W) \otimes V \to W$, $L(v)$ replaced by $\text{ev}(L \otimes v)$
- $\bullet : \text{hom}(V, W) \otimes \text{hom}(X, V) \to \text{hom}(X, W)$, $L \circ L'$ replaced by $L \bullet L'$
- $\otimes : \text{hom}(V, W) \otimes \text{hom}(X, Y) \to \text{hom}(V \otimes X, W \otimes Y)$, $L \otimes L'$ replaced by $L \otimes L'$
- $\eta : k \to \text{end}(V)$

morphisms for internal hom-objects.

Results:
1. $(\text{end}(V), \bullet, \eta)$ is an algebra object in $H\mathcal{M}$
2. Defining $[\cdot, \cdot] := \bullet - \bullet \circ \tau$, $(\text{end}(V), [\cdot, \cdot])$ is a Lie algebra object in $H\mathcal{M}$
Derivations of the commutative algebra object in $^H_M$

**Guiding principle:** Define geometry via universal constructions, e.g. equalisers (sensible since geometrical concepts are ‘universal’ in the sense that we can speak of the Leibniz rule for e.g.)

**Leibniz rule for derivation** $X$: $X(ab) = (-1)^{|a||X|}a X(b) + X(a)b$ for all $a, b \in A$

**Observation:** $X(a-) - (-1)^{|a||X|}a X(-) = X(a) -$. Content of Leibniz rule can be captured by two parallel morphisms

\[
\begin{align*}
\text{end}(A) \otimes A & \xrightarrow{[\cdot, \cdot] \circ (\text{id} \otimes \zeta(\mu))} \text{end}(A) \\
& \xrightarrow{\zeta(\mu) \circ \text{ev}} \text{end}(A)
\end{align*}
\]

$(a \rightarrow \zeta(\mu)(a) \in \text{end}(A))$. Then we can use the currying natural bijection to obtain the derivations as the equaliser

\[
\begin{align*}
\text{der}(A) & \xrightarrow{\zeta([\cdot, \cdot])} \text{end}(A) \\
& \xrightarrow{\zeta(\hat{\mu} \circ \text{ev})} \text{hom}(A, \text{end}(A))
\end{align*}
\]

**Result:** $(\text{der}(A), [\cdot, \cdot])$ is a Lie algebra object in $^H_M$
Building up to the notion of connection

Need

- exterior derivative (differential calculus)
- configuration space for affine space of connections
Differential calculus in $^HM$

Need an $H$-invariant, nilpotent derivation

- $k[1] = \bigoplus_n k[1]_n \in \text{Obj}(^HM), k[1]_1 = k, k[1]_n = 0$ for all $n \neq 1$
- $d : k[1] \rightarrow \text{der}(A)$ (H-invariant)
- $k[1] \otimes k[1] \xrightarrow{d \otimes d} \text{der}(A) \otimes \text{der}(A) \xrightarrow{[\cdot, \cdot]} \text{der}(A)$ is 0 (nilpotent)

In summary $d(c) \in \text{der}(A)$ for any $c \in k$, and $d(c) \cdot d(c) = 0$. Note $d(c) = c d(1)$.

Differential calculus

$(A, d(1))$
for the configuration space of the affine space of connections

Idea: Ought to be the ‘internal version of $A$-bimodule morphisms’

$$L(a v) = (-1)^{|a||L|} a L(v)$$

becomes

$$L \cdot \hat{i}_W(a) - (-1)^{|a||L|} R^{(2)} \circ \hat{i}_V(a) \cdot R^{(1)} \triangleright L = 0$$

Define

$$[\cdot, \cdot] := \bullet \circ (id \otimes \hat{i}) - \bullet \circ (\hat{i} \otimes id) \circ \tau : \text{hom}(V, W) \otimes A \longrightarrow \text{hom}(V, W)$$

and formulate property in terms of an equaliser:

$$\text{hom}_A(V, W) \longrightarrow \text{hom}(V, W) \longrightarrow \text{hom}(A, \text{hom}(V, W))$$

$$\zeta([\cdot, \cdot]) \quad 0$$

**Result:** $U \subset \text{hom}_A(V, W)$ if and only if $[L, a] = 0 \forall L \in U$
Connections on noncommutative and nonassociative vector bundles...

$$\nabla(a \, v) = (-1)^{|a|} a \nabla(v) + d(a) \otimes v \cong (d(a) \, v)$$

Connections form an affine space, but our category is one of linear spaces. Obtain linear space of which connections is a subcollection in much same way as for derivations

$$(\text{end}(V) \times k[1]) \otimes A \xrightarrow{[\cdot, \cdot] \circ (pr_1 \otimes \text{id})} \text{end}(V),$$

Define the continuous connections to be the equaliser

$$\text{con}(V) \xrightarrow{\zeta([\cdot, \cdot] \circ (pr_1 \otimes \text{id}))} \text{hom}(A, \text{end}(V))$$

Result: $\{ (L, 1) \} \subset \text{con}(V)$ is an affine space over $\{ (L, 0) \} \subset \text{con}(V) = \text{end}_A(V)$, “the usual affine space of connections”
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Products of $k$-connections in $\mathcal{H}M$

$\otimes_A$ defined via coequaliser

\[
\begin{array}{ccc}
(V \otimes A) \otimes W & \xrightarrow{(id_V \otimes l_W) \circ \Phi_{V,A,W}} & V \otimes W & \xrightarrow{r_V \otimes id_W} & V \otimes_A W.
\end{array}
\]

Can we define a connection on $V \otimes_A W$ from connections on $V$ and $W$?

subtlety: require a fibered product:

\[
\begin{array}{ccc}
\text{con}(V) \times_{k[1]} \text{con}(W) & \xrightarrow{} & \text{con}(W) \\
\downarrow & & \downarrow \text{pr}_2 \\
\text{con}(V) & \xrightarrow{\text{pr}_2} & k[1]
\end{array}
\]

Theorem (GEB, Schenkel, Szabo) (sum of $k$-connections)

Given two $A$-bimodule objects $V$, $W$ in $\mathcal{H}M$ there is an $\mathcal{H}M$-morphism

\[
\boxdot : \text{con}(V) \times_{k[1]} \text{con}(W) \xrightarrow{} \text{con}(V \otimes_A W),
\]

\[
((\nabla_V, c), (\nabla_W, c)) \xmapsto{} (\nabla_V \otimes 1 + 1 \otimes \nabla_W, c).
\]
**k-connections on dual modules in $H \mathcal{M}$**

Can we define a connection on $\text{hom}_A(V, W)$ from connections on $V$ and $W$?

**Theorem (GEB, Schenkel, Szabo)**

Given two $A$-bimodule objects $V, W$ in $H \mathcal{M}$ there is an $H \mathcal{M}$-morphism

$$\text{ad}_\bullet : \text{con}(W) \times_{k[1]} \text{con}(V) \rightarrow \text{con}(\text{hom}_A(V, W)),$$

$$((\nabla W, c), (\nabla V, c)) \mapsto (\mathcal{L}(\nabla W) - \mathcal{R}(\nabla V), c).$$

**Cor:** For dual modules $V^\vee := \text{hom}_A(V, A)$ we can define the dual connection

$$\nabla^\vee := \text{ad}_\bullet((\nabla, 1), (d(1), 1)) \in \text{con}(V^\vee)$$
We can build such a toolbox but does this toolbox exist?

Result: There is an equivalence of closed braided monoidal categories

\[ \mathcal{F} : H \mathcal{M} \rightarrow H_{F} \mathcal{M} \]

where \( H_{F} \) is obtained from \( H \) using a cochain twist, \( F \) (Kassel).

- Interested in the case \( H \) is a Hopf algebra \((R = 1 \otimes 1, \phi = 1 \otimes 1 \otimes 1)\) where tools exist.

- Note: \( F \) is a cochain twist in motivating example.
Reflections

- Framework for noncommutative and nonassociative geometry.
- Description fits naturally into the context of a certain closed braided monoidal category, the representation category of a quasi-Hopf algebra.
- Twist deformation quantisation explains the present noncommutativity and nonassociativity and is simply a functor between two such categories.
- Exploring the syntax of category theory enables one to express well known geometrical concepts in terms of universal constructions internal to the category.

Further results:

- Able to provide action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces
- Provides a local description. For global description use functor of points which also offers interpretation for what noncommutative and nonassociative space is.
Summary

- Why nonassociativity (flux compactifications of string theory)
- Actually associative (in the correct category)
- Toolbox for an arbitrary triangular quasi-Hopf algebra (derivations, connections)
- Existence due to an equivalence of categories
- Reflections
Thank you

questions and comments