COLLECTIVE QUANTUM GAMES

Ramón Alonso-Sanz

ramon.alonso@upm.es

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CLASSICAL GAME THEORY

Two-person, Non-zero sum games

		B: C	B: D
A	C	R, R	S, $\mathfrak{T}$
A	D	$\mathfrak{T}$, S	P, P

$\mathfrak{T} > R > P > S$

INDEPENDENT players

	y	1-y
x	xy	x(1-y)
1-x	(1-x)y	(1-x)(1-y)

CORRELATED games

	π_{11}	π_{12}
	π_{21}	π_{22}

Prisoner's Dilemma (PD)

Fair and Symmetric

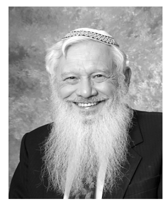
INDEPENDENT probabilistic strategies: $\mathbf{x} = (x, 1-x)'$, $\mathbf{y} = (y, 1-y)' \to \mathbf{\Pi} = \mathbf{x}\mathbf{y}'$

Expected payoffs: $p_A(x,y) = Rxy + Sx(1-y) + \mathfrak{T}(1-x)y + P(1-x)(1-y)$

$p_B(x,y) = Rxy + \mathfrak{T}x(1-y) + S(1-x)y + P(1-x)(1-y)$

(x,y) in **Nash Equilibrium (NE)**: x **is the best response to** y, y **is the best response to** x.

PD : (0,0) ≡ (D,D) **unique NE**



CORRELATED games : **JOINT probability distribution** $\mathbf{\Pi} = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$

Expected payoffs: $p_A = R\pi_{11} + S\pi_{12} + \mathfrak{T}\pi_{21} + P\pi_{22}$

$p_B = R\pi_{11} + \mathfrak{T}\pi_{12} + S\pi_{21} + P\pi_{22}$

Social Welfare Solution (SWS) : Maximization of the sum of the payoffs of both players.

PD : (1,1) ≡ (C,C), $\pi_{11} = 1$ **unique SWS** $\to p_A + p_B = 2R \; (\leftarrow \mathfrak{T} + S < 2R)$

QUANTUM approach EWL model[1]

PROBABILITY AMPLITUDES $\in \mathbb{C}$

$$\Pi = \begin{pmatrix} |\Psi_1|^2 & |\Psi_2|^2 \\ |\Psi_3|^2 & |\Psi_4|^2 \end{pmatrix} \overset{\text{Born}}{\longleftarrow} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \equiv |\Psi\rangle = \hat{J}^\dagger(\hat{U}_A \otimes \hat{U}_B)\hat{J}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

INDEPENDENT Quantum Strategies ($\hat{U}\in$SU(2), Special Unitary operators: $|\hat{U}| = 1$, $\hat{U}^\dagger\hat{U} = I$)

$$\hat{U}(\theta,\alpha) = \begin{pmatrix} e^{i\alpha}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\alpha}\cos\frac{\theta}{2} \end{pmatrix} \qquad \begin{matrix} \theta \in [0,\pi] \\ \alpha \in [0,\pi/2] \end{matrix}$$

$$\mathbf{\hat{C}} = \hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad \boxed{\mathbf{\hat{D}} = \hat{U}(\pi,\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = i\sigma_y$$

ENTANGLED by: $\hat{J}(\gamma) = \cos\frac{\gamma}{2}\hat{C}\otimes\hat{C} + i\sin\frac{\gamma}{2}\hat{D}\otimes\hat{D}$ ***Entanglement Factor*** $\gamma \in [0,\pi/2]$

- $\gamma = 0$ **or** $\alpha_A = \alpha_B = 0$ **or** $\theta_A = \theta_B = \pi\ (\forall\gamma)$**:** $\mathbf{\Pi} = \begin{pmatrix} x \\ 1-x \end{pmatrix}\begin{pmatrix} y & 1-y \end{pmatrix}$ **FACTORIZABLE (SEPARABLE)**

$$x = \cos^2\frac{\theta_A}{2},\ y = \cos^2\frac{\theta_B}{2} \qquad \theta \textbf{ is the } \textbf{\textit{classical}} \textbf{ parameter}$$

$$\theta_A = \theta_B = \alpha_A = \alpha_B = 0 \to U_A = U_B = \hat{U}(0,0) = I = \hat{C} \to |\Psi\rangle = \hat{J}^\dagger(\hat{I}\otimes\hat{I})\hat{J}|00\rangle\ |00\rangle = \hat{J}^\dagger\hat{J}|00\rangle = \hat{I}|00\rangle = |00\rangle \to \Pi^{CC} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\theta_A = \theta_B = \pi \to U_A = U_B = \hat{D} = \hat{U}(\pi,\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to |\Psi\rangle = |11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}' \to \boxed{\Pi^{DD} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ \forall\gamma}$$

- $\theta_A = \theta_B = \mathbf{0}$**:** $\mathbf{\Pi} = \begin{pmatrix} 1-\pi_{22} & 0 \\ 0 & \pi_{22} = \sin^2(\alpha_A+\alpha_B)\sin^2\gamma \end{pmatrix}$ **NON-FACTORIZABLE** (**diagonal**$\to |lcc| = 1$)

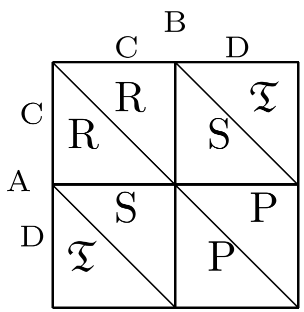
$$\star \quad \alpha_A = \alpha_B = \pi/2 \quad \hat{U}_A = \hat{U}_B = \boxed{\hat{U}(0,\frac{\pi}{2}) = \hat{Q}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \to \boxed{\mathbf{\Pi}^{QQ} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ \forall\gamma}$$

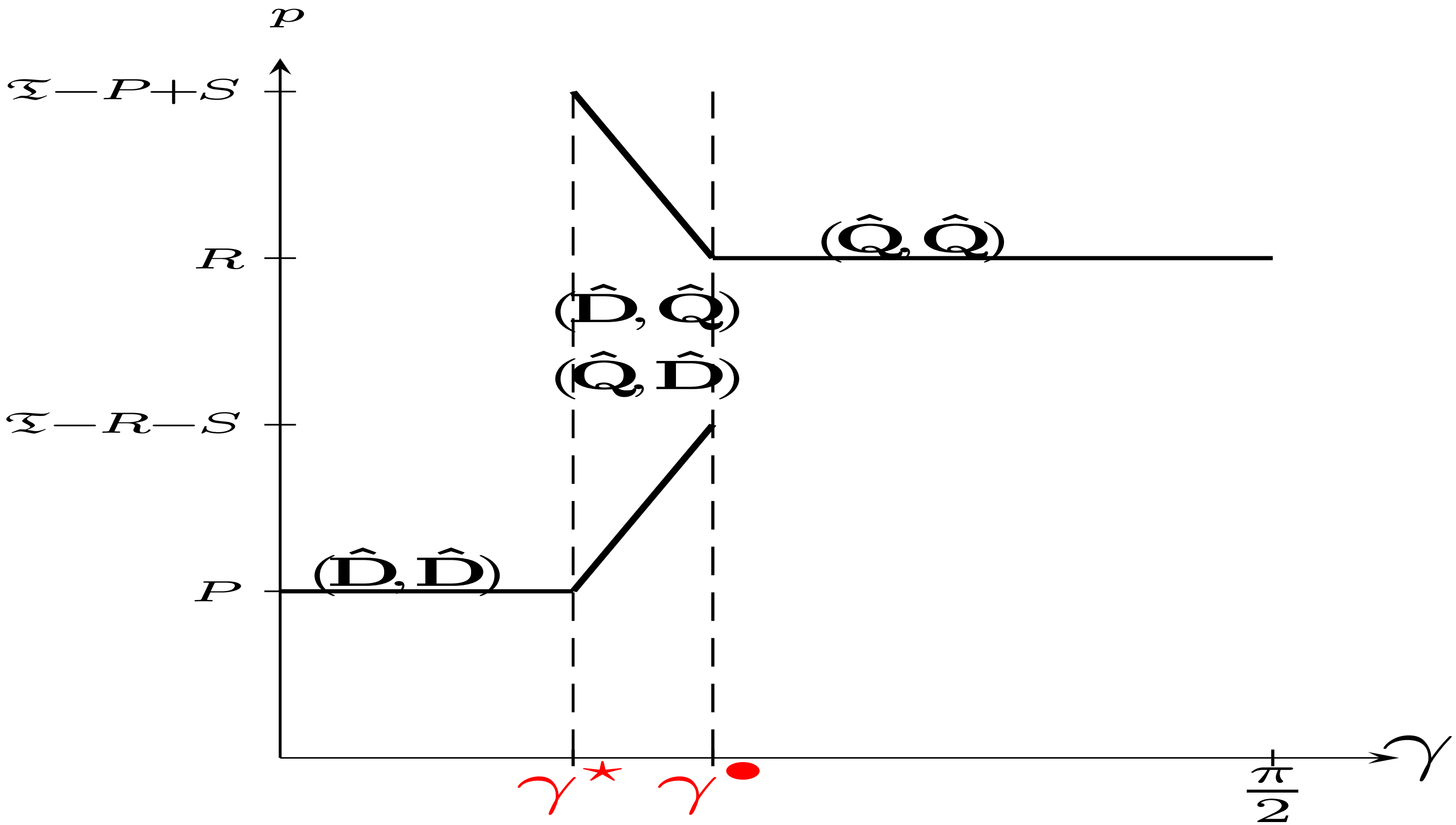
[1] **Eisert,J.,Wilkens,M.,Lewenstein,M.(1999). Quantum games and Quantum Strategies. Phys. Rev. Lett., 83,15,3077-3080.**

Nash Equilibrium (NE) in the QPD **EWL model**

$(\hat{Q},\hat{Q})$ unique NE for $\gamma > \gamma^{\bullet}$

$$\hat{Q} = \hat{U}(0, \frac{\pi}{2}), \quad \Pi^{QQ} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to p^{QQ}_{A,B} = \mathbf{R}$$





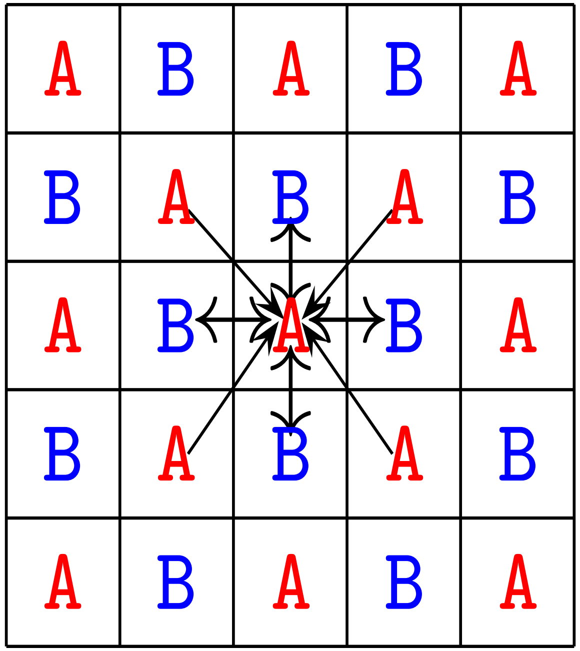
$$\gamma^{*} = \arcsin\sqrt{\frac{P-S}{\mathfrak{T}-S}} \qquad \gamma^{\bullet} = \arcsin\sqrt{\frac{\mathfrak{T}-R}{\mathfrak{T}-S}}$$

ITERATIVE COLLECTIVE QUANTUM Games

• Spatial Games (CA)

Each player occupies a cell (i, j) in a 2D $N \times N$ lattice.

A and B alternate in the site occupation (chessboard). Every player surrounded by four partners (A-B, B-A), and four mates (A-A, B-B) .

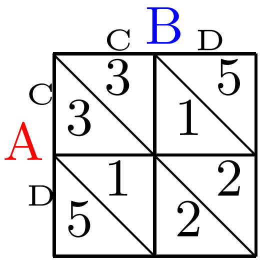


• Games on random Networks (NW)

Each node is connected at random with four mates and four partners.

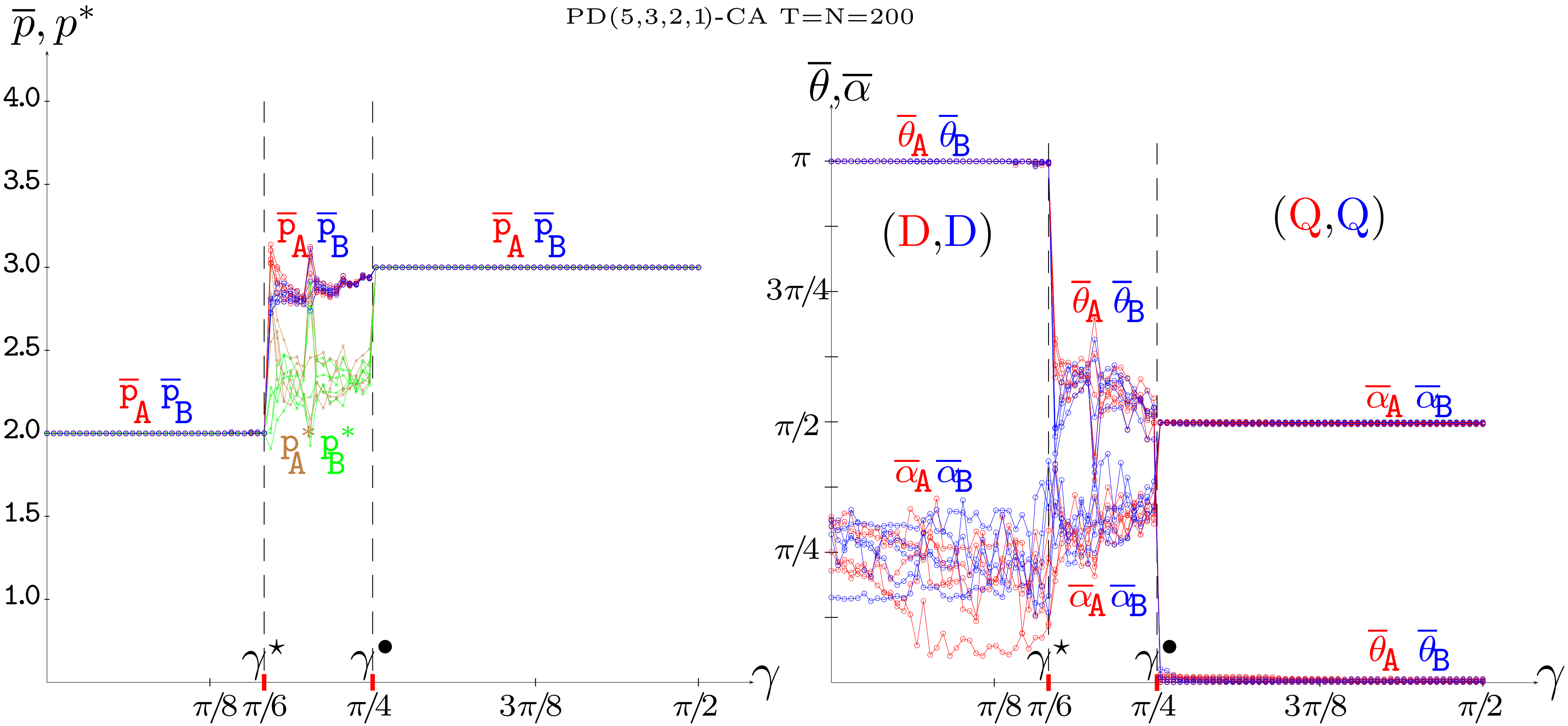
In each round (T) every player (i, j) :

- plays with his four adjacent partners. His payoff $p_{i,j}^{(T)}$ is the sum over these four games.
- Adopts the strategy parameters $(\theta_{k,l}^{(T)}, \alpha_{k,l}^{(T)})$ of the adjacent mate (including himself) with the highest $p^{(T)}$

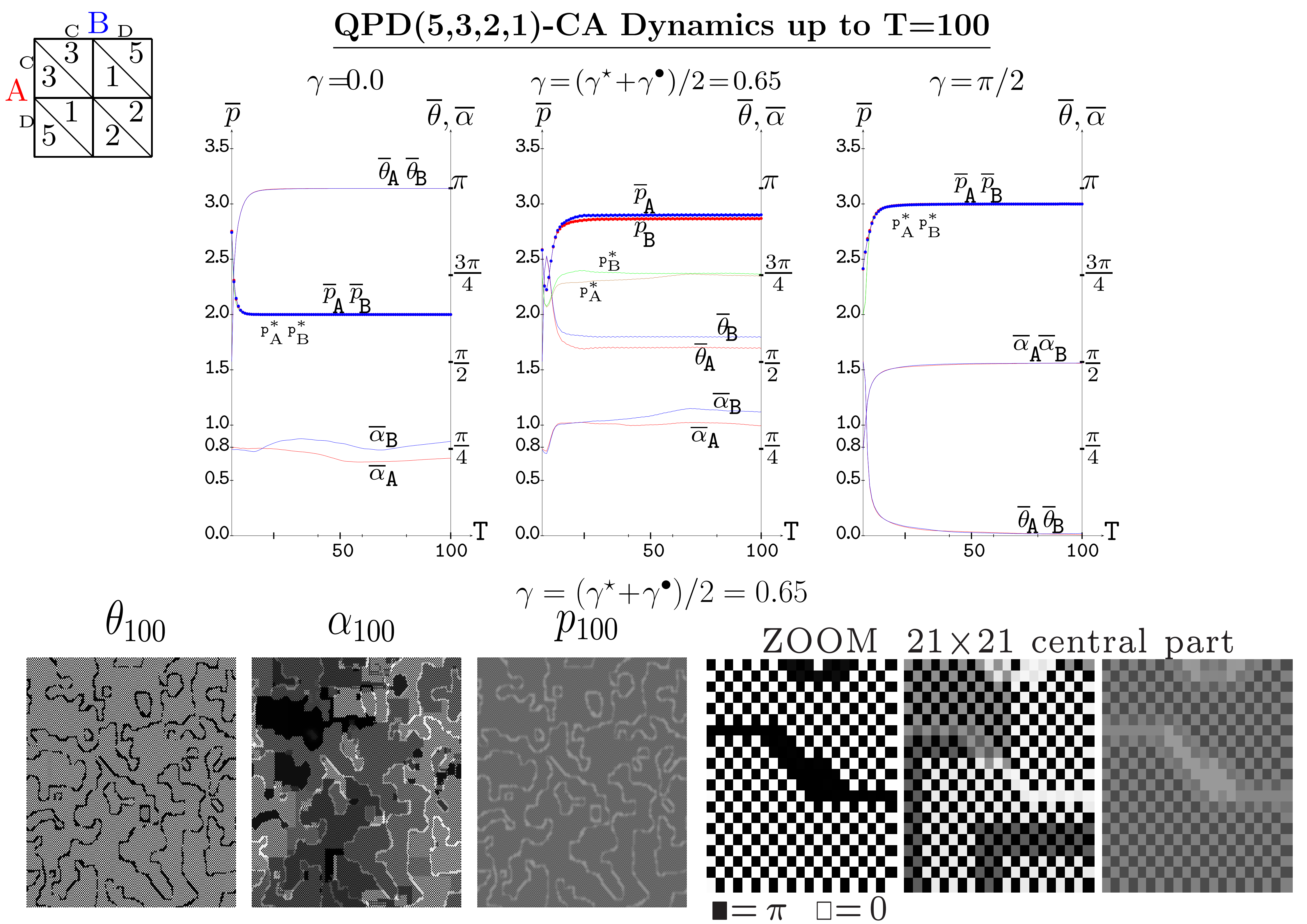


Spatial QPD(5,3,2,1)-CA

Five random initial assignments of the (θ, α) parameter values.
200×200 lattice with periodic boundary conditions. T=200.



$$U_A^* = \begin{pmatrix} e^{i\overline{\alpha}_A}\cos\frac{\overline{\theta}_A}{2} & \sin\frac{\overline{\theta}_A}{2} \\ -\sin\frac{\overline{\theta}_A}{2} & e^{-i\overline{\alpha}_A}\cos\frac{\overline{\theta}_A}{2} \end{pmatrix}, \quad U_B^* = \begin{pmatrix} e^{i\overline{\alpha}_B}\cos\frac{\overline{\theta}_B}{2} & \sin\frac{\overline{\theta}_B}{2} \\ -\sin\frac{\overline{\theta}_B}{2} & e^{-i\overline{\alpha}_B}\cos\frac{\overline{\theta}_B}{2} \end{pmatrix}$$


QPD(5,3,2,1)-CA Dynamics up to T=100
$\gamma = 0.0$
$\gamma = (\gamma^\star + \gamma^\bullet)/2 = 0.65$
$\gamma = \pi/2$
$\overline{p}$
$\overline{\theta}, \overline{\alpha}$
$\overline{\theta}_A \; \overline{\theta}_B$
$\overline{p}_A \; \overline{p}_B$
$p^*_A \; p^*_B$
$\overline{\alpha}_A \; \overline{\alpha}_B$
T
$\gamma = (\gamma^\star + \gamma^\bullet)/2 = 0.65$
θ_{100}
α_{100}
p_{100}
ZOOM 21×21 central part
$\blacksquare = \pi$ $\square = 0$

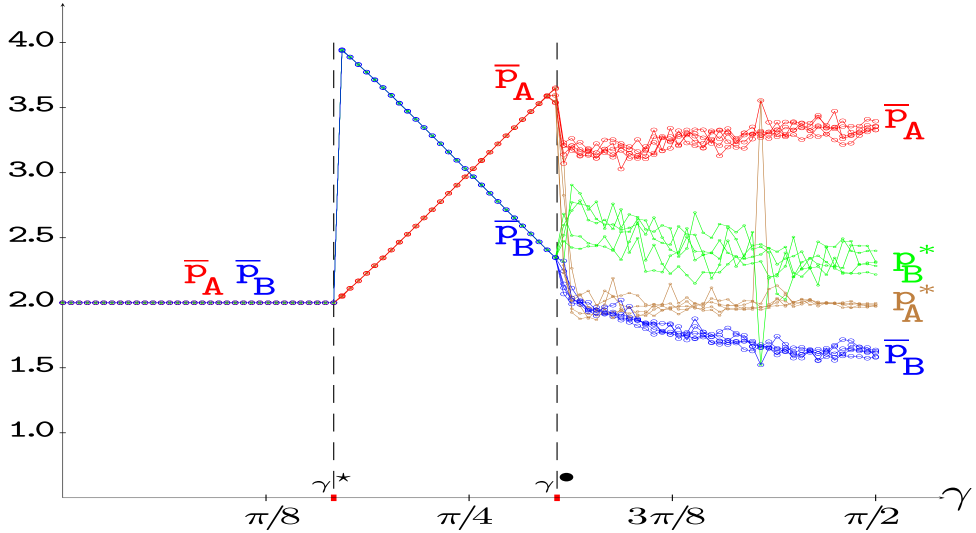
UNFAIR QPD-CA

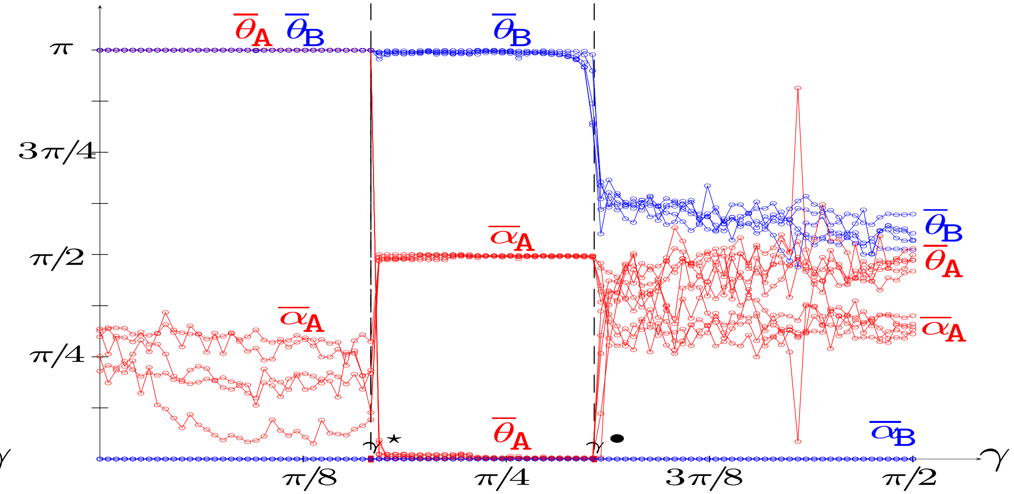
		B: C	B: D
A	C	3, 3	1, 5
	D	5, 1	2, 2

A may use Quantum $\hat{U}(\theta,\alpha)$

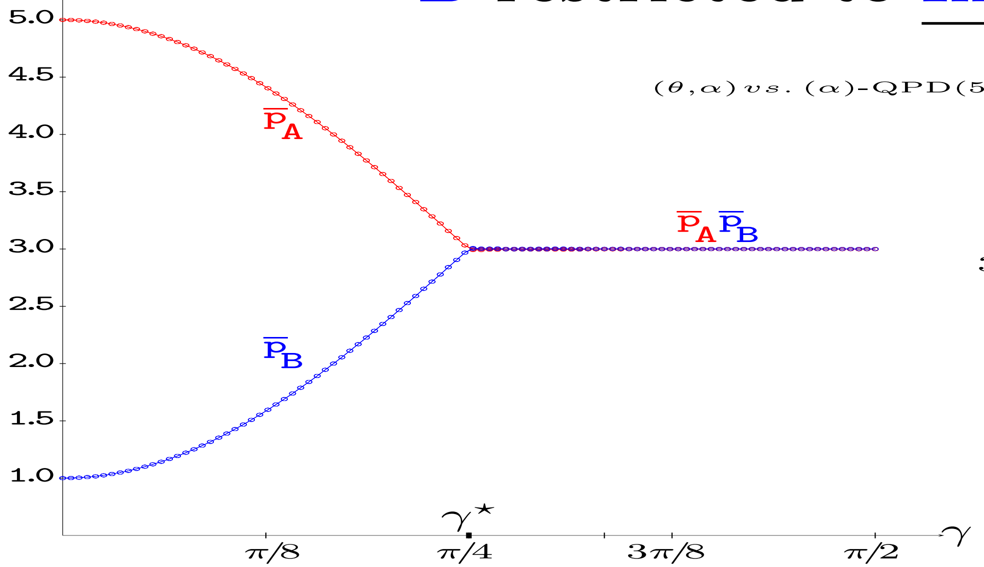
B restricted to Classical $\hat{U}(\theta,0)$

(θ,α)vs.(θ)-QPD(5,3,2,1)-CA T=N=200

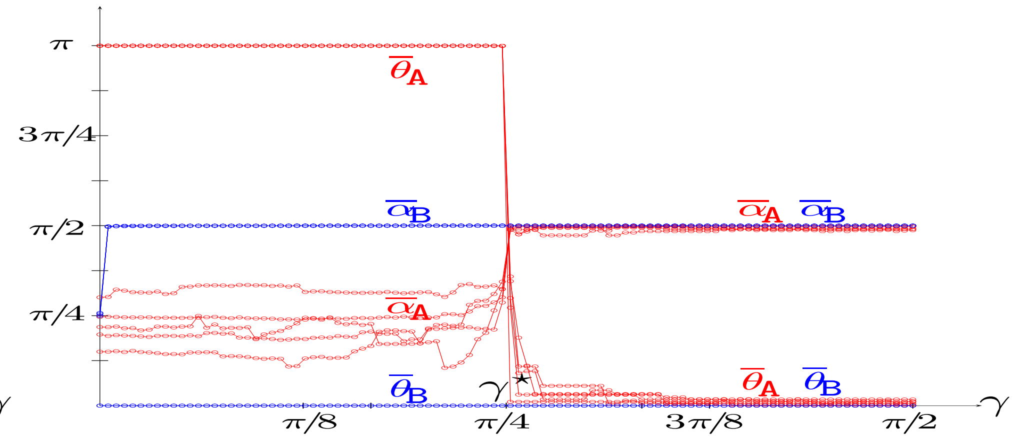




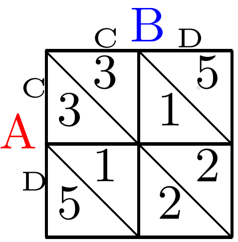
B restricted to Imaginary $\hat{U}(0,\alpha)$

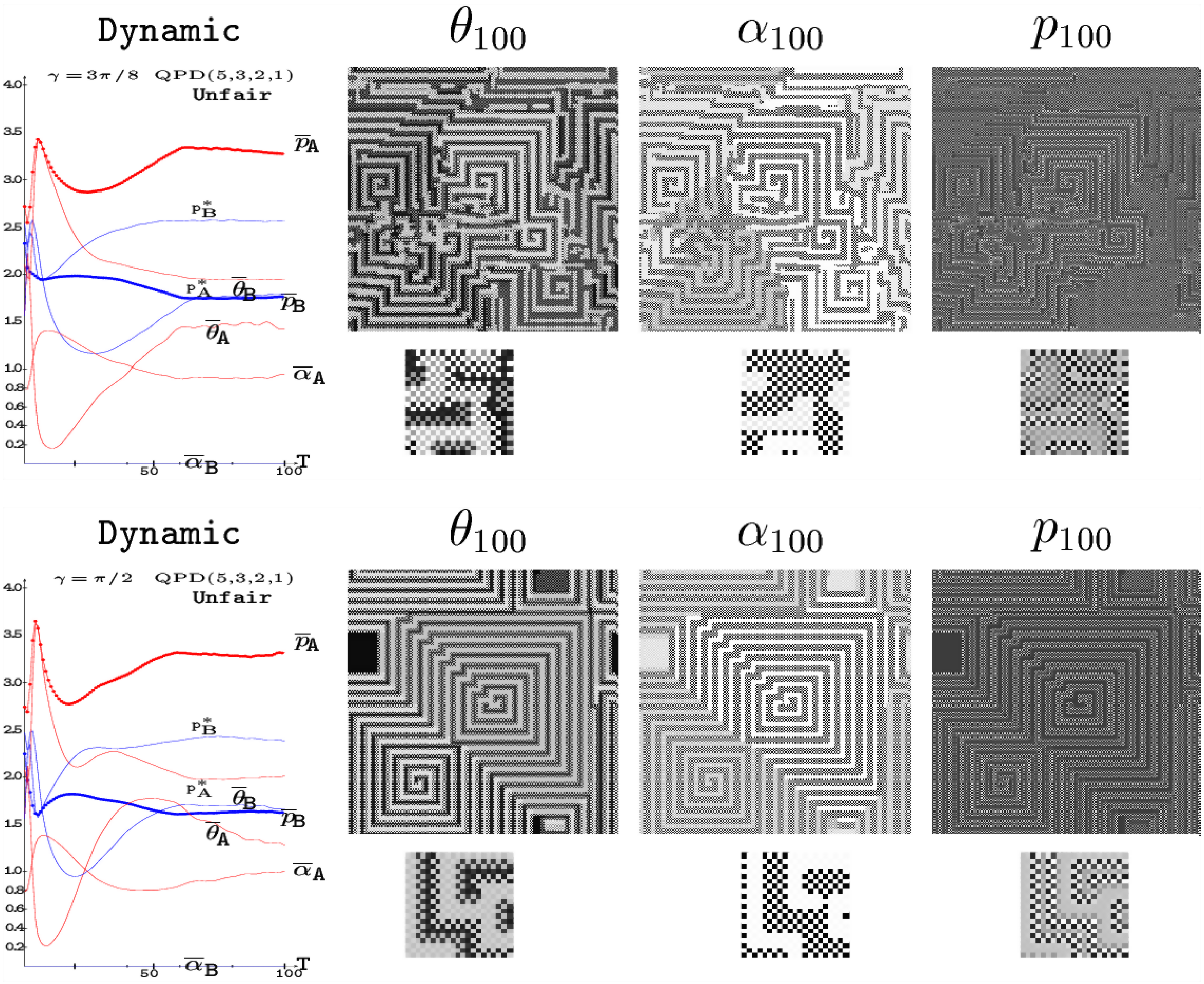


(θ,α)vs.(α)-QPD(5,3,2,1)-CA T=N=200

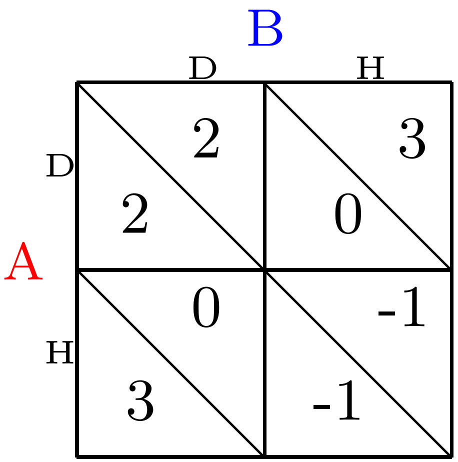


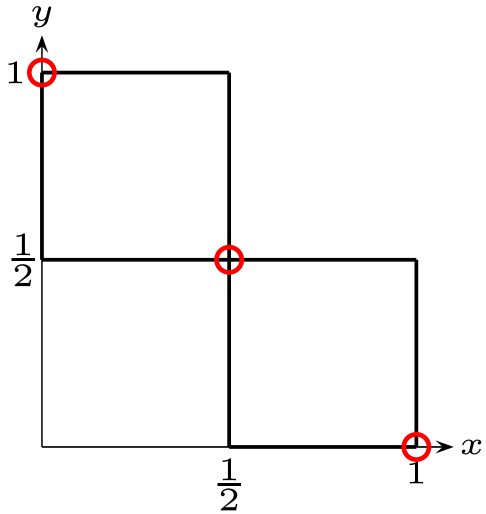
UNFAIR Quantum *vs.* Classic PD-CA

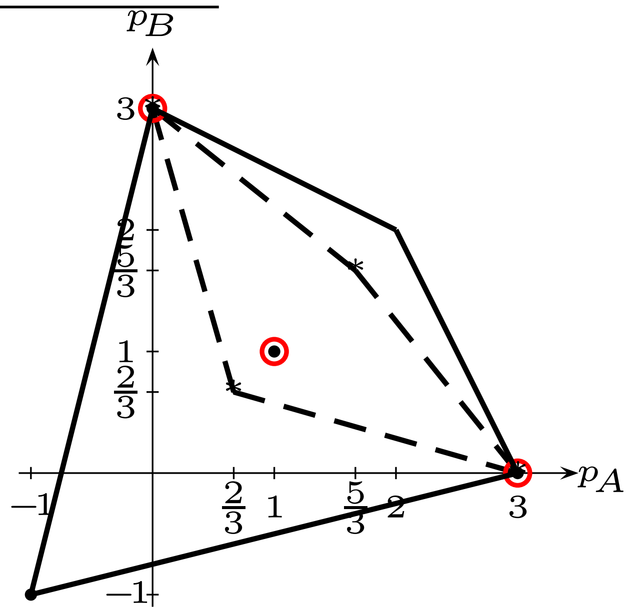




HAWK-DOVE (HD)

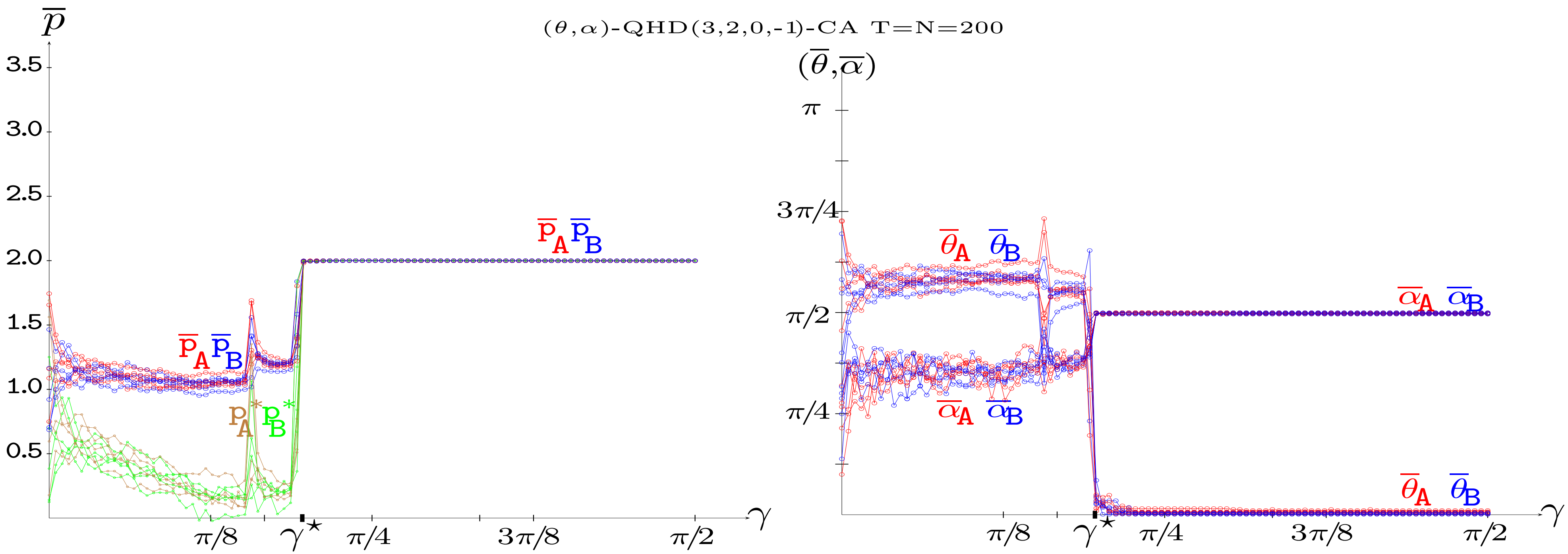




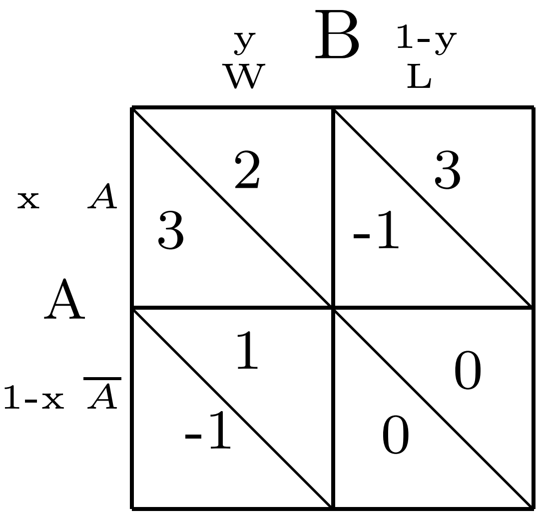


NE: • $(1,0) \equiv (D,H) \rightarrow p_A = 0\,,\ p_B = 3$ • $(0,1) \equiv (H,D) \rightarrow p_A = 3\,,\ p_B = 0$ • $(1/2,1/2) \rightarrow p_{AB} = 1$

SWS : (1,1) ≡ (D,D), $\pi_{11} = 1 \rightarrow p_A + p_B = 4$ $\star (1,1) \equiv$ **(H,H) NO NE**

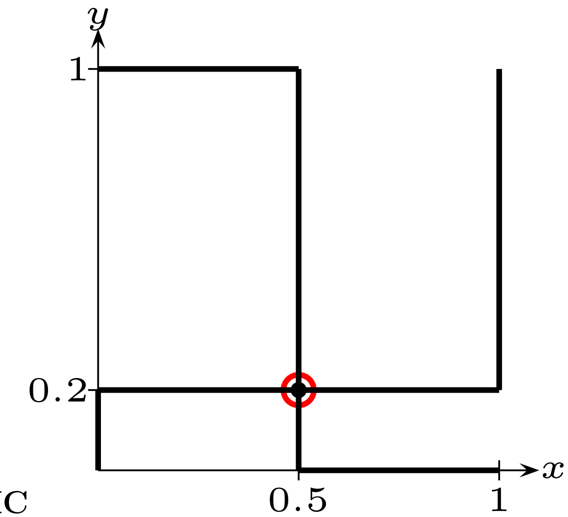


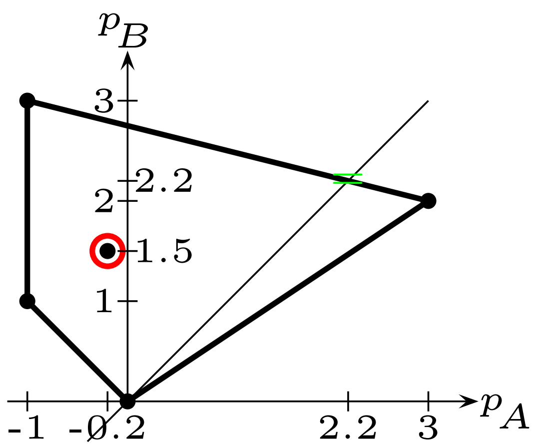
The SAMARITAN'S DILEMMA (SD)



UNFAIR, ASYMMETRIC

DISCOORDINATION



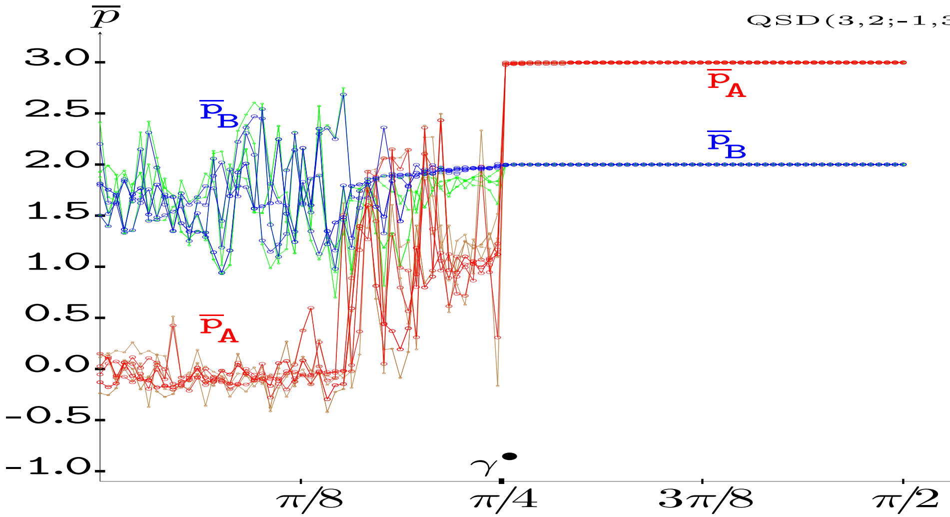


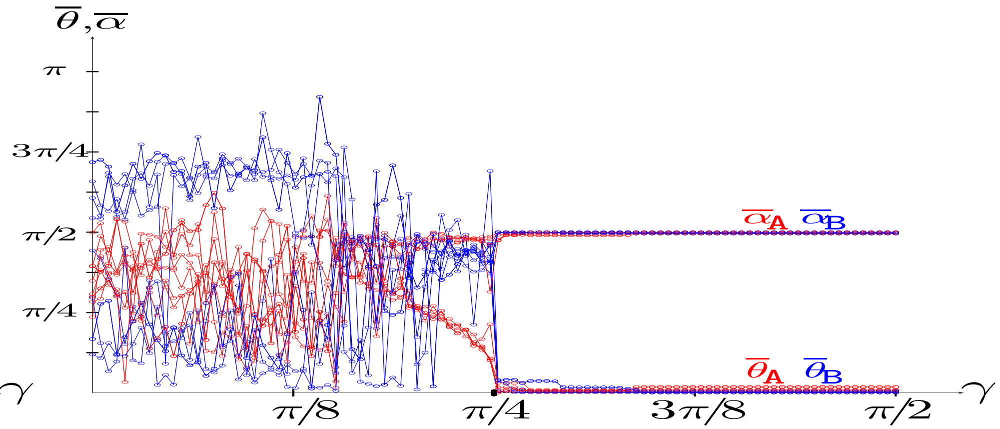
$\frac{3-1-1+0}{4}=0.25$ $\frac{2+3+1+0}{4}=1.5$

NE : $(\mathbf{x^* = 1/2}, \mathbf{y^* = 1/5})$, $(\mathbf{p_A = -0.2}, \mathbf{p_B = 1.5})$ (1,1) ≡ (A,W), $\pi_{11}=1$ **unique SWS** → $p_A + p_B = 5$

Spatial QSD-CA

QSD(3,2;-1,3;-1,1)-CA T=N=200

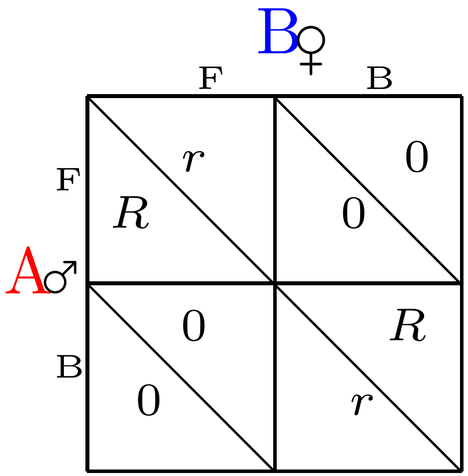


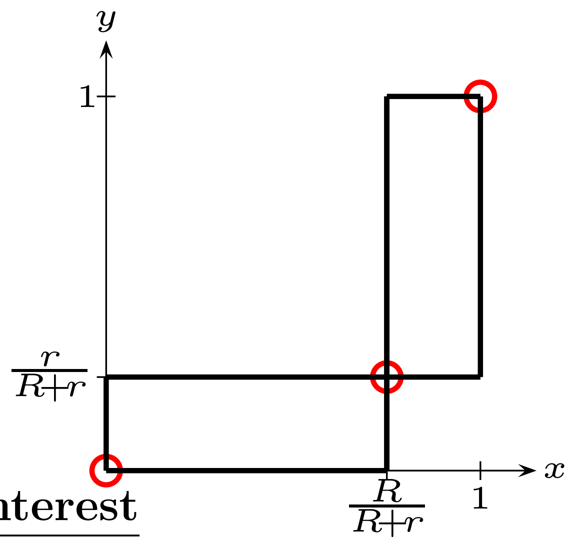


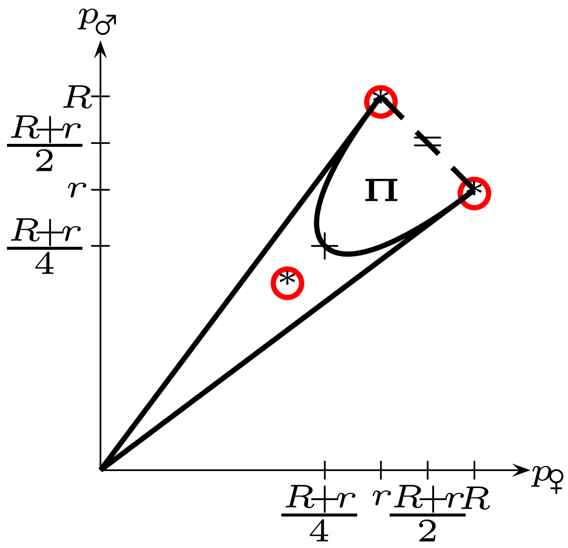
Alonso-Sanz,R.,Situ,H.(2017). A quantum Samaritan's dilemma cellular automaton. *Royal Society Open Science*, 4,6,160669.

BATTLE OF THE SEXES (BOS) $R > r > 0$

Fair, Asymmetric







Coordination, conflicting interest

INDEPENDENT probabilistic strategies

$$p_{♂}(x,y) = Rxy + r(1-x)(1-y)$$

$$p_{♀}(x,y) = rxy + R(1-x)(1-y)$$

$$y = 1-x \to p_{♂} = p_{♀} = (R+r)(1-x)x$$

$$x = y = 1/2 \to p^{+}_{max} = (R+r)/4$$

NE: $\bullet\ (1,1) \equiv (F,F) \to p_{♂} = R\,,\ p_{♀} = r$ $\quad \bullet\ (0,0) \equiv (B,B) \to p_{♂} = r\ ,\ p_{♀} = R$

$$\bullet\ x^* = \frac{R}{R+r},\, y^* = \frac{r}{R+r} \to p_{♂,♀} = \frac{Rr}{R+r} < r$$

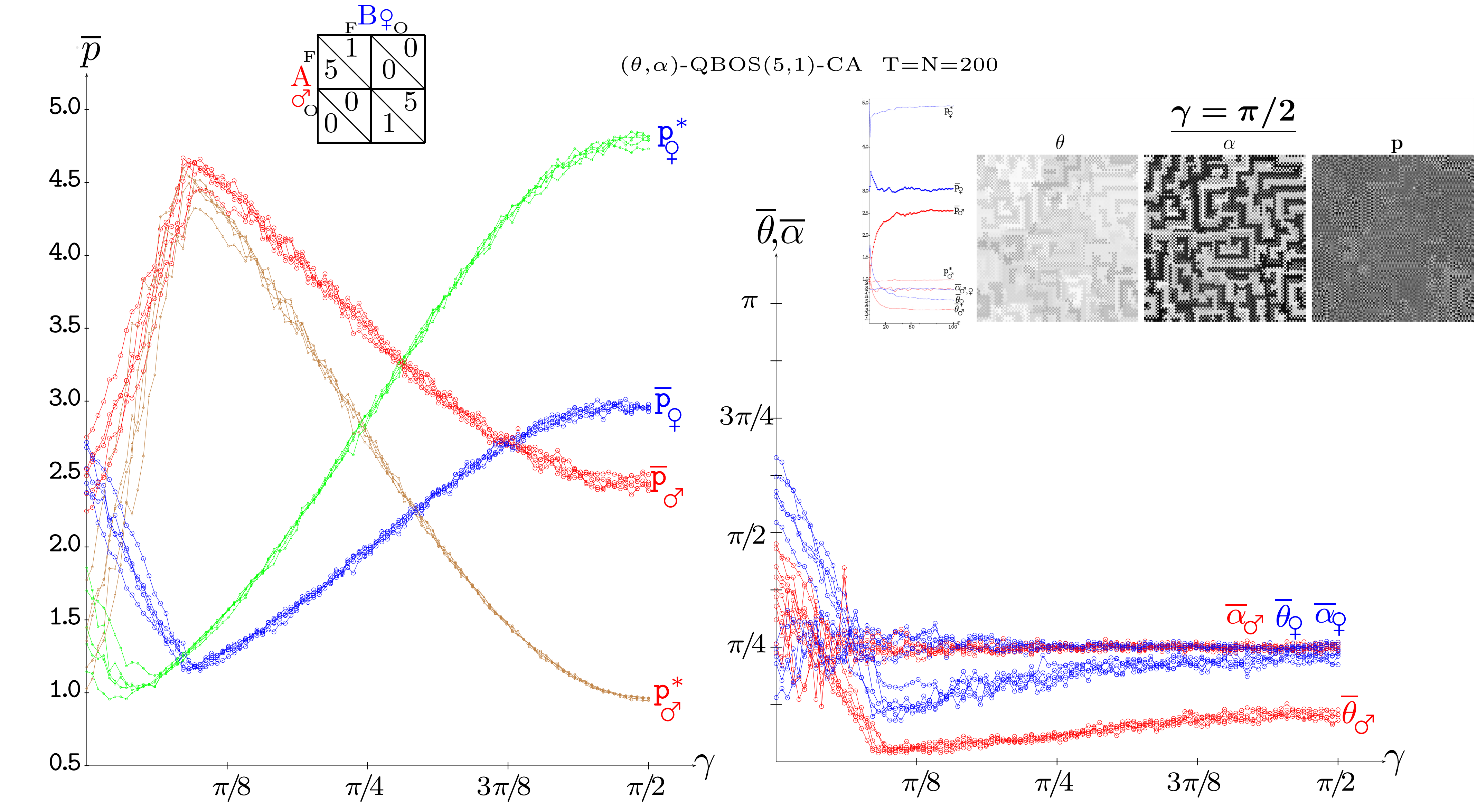
CORRELATED games

$$p_{♂} = \pi_{11}R + \pi_{22}r$$

$$p_{♀} = \pi_{11}r + \pi_{22}R$$

$$\pi_{11} = \pi_{22} = \pi \to p_{♂} = p_{♀} = (R+r)\pi \qquad \pi = 1/2 \to p^{=}_{max} = (R+r)/2$$

SWS : (1,1) $\equiv$ (F,F), (0,0) $\equiv$ (B,B), $\pi_{11} + \pi_{22} = 1$, e.g., $\pi_{11} = 1, \pi_{22} = 1$ $\to p_{A} + p_{B} = R + r$

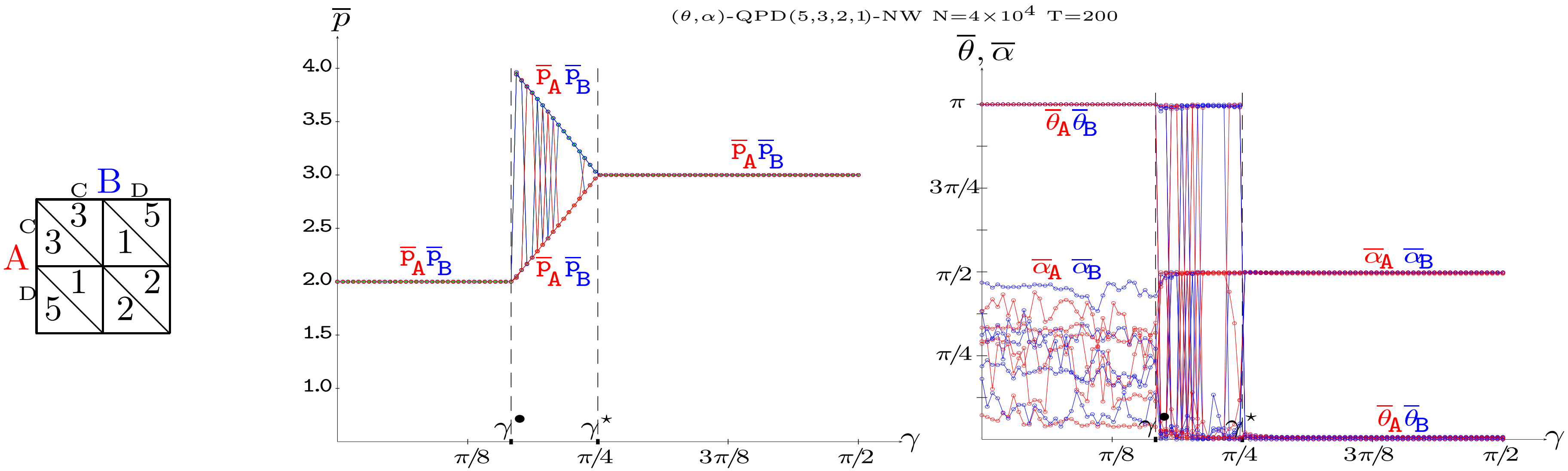


Alonso-Sanz,R.(2014). Variable entangling in a quantum battle of the sexes cellular automaton. Was,J.,Sirakoulis,G.C.,Bandini,S.(Eds.): ACRI-2014,LNCS,8751,125-135.

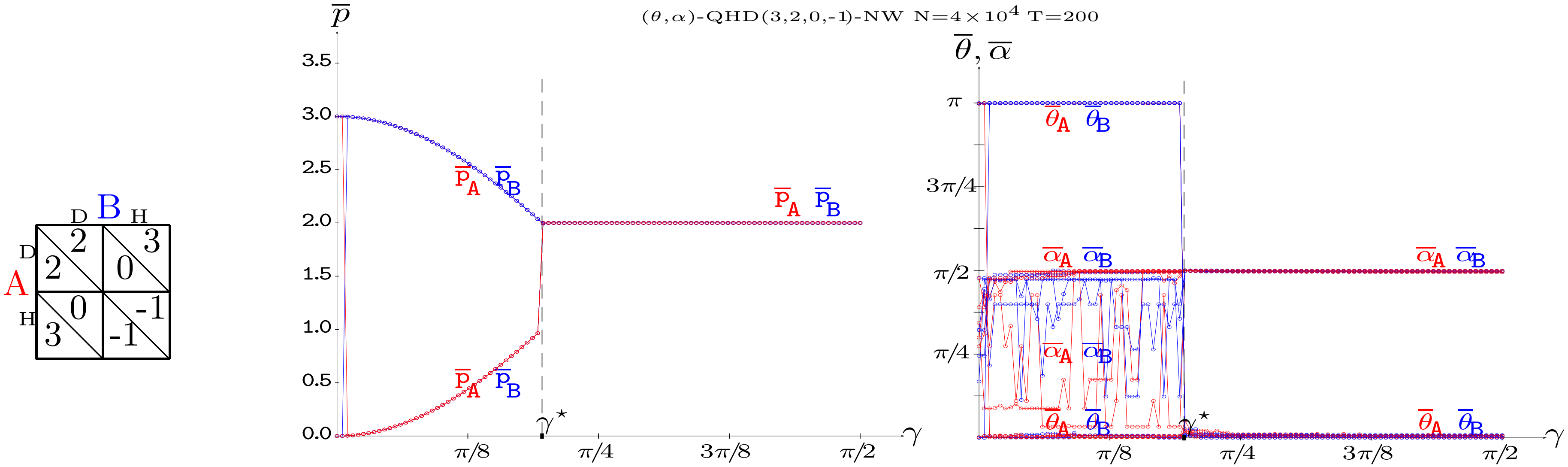
Alonso-Sanz,R.(2013). On a three-parameter quantum battle of the sexes cellular automaton. *Quantum Information Processing*, 12,5,1835-1850.

Alonso-Sanz,R.(2012). A quantum battle of the sexes cellular automaton. Proc. R. Soc. A, 468,3370-3383 .

QPD on NETWORK

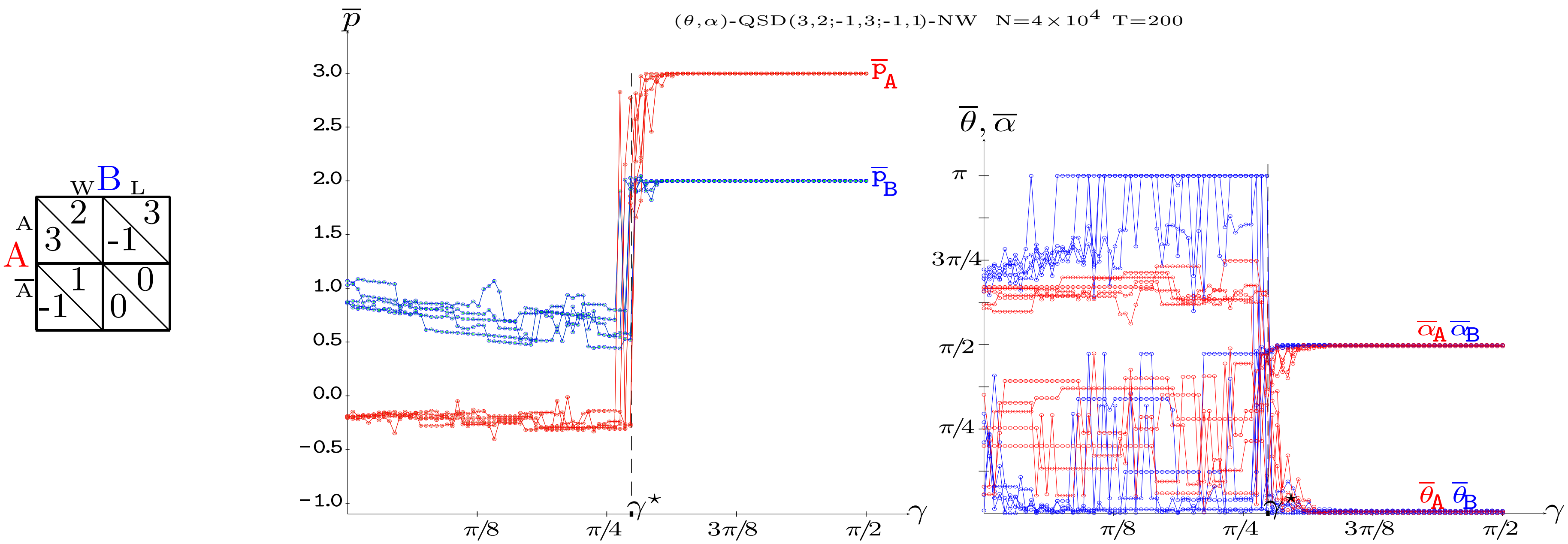


QHD on NETWORK

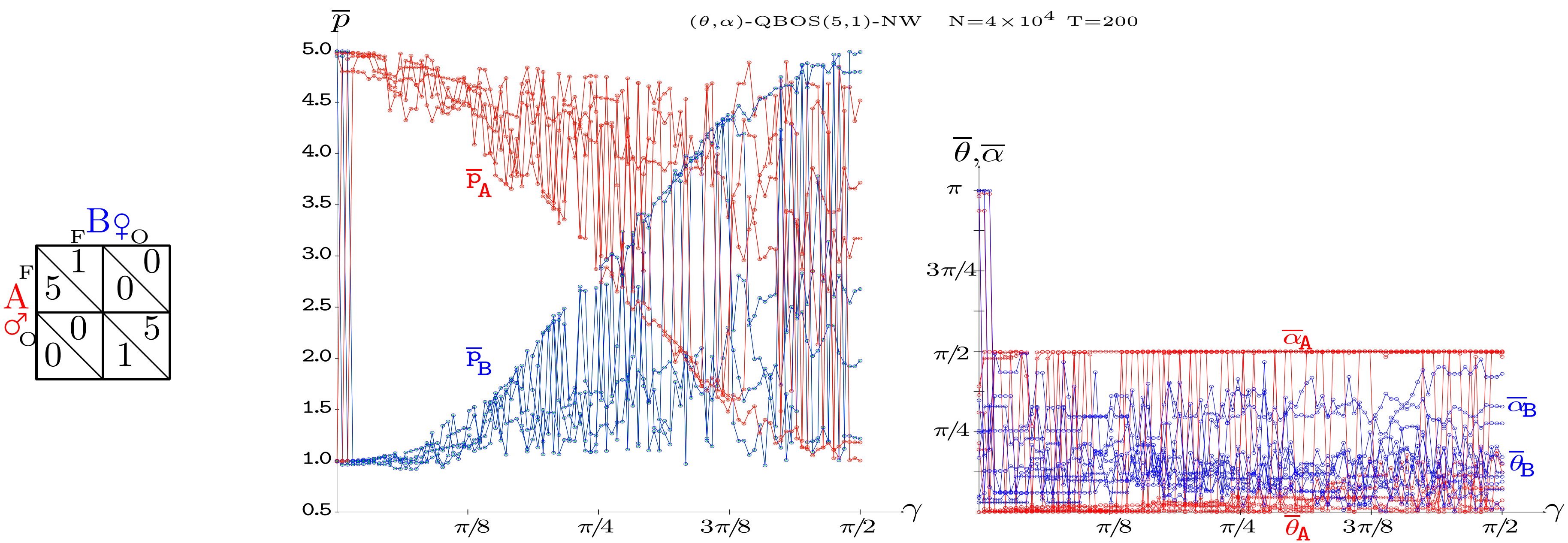


$$p_A^{DQ} = \mathfrak{T}\cos^2\gamma + S\sin^2\gamma == p^{QQ} = R \rightarrow \gamma^\star = \arcsin\left(\sqrt{\frac{\mathfrak{T}-R}{\mathfrak{T}-S}} = \frac{1}{3}\right) = 0.616\,.$$

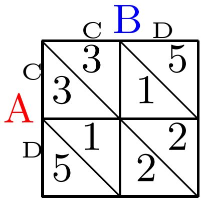
QSD on NETWORK

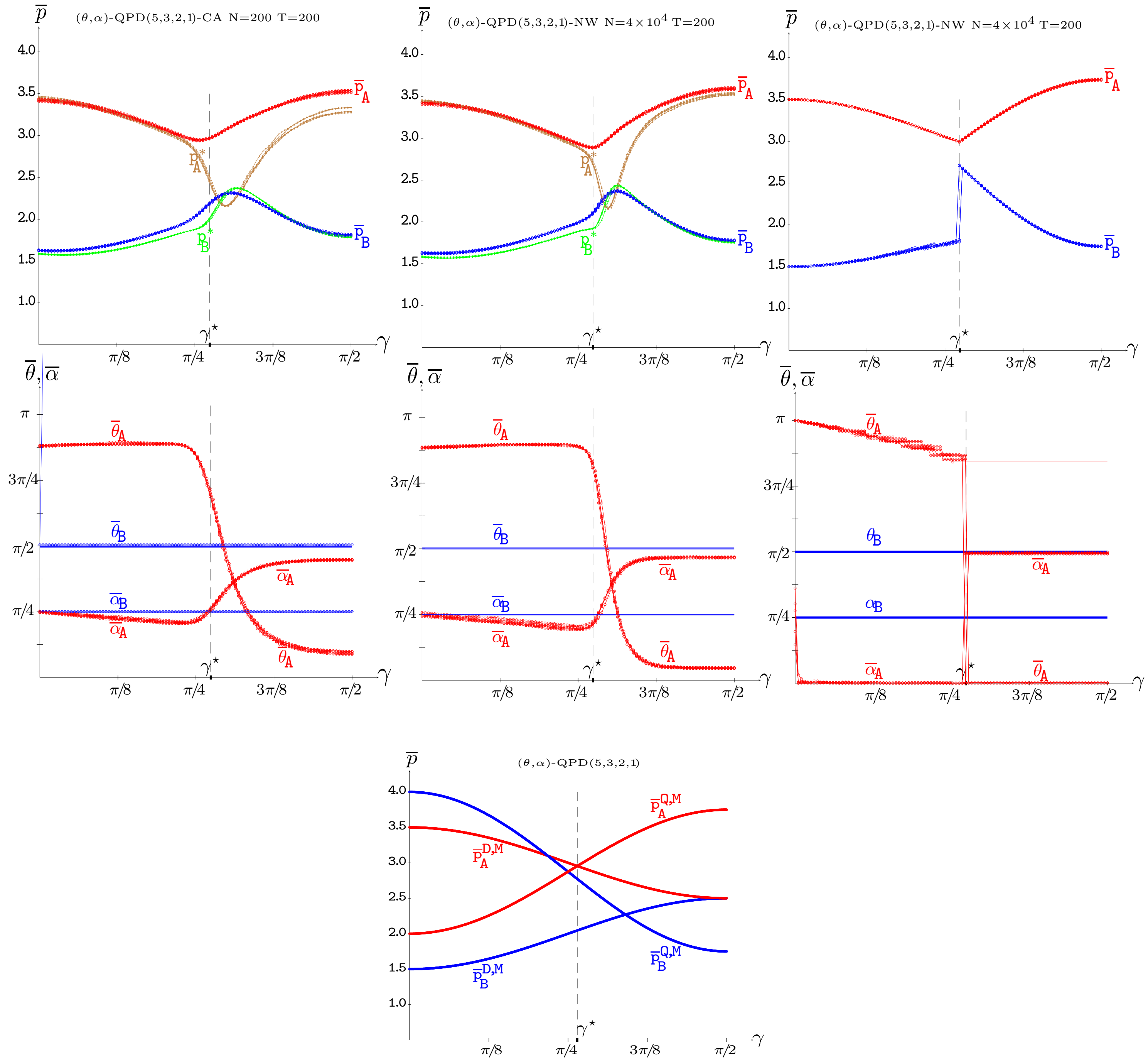


QBOS on NETWORK

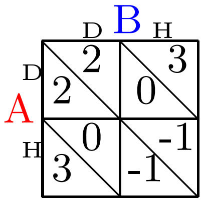


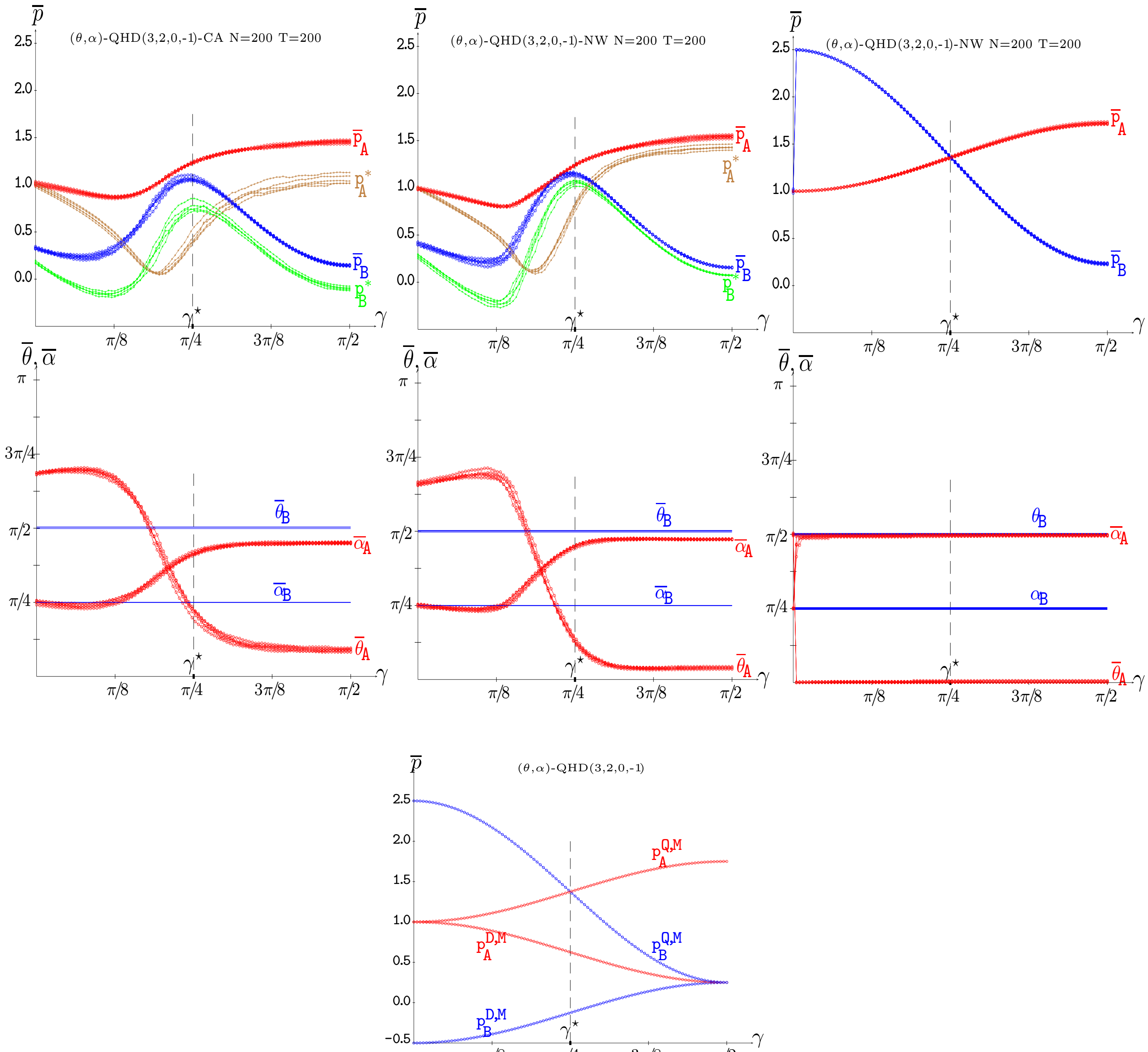
ONLY A UPDATES STRATEGY QPD



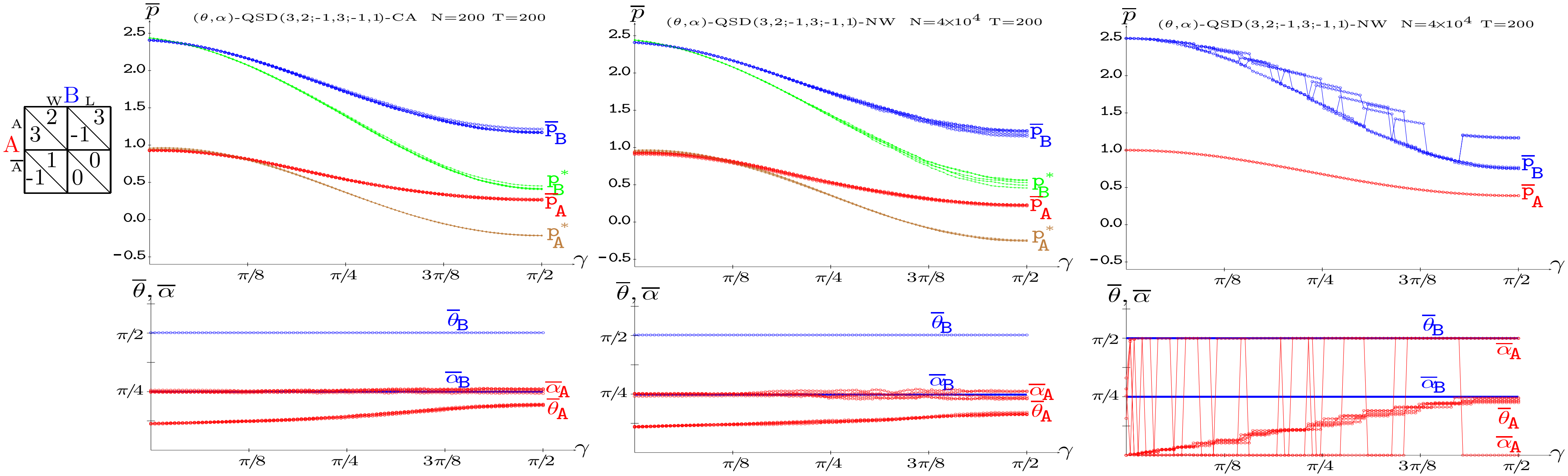


ONLY A UPDATES STRATEGY QHD

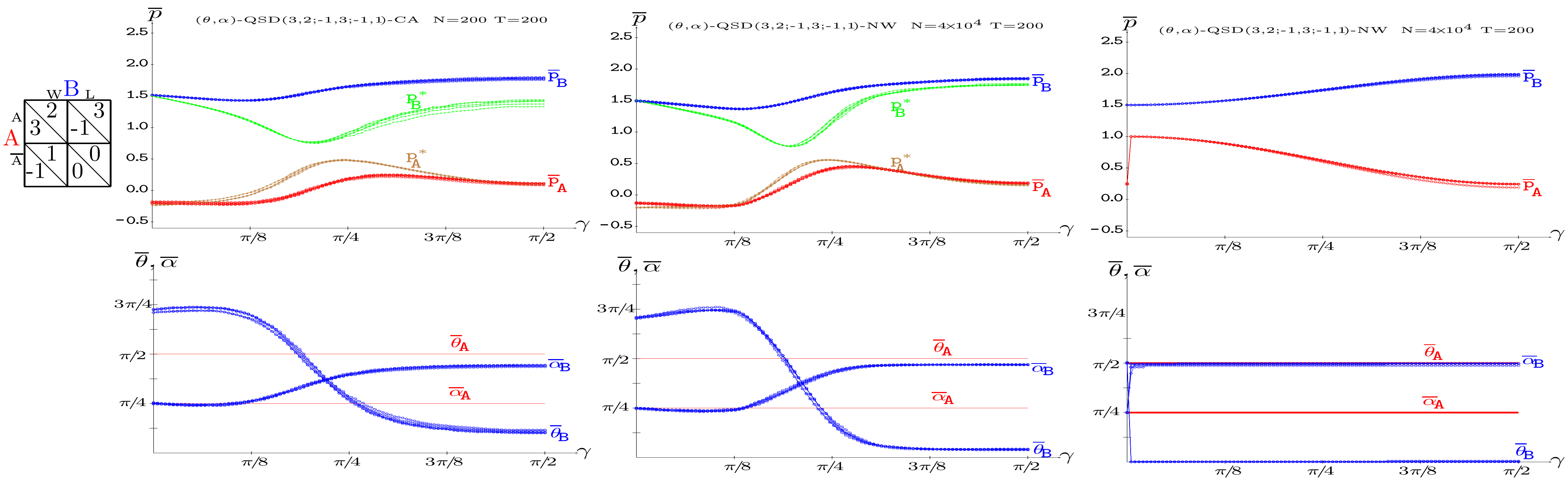




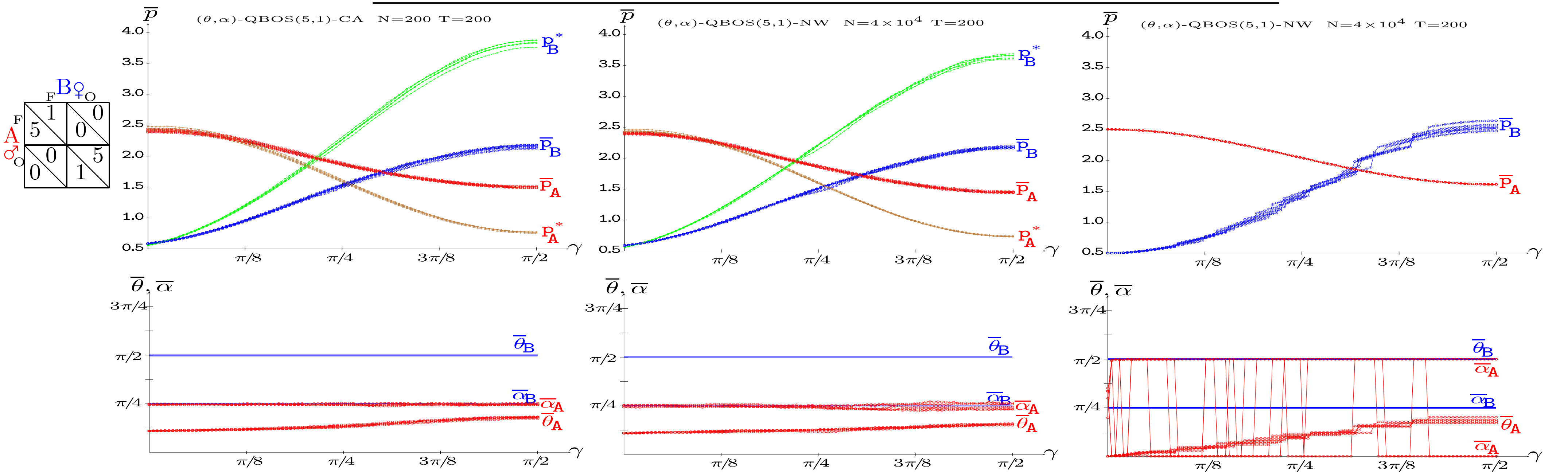
ONLY A UPDATES STRATEGY QSD(3,2,1,-1)



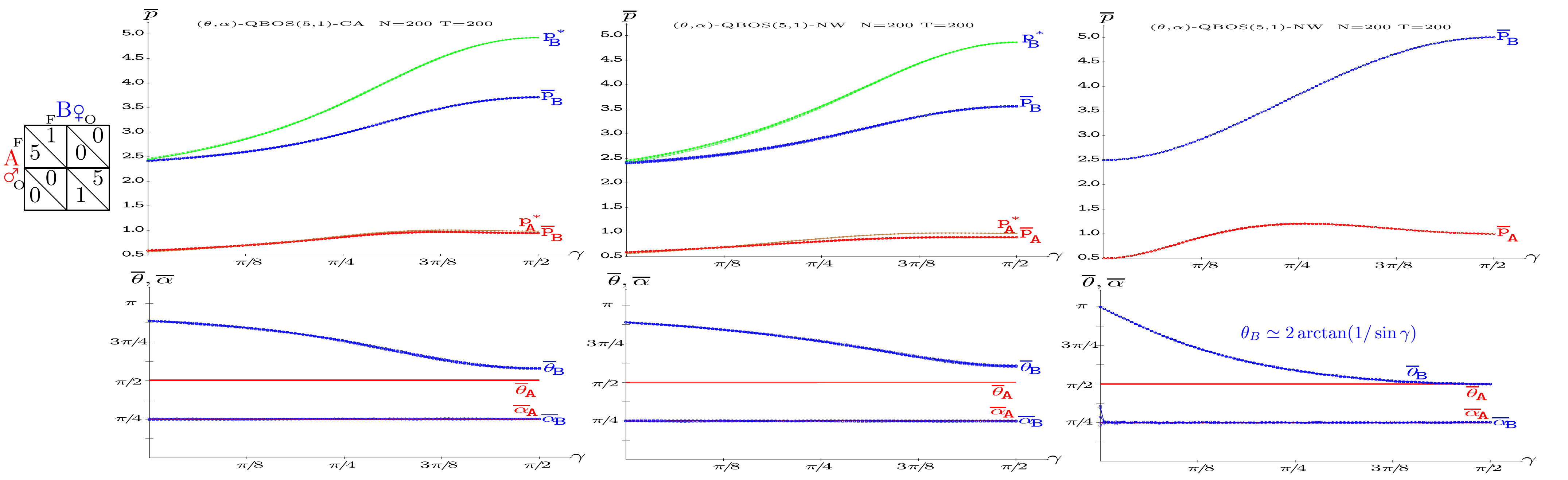
ONLY B UPDATES STRATEGY QSD(3,2,1,-1)



ONLY A UPDATES STRATEGY QBOS(5,1)

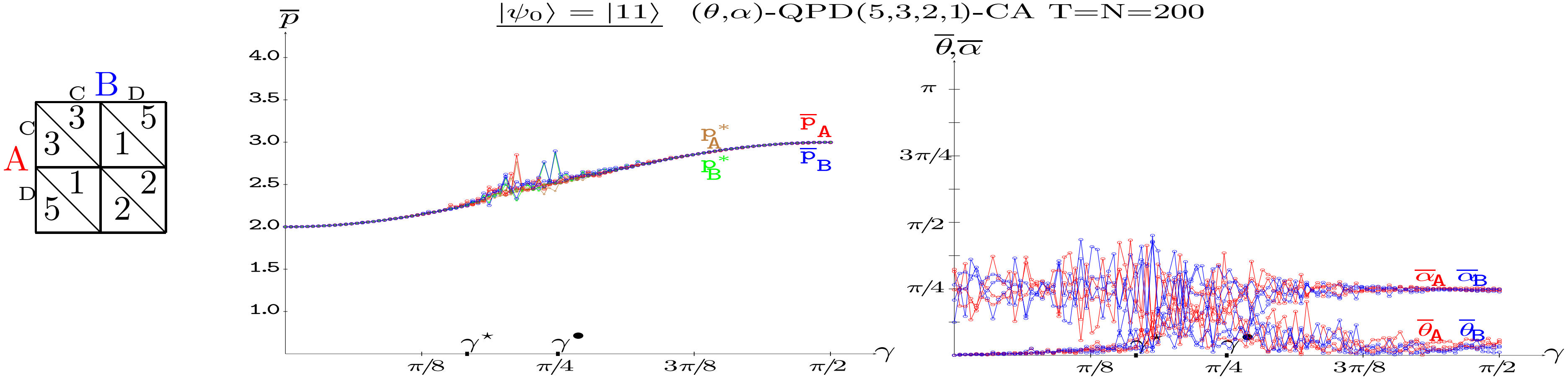


ONLY B UPDATES STRATEGY QBOS(5,1)



$$\underline{|\Psi\rangle = \hat{J}^{\dagger}(\hat{U}_A \otimes \hat{U}_B)\hat{J}|\Psi_0\rangle}$$

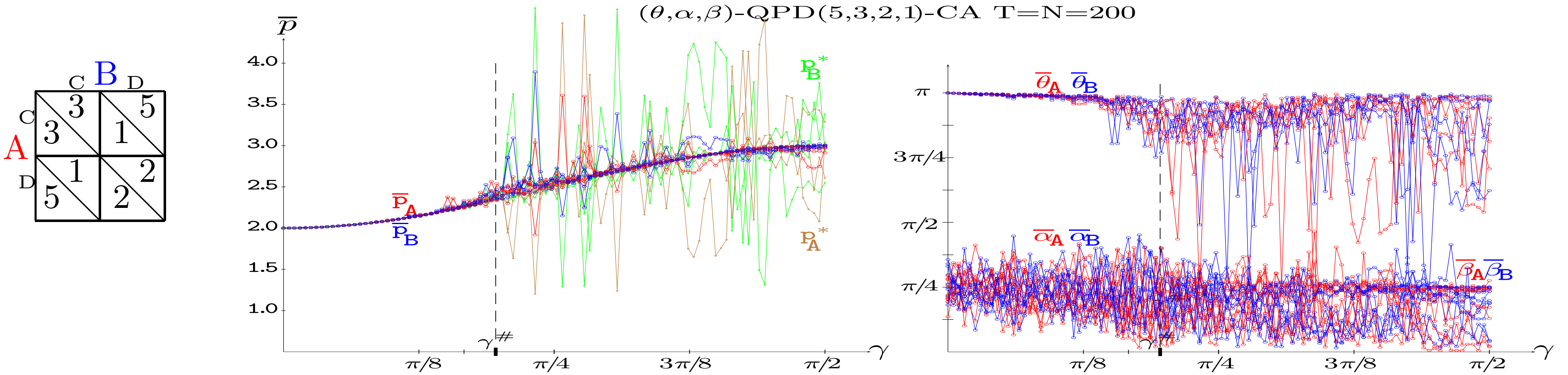
$$\text{EWL:}\quad |\Psi_0\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \text{Marinatto-Weber:}\quad |\Psi_0\rangle = |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$



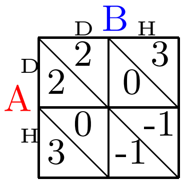
THREE-PARAMETER strategies

$$\hat{U}(\theta,\alpha,\beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & e^{i\beta}\sin(\theta/2) \\ -e^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2)\end{pmatrix} \qquad \begin{array}{c}\theta \in [0,\pi] \\ \alpha,\beta \in [0,\pi/2]\end{array}$$

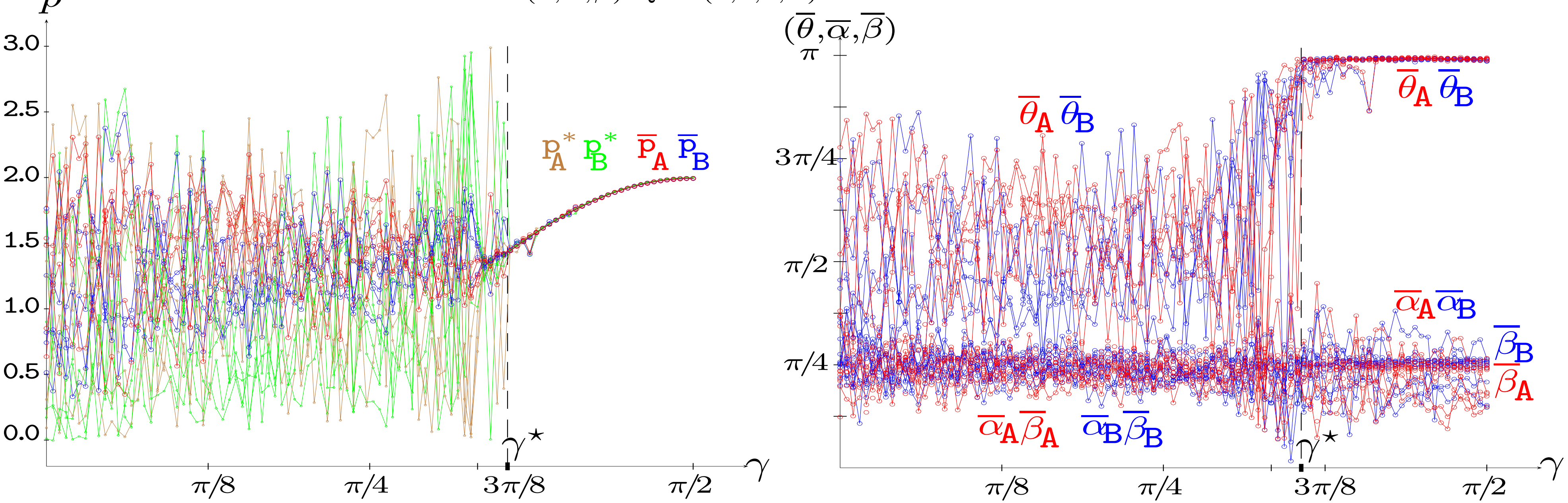
$$\gamma^{\#} = \arcsin\sqrt{\frac{P-S}{P+\mathfrak{T}-R-S}}:\quad \gamma < \gamma^{\#} \rightarrow \text{NE}:\ \mathtt{p}_A = \mathtt{p}_B = P + (R-P)\sin^2\gamma\,,\quad \gamma > \gamma^{\#} \rightarrow \text{NO NE}$$



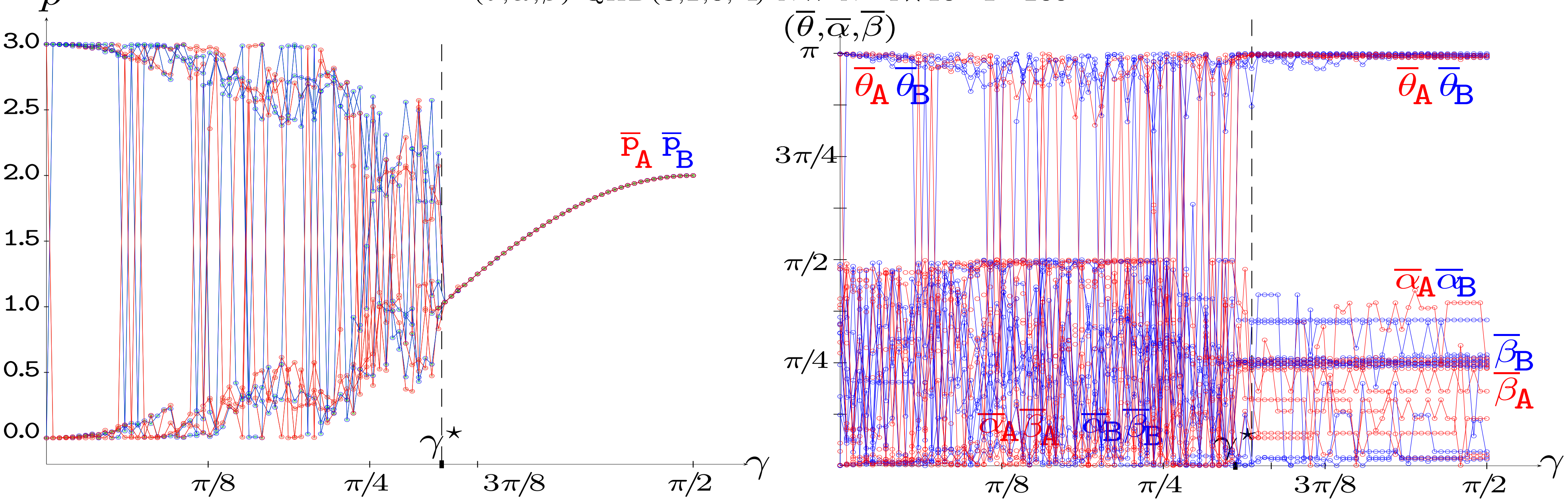
HAWK-DOVE 3P-QHD(3,2,0,-1)



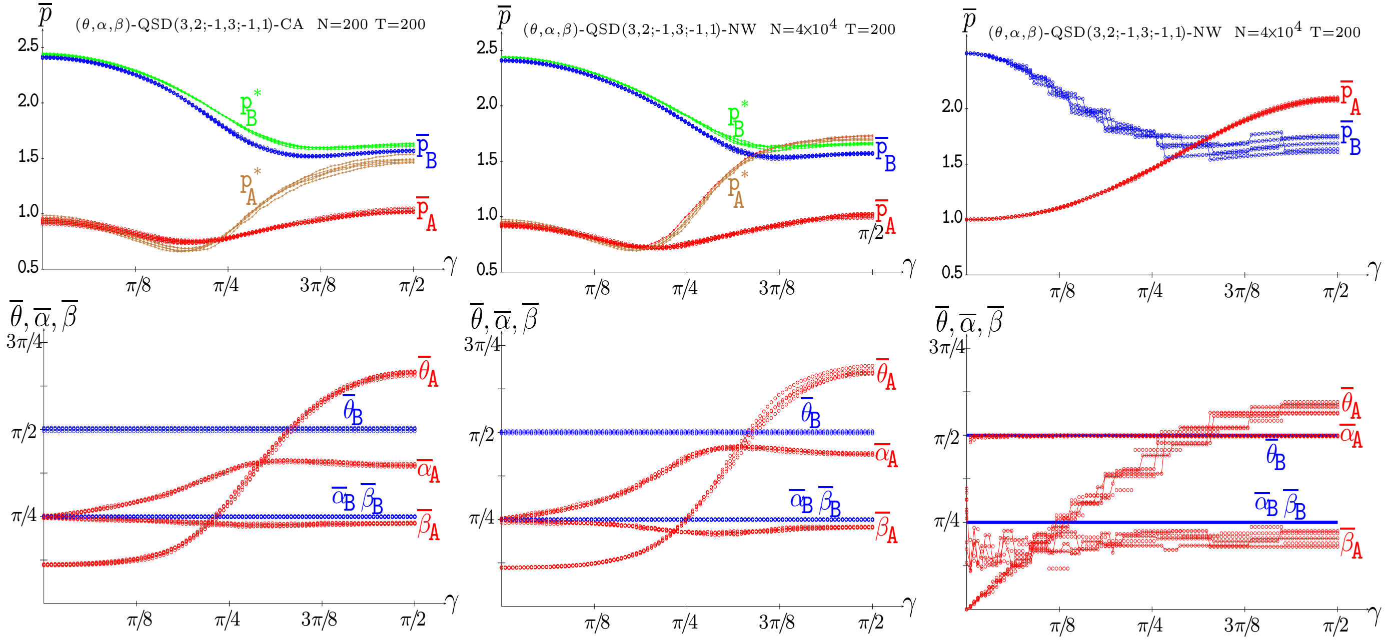
(θ,α,β)-QHD(3,2,0,-1)-CA T=N=200



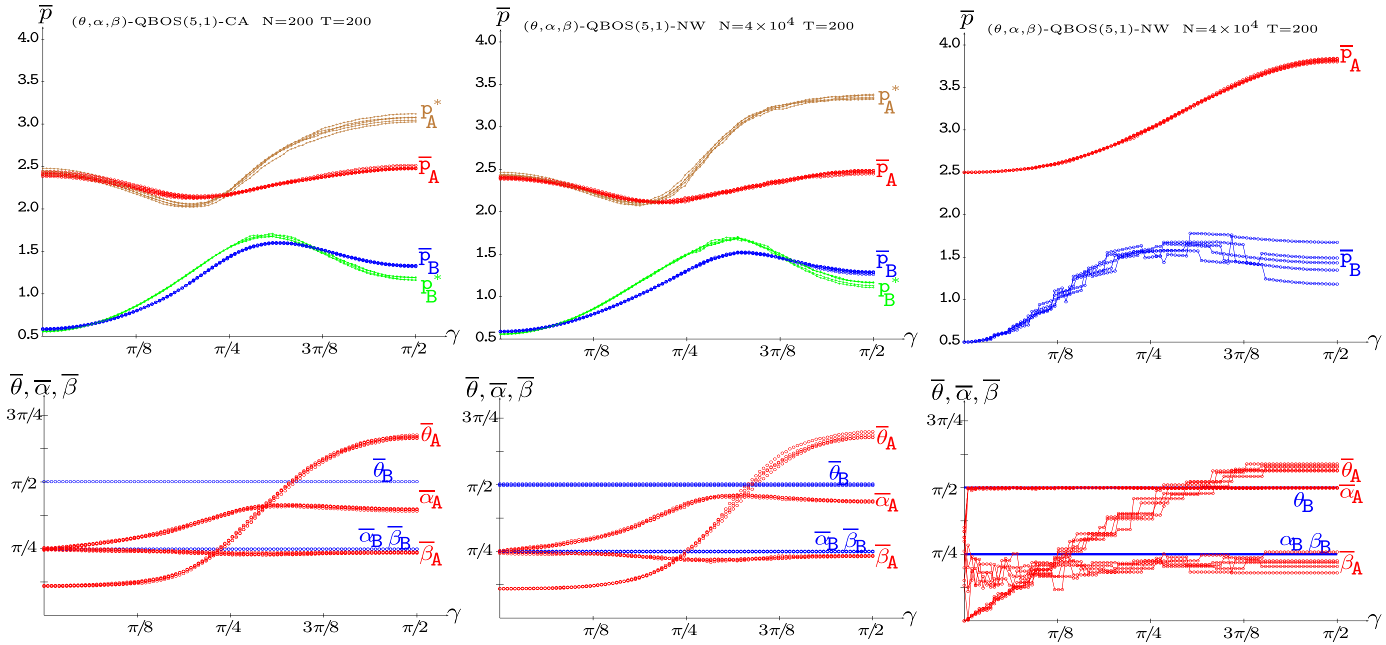
(θ,α,β)-QHD(3,2,0,-1)-NW $N=4\times10^4$ T=200

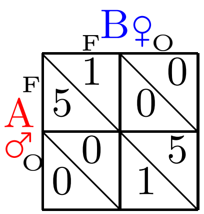


ONLY A UPDATES STRATEGY 3P-QSD

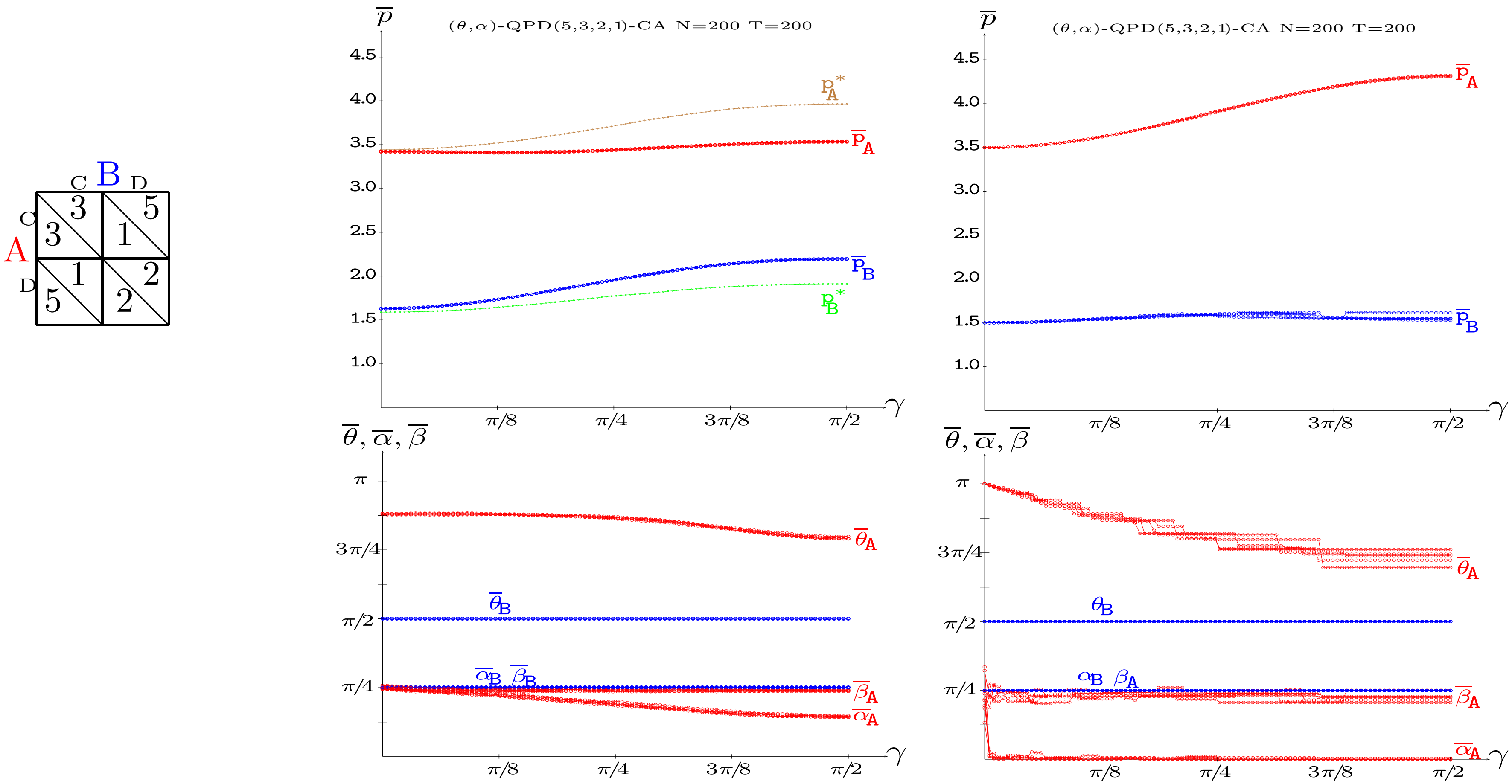


ONLY A UPDATES STRATEGY 3P-QBOS





ONLY A UPDATES STRATEGY 3P-QPD



CONCLUSSIONS

- The collective simulation of iterated games on spatial lattices and networks is a very useful tool for the study of two-person games, providing interesting additional information on their features.

- In the quantum Prisoner Dilemma, Hawk-Dove and Samaritan's Dilemma games, collective simulations detect the emergence in Nash equilibrium of the pair (Q,Q), which renders the payment of mutual cooperation, when the degree of entanglement is sufficiently high.

- In the quantum Battle of the Sexes game, said pair (Q,Q) is not in Nash equilibrium, being replaced by pairs of strategies that the collective simulation detects in a crisp way.

$\mathrm{D}(\boldsymbol{\pi}, \mathbf{0})$ *vs.* $\mathrm{Middle}(\boldsymbol{\pi/2}, \boldsymbol{\pi/4})$

$$\Pi^{D,M} = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{2}\sin^2\gamma \\ 1-\frac{1}{2}\sin^2\gamma & 1 \end{pmatrix}$$

$$P_A^{D,M} = \tfrac{1}{2}\big(T+P-\tfrac{1}{2}\sin^2\gamma(T-S)\big) \quad = (T+P)/2 = (5+2)/2 = 3.50 \leftarrow \gamma = 0$$
$$P_B^{D,M} = \tfrac{1}{2}\big(P+S+\tfrac{1}{2}\sin^2\gamma(T-S)\big) \quad = (P+S)/2 = (2+1)/2 = 1.50 \leftarrow \gamma = 0$$

$\mathrm{Q}(\mathbf{0}, \boldsymbol{\pi/2})$ *vs.* $\mathrm{Middle}(\boldsymbol{\pi/2}, \boldsymbol{\pi/4})$

$$\Pi^{Q,M} = \frac{1}{2}\begin{pmatrix} 1-\frac{1}{2}\sin^2\gamma & 1-\sin^2\gamma \\ \sin^2\gamma & \frac{1}{2}\sin^2\gamma \end{pmatrix}$$

$$P_A^{Q,M} = \frac{1}{2}[R+S+\sin^2\gamma(T-R\frac{1}{2}+P\frac{1}{2}-S)]$$
$$P_B^{D,M} = \frac{1}{2}[T+R+\sin^2\gamma(-T-R\frac{1}{2}+P\frac{1}{2}+S)]$$

$$P_A^{Q,M}(\gamma=0) = \frac{1}{2}[R+S] \quad \rightarrow \frac{1}{2}[3+1] = 2.00 \; P_A^{Q,M}(\gamma=\pi/2) = \frac{1}{2}[T+\frac{1}{2}(R+P))] \quad \rightarrow \frac{1}{2}[5+\frac{1}{2}(3+2))] = 15/4 = 3.75$$
$$P_B^{D,M}(\gamma=0) = \frac{1}{2}[T+R] \quad \rightarrow \frac{1}{2}[5+3] = 4.00 \; P_B^{D,M}(\gamma=\pi/2) = \frac{1}{2}[R\frac{1}{2}+P\frac{1}{2}+S] \quad \rightarrow \frac{1}{2}[3/2+2/2+1] = 7/4 = 1.75$$

$$P_A^{Q,M} = P_A^{D,M}$$
$$R+S+\sin^2\gamma(T-R\frac{1}{2}+P\frac{1}{2}-S) = T+P-\frac{1}{2}\sin^2\gamma(T-S)\big)$$
$$\sin^2\gamma(\frac{3}{2}T-R\frac{1}{2}+P\frac{1}{2}-\frac{3}{2}S) = T-R+P-S \rightarrow \gamma^\star = \arcsin\sqrt{\frac{2(T-R+P-S)}{3T-R+P-3S}}$$
$$\gamma^\star(5,3,2,1) = \arcsin\sqrt{\frac{2(T-R+P-S)}{3T-R+P-3S} = \frac{2(5-3+2-1)}{15-3+2-3} = \frac{6}{11}} = 0.831$$
$$\gamma^\star(3,2,0,-1) = \arcsin\sqrt{\frac{2(T-R+P-S)}{3T-R+P-3S} = \frac{2(3-2-1-0)}{9-2-1-0} = \frac{0}{6}} = \pi/2$$

CLASSICAL CORRELATED games FROM INDEPENDENT players

$$\begin{array}{c} \\ \Pi = \end{array} \begin{array}{c} \begin{array}{cc} y & 1-y \end{array} \\ \begin{pmatrix} (2k-1)^2 xy & (1-k)x(1-y)+k(1-x)y \\ (1-k)(1-x)y + kx(1-y) & (1-x)(1-y)+4k(1-k)xy \end{pmatrix} \end{array} \begin{array}{c} \\ x \\ 1-x \end{array}$$

$$0 \le k \le 1$$

$\underline{k = 0.0}$

$$\Pi = \begin{pmatrix} xy & x(1-y) \\ (1-x)y & (1-x)(1-y) \end{pmatrix}$$

$\rho = 0.0$

$\underline{k = 0.5}$

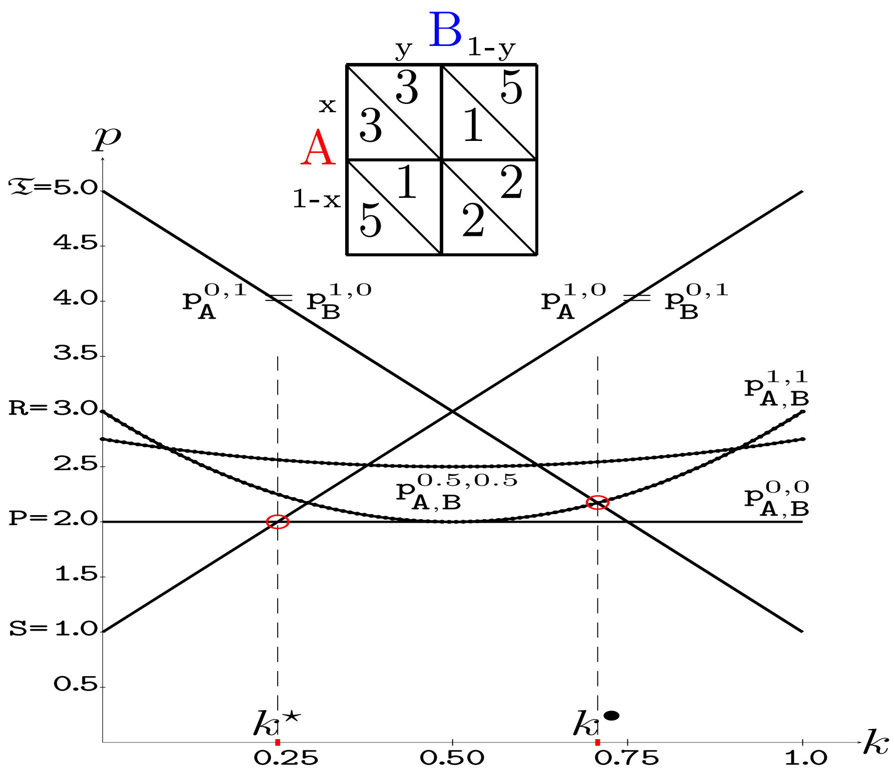
$$\Pi = \begin{pmatrix} 0 & (x+y-2xy)/2 \\ (x+y-2xy)/2 & 1-(x+y-2xy) \end{pmatrix}$$

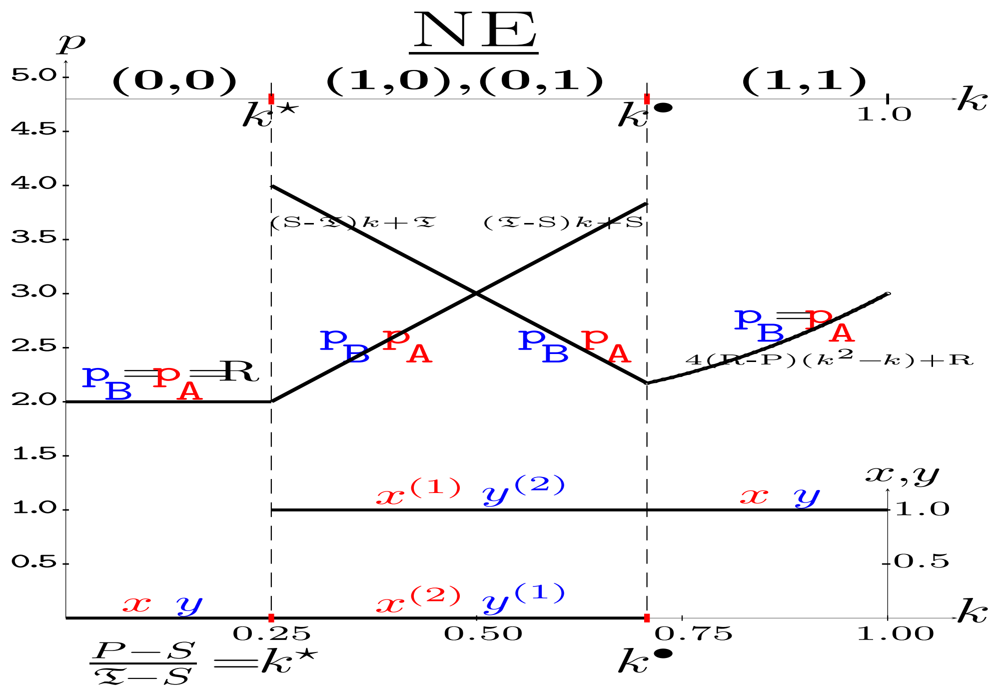
$\rho = p/(1-p)$

$\underline{k = 1.0}$

$$\Pi = \begin{pmatrix} xy & (1-x)y \\ x(1-y) & (1-x)(1-y) \end{pmatrix}$$

$\rho = 0.0$





Iqbal,A.,Chappell,J.M.,Abbott,D.(2016). On the equivalence between non-factorizable mixed-strategy classical games and quantum games. *Royal Society Open Science*, 3,150477.

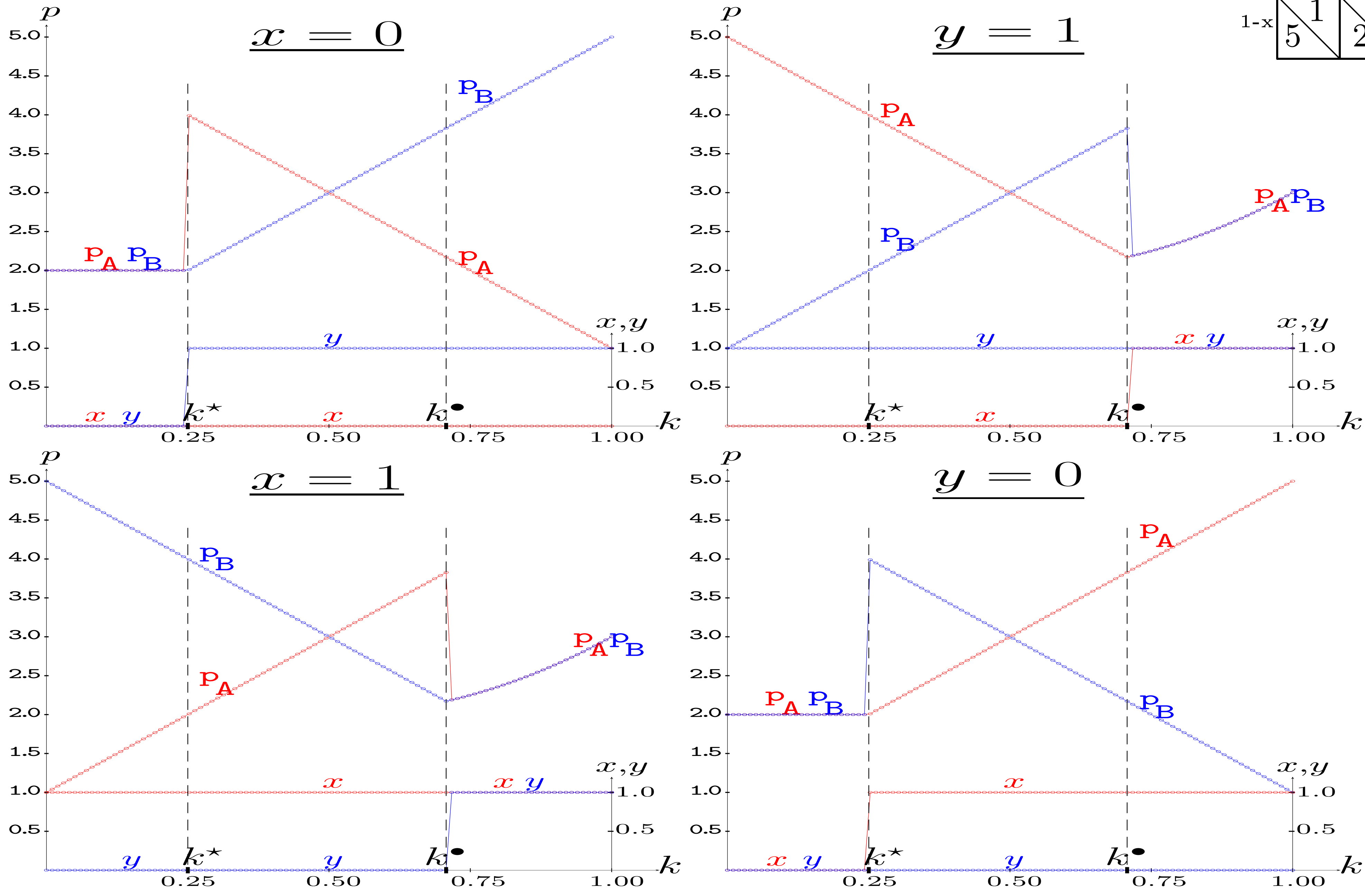
$$\mathbf{\Pi} = \begin{pmatrix} 0 & (x+y-2xy)/2 = p \\ (x+y-2xy)/2 = p & 1-(x+y-2xy) = 1-2p \end{pmatrix}$$

$\sigma_X^2 = x_1^2 p + x_2^2(1-p) - [(x_1 p + x_2(1-p))^2 = x_1^2 p^2 + x_2^2(1-p)^2 + 2x_1x_2p(1-p)] =$
$x_1^2(p-p^2) + x_2^2((1-p)-(1-p)^2) - 2x_1x_2p(1-p) = (x_1^2 - 2x_1x_2)(p-p^2) + x_2^2[(1-p-1+2p-p^2) = p(1-p)] = (x_1-x_2)^2 p(1-p)$
$\sigma_{X,Y} = x_1y_2p + x_2y_1p + x_2y_2(1-2p) - (x_1p + x_2(1-p))(y_1p + y_2(1-p)) = x_1y_2p + x_2y_1p + x_2y_2(1-2p) - (x_1p + x_2(1-p))(y_1p + y_2(1-p)) =$
$x_1y_2(p - p(1-p)) + x_2y_1(p - p(1-p)) + x_2y_2((1-2p) - (1-p)^2) - x_1y_1p^2 =$
$x_1y_2p^2 + x_2y_1p^2 + x_2y_2((1-2p) - (1-p)^2 = -p^2) - x_1y_1p^2 = (x_1y_2 + x_2y_1 - x_2y_2 - x_1y_1)p^2 = [x_1(y_2-y_1) + x_2(y_1-y_2)]p^2 = [(x_1-x_2)(y_2-y_1)]p^2$

$$\rho = \frac{p}{1-p}$$

BEST RESPONSEs - KPD(5,3,2,1)

		B: y	B: $1-y$
A	x	3, 3	1, 5
	$1-x$	5, 1	2, 2



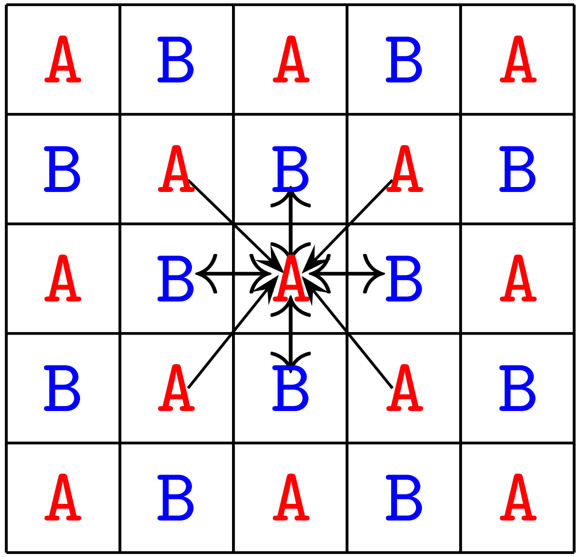
$$p^{(0,0)} = 2 = 1 - 4k = p_A^{(1,0)} \to k^\star = \frac{1}{4} \qquad p_A^{(0,1)} = 5 - 4k = 4k^2 - 4k + 3 = p^{(1,1)} \to k^\bullet = \frac{1}{\sqrt{2}}$$

ITERATIVE COLLECTIVE GAMES

- ## Spatial Games (CA)

Each player occupies a cell (i, j) in a 2D $N \times N$ lattice.

A and B alternate in the site occupation (chessboard). Every player surrounded by four partners (A-B, B-A), and four mates (A-A, B-B) .



- ## Games on random Networks (NW)

Each node is connected at random with four mates and four partners.

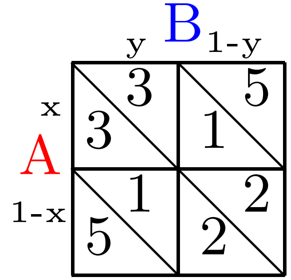
In each round (T) every player (i, j) :

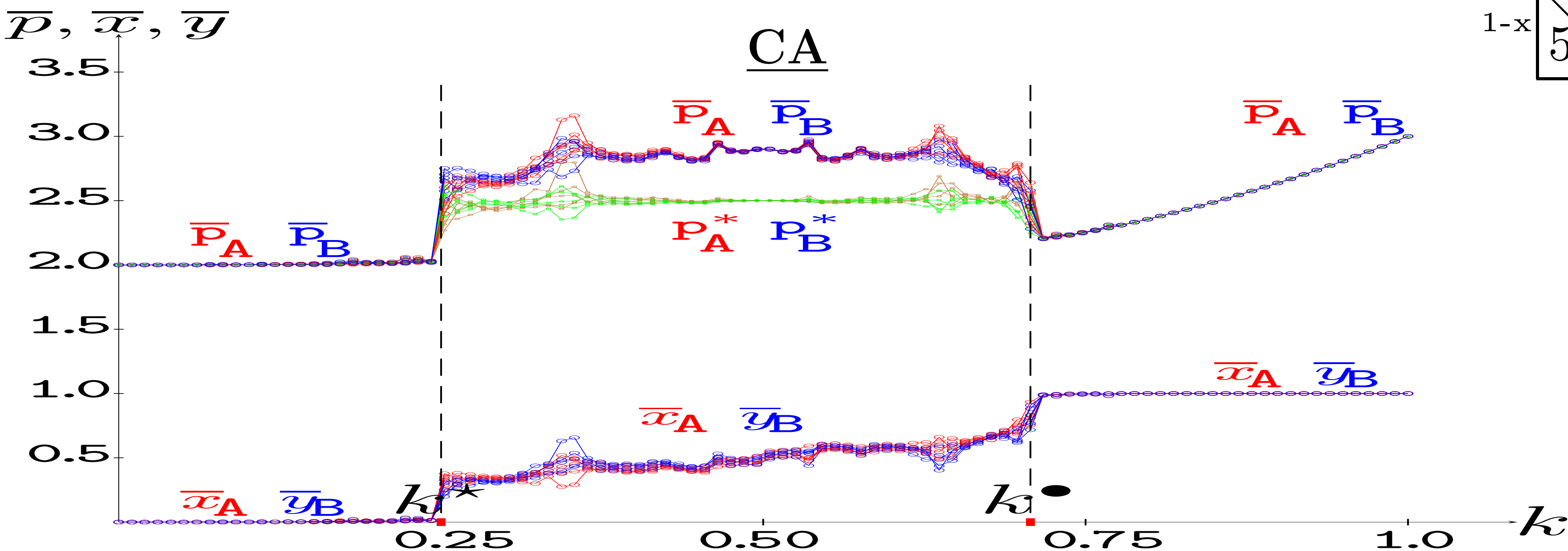
- plays with his four partners. His payoff $p_{i,j}^{(T)}$ is the sum over these four games.
- Adopts the probabilty (x or y) of his best paid mate (including himself), i.e., with the highest $p^{(T)}$.

Alonso-Sanz,R.(2017). Spatial correlated games. *Royal Society Open Science*, 4,11,171361.

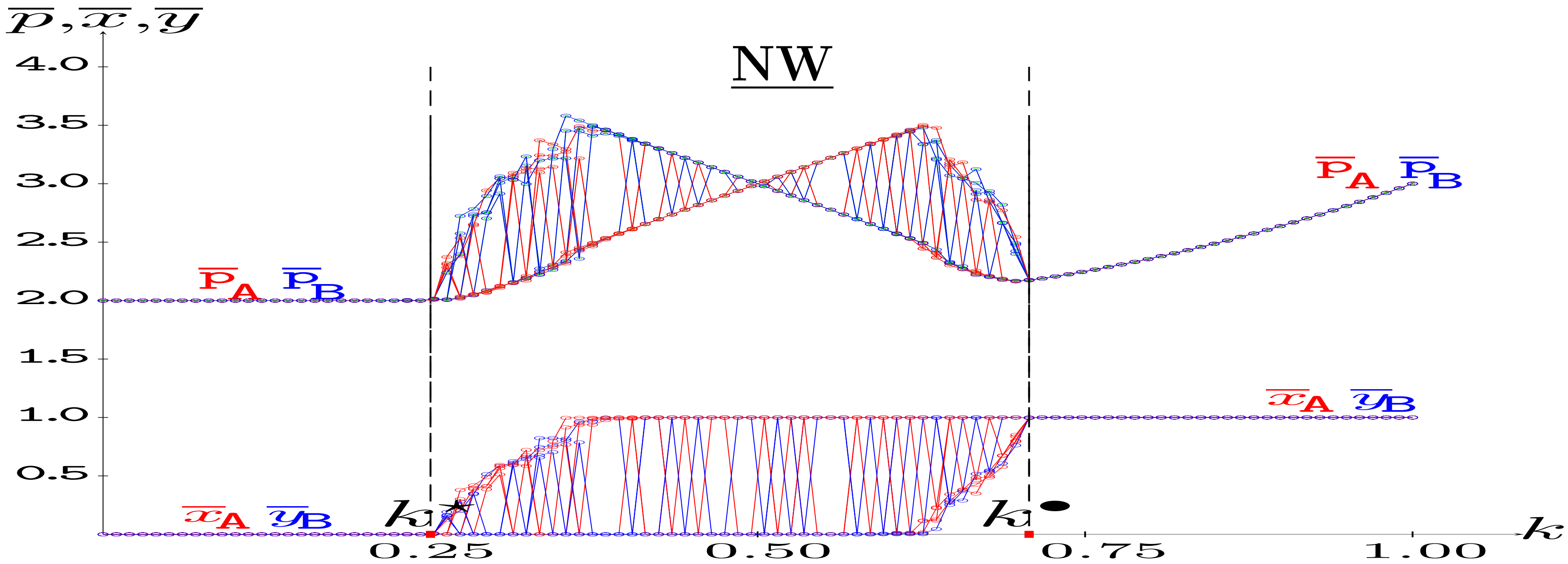
KPD(5,3,2,1) 200 × 200 players T=200

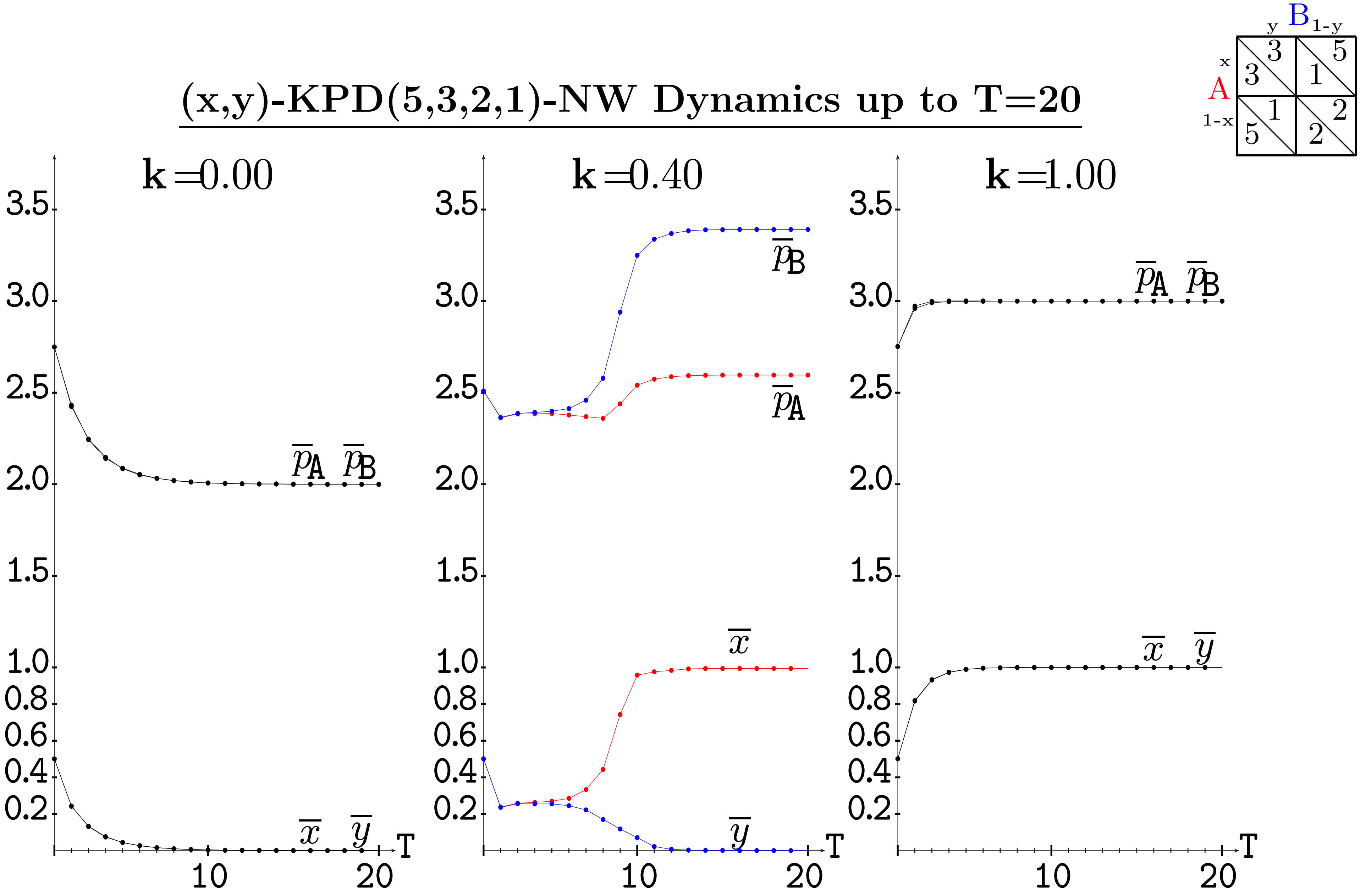
Five initial random probability configurations



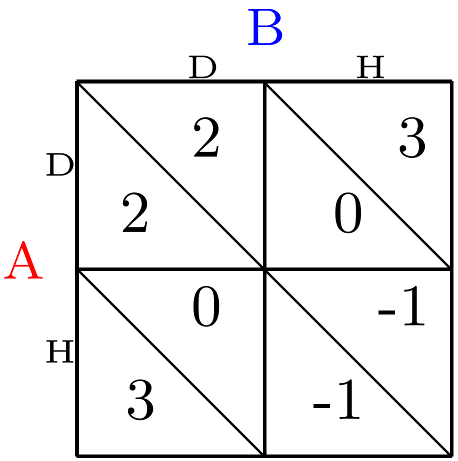


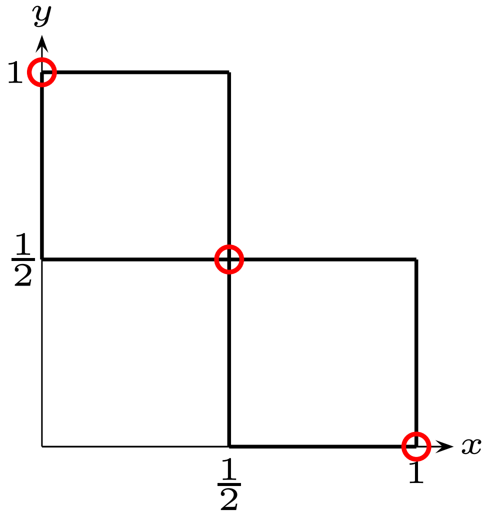
$$\Pi^{\star} = \begin{pmatrix} (2k-1)^2\overline{x}\,\overline{y} & (1-k)\overline{x}(1-\overline{y}) + k(1-\overline{x})\overline{y} \\ (1-k)(1-\overline{x})\overline{y} + k\overline{x}(1-\overline{y}) & (1-\overline{x})(1-\overline{y}) + 4k(1-k)\overline{x}\,\overline{y} \end{pmatrix}$$

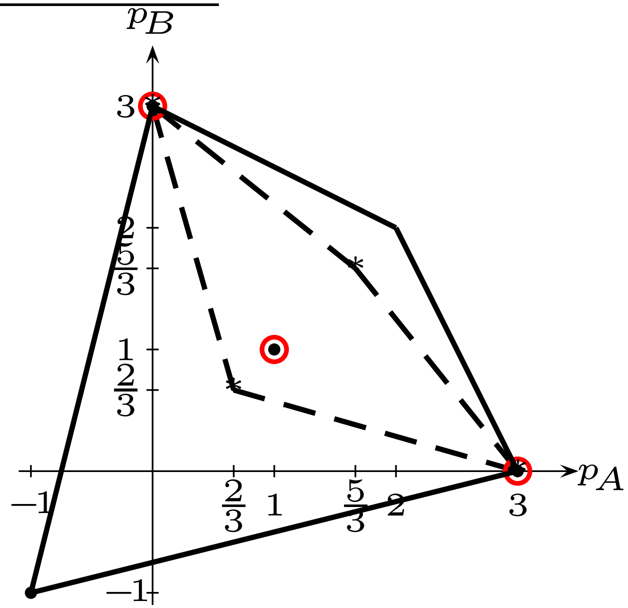



(x,y)-KPD(5,3,2,1)-NW Dynamics up to T=20
k=0.00
k=0.40
k=1.00
A
B
x
1-x
y
1-y
3
3
1
5
5
1
2
2
$\overline{p}_A$ $\overline{p}_B$
$\overline{x}$ $\overline{y}$
3.5
3.0
2.5
2.0
1.5
1.0
0.8
0.6
0.4
0.2
10
20
T

HAWK-DOVE (HD)



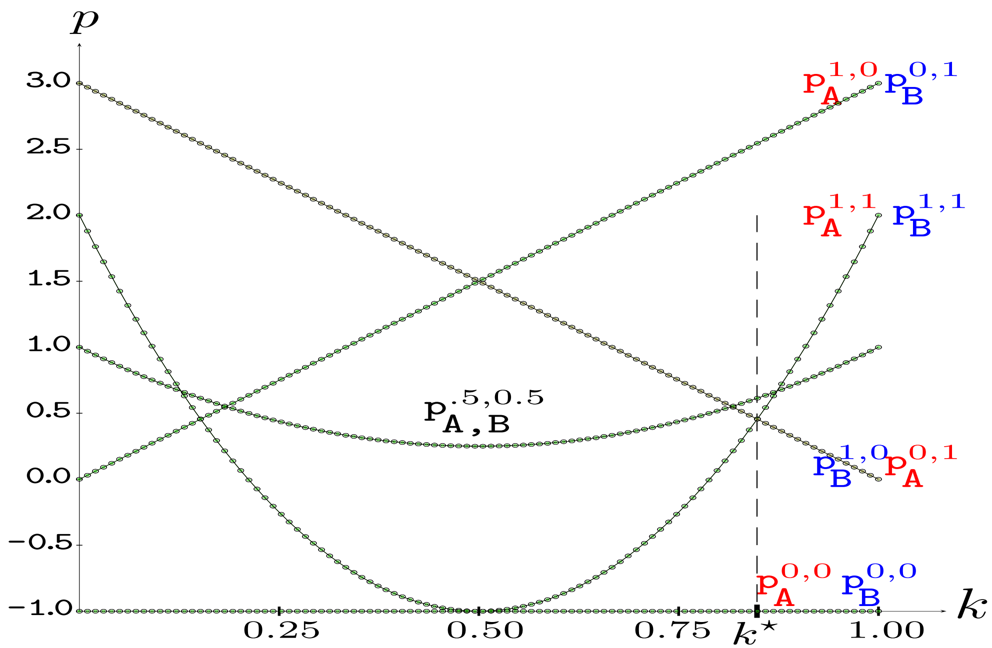


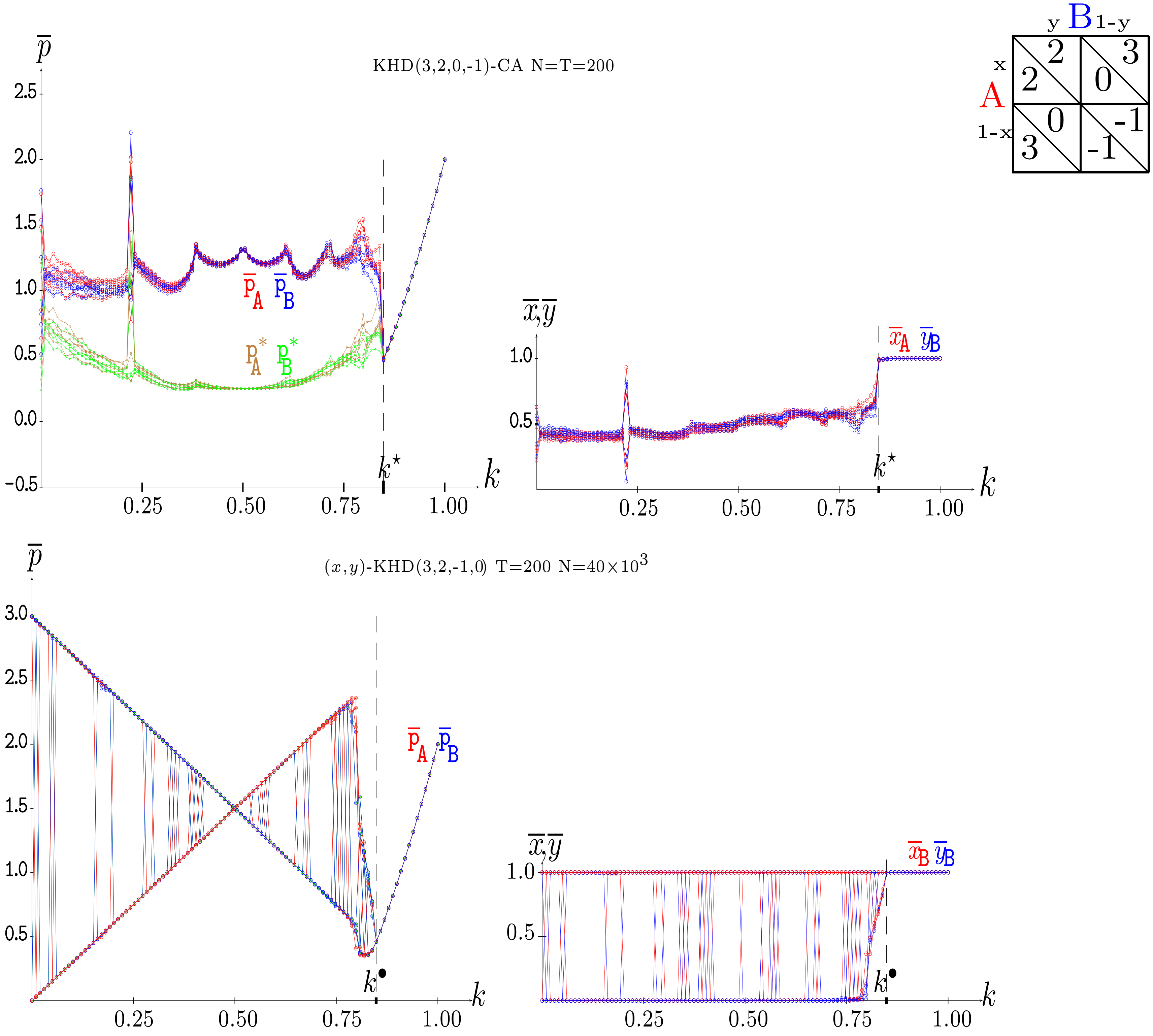


NE: $\bullet$ (1, 0) $\equiv (D, H) \to p_A = 0\,,\ p_B = 3$ $\quad\bullet$ (0, 1) $\equiv (H, D) \to p_A = 3\,,\ p_B = 0$ $\quad\bullet$(1/2, 1/2) $\to p_{AB} = 1$

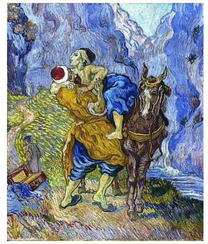
$\star\,(1, 1) \equiv$(H,H) NO NE

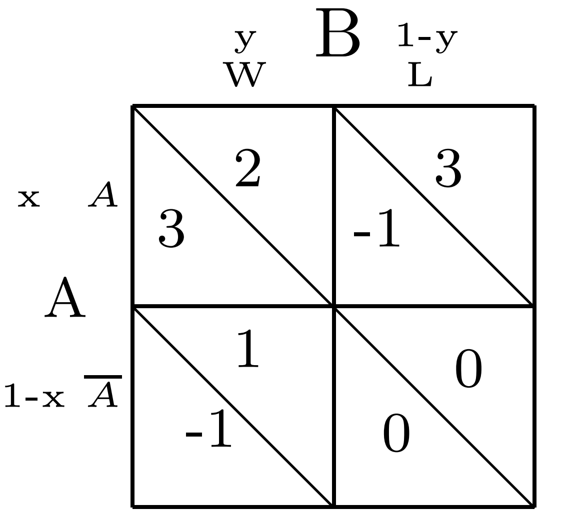
SWS : (1,1) $\equiv$ (D,D), $\pi_{11} = 1 \to p_A + p_B = 4$




KHD(3,2,0,-1)-CA N=T=200
(x,y)-KHD(3,2,-1,0) T=200 N=40×10^3
$\overline{p}_A$ $\overline{p}_B$
p_A^* p_B^*
$\overline{x}_A$ $\overline{y}_B$
$\overline{x}_B$ $\overline{y}_B$
$\overline{x},\overline{y}$
$\overline{p}$
$k^\star$
$k^\bullet$
k
A
B
x
1-x
y
1-y

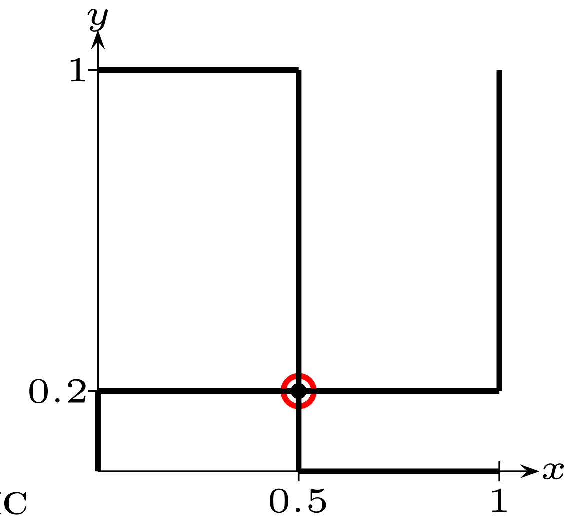
SAMARITAN'S DILEMMA (SD)

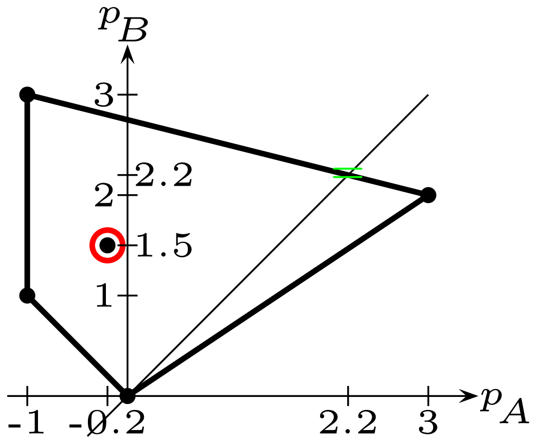




UNFAIR, ASYMMETRIC

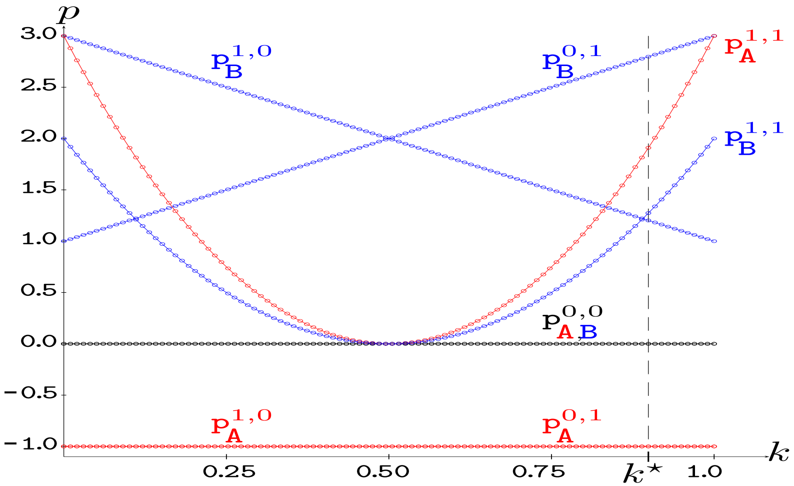
DISCOORDINATION

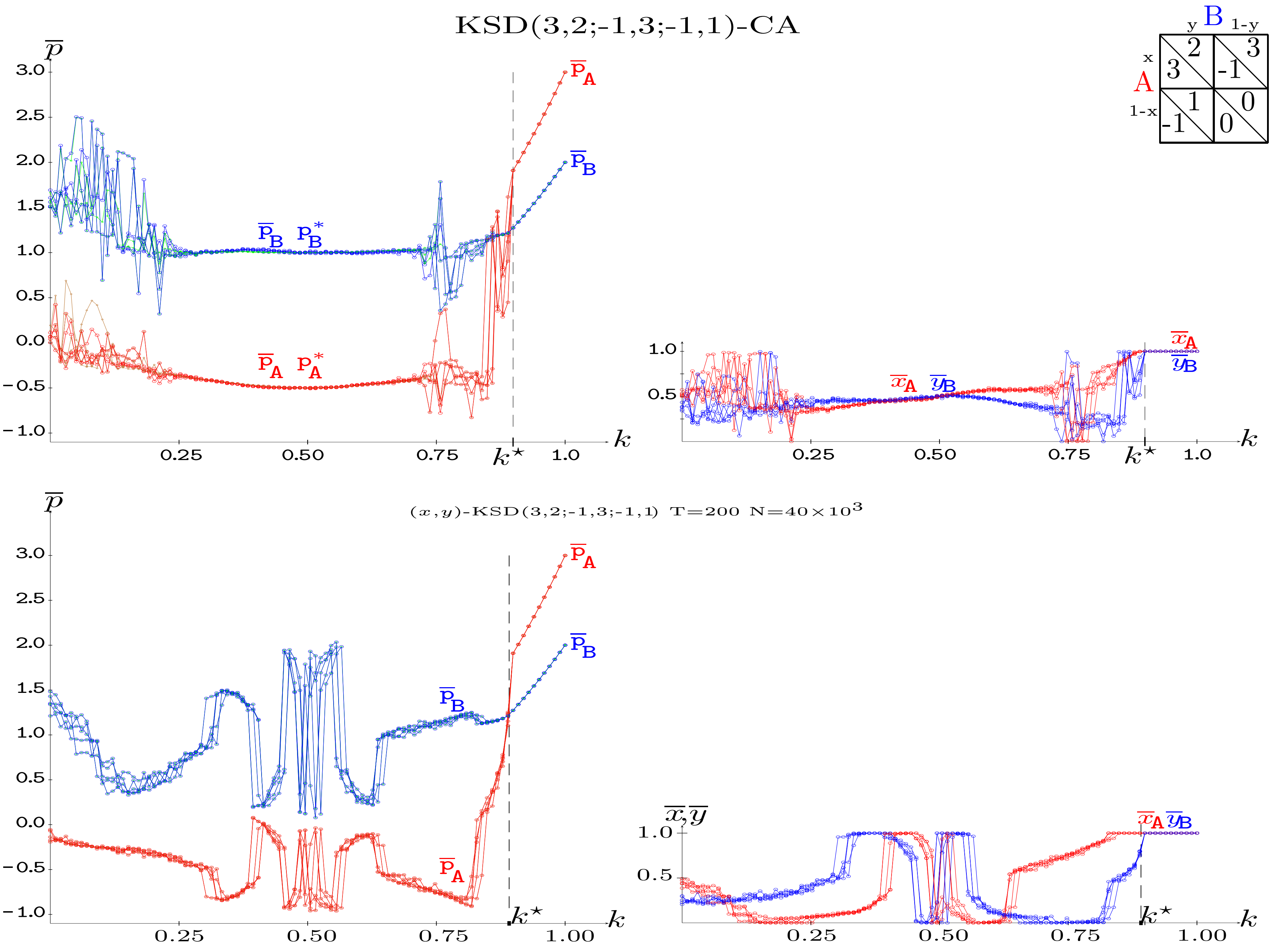




$\frac{3-1-1+0}{4}=0.25$ $\qquad$ $\frac{2+3+1+0}{4}=1.5$

NE : $(\mathbf{x^* = 1/2}, \mathbf{y^* = 1/5})$, $(\mathbf{p_A = -0.2}, \mathbf{p_B = 1.5})$ $\qquad$ (1,1) $\equiv$ (A,W), $\pi_{11} = 1$ **unique SWS** $\rightarrow p_A + p_B = 5$

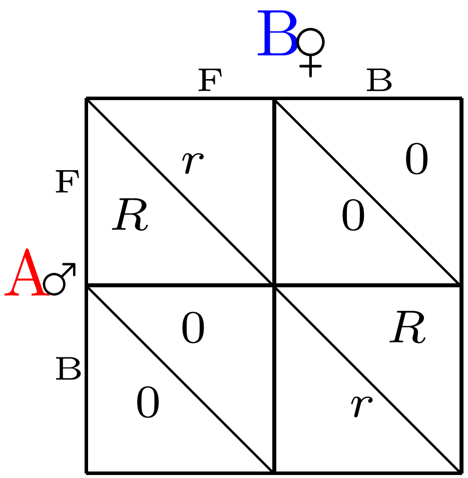


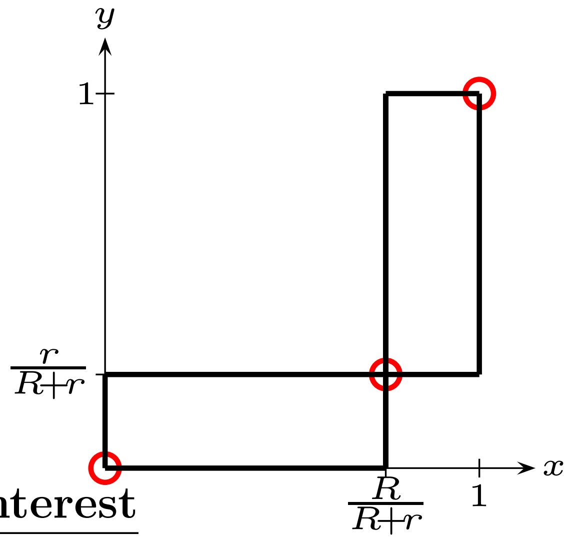


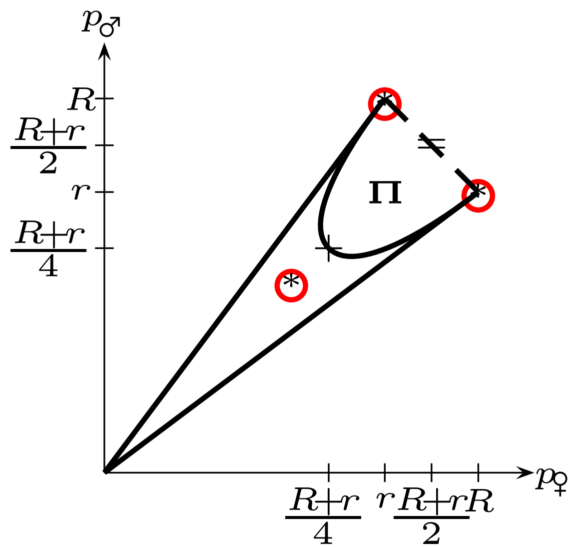
Alonso-Sanz,R.,Situ,H.(2017). A quantum Samaritan's dilemma cellular automaton. *Royal Society Open Science*, 4,6,160669.

BATTLE OF THE SEXES (BOS) $R > r > 0$

Fair, Asymmetric







Coordination, conflicting interest

INDEPENDENT probabilistic strategies

$$p_{♂}(x,y) = Rxy + r(1-x)(1-y)$$
$$p_{♀}(x,y) = rxy + R(1-x)(1-y)$$

$$y = 1-x \rightarrow p_{♂} = p_{♀} = (R+r)(1-x)x$$
$$x = y = 1/2 \rightarrow p^{+}_{max} = (R+r)/4$$

NE: $\bullet\ (1,1) \equiv (F,F) \rightarrow p_{♂} = R\,,\ p_{♀} = r \qquad \bullet\ (0,0) \equiv (B,B) \rightarrow p_{♂} = r\,,\ p_{♀} = R$

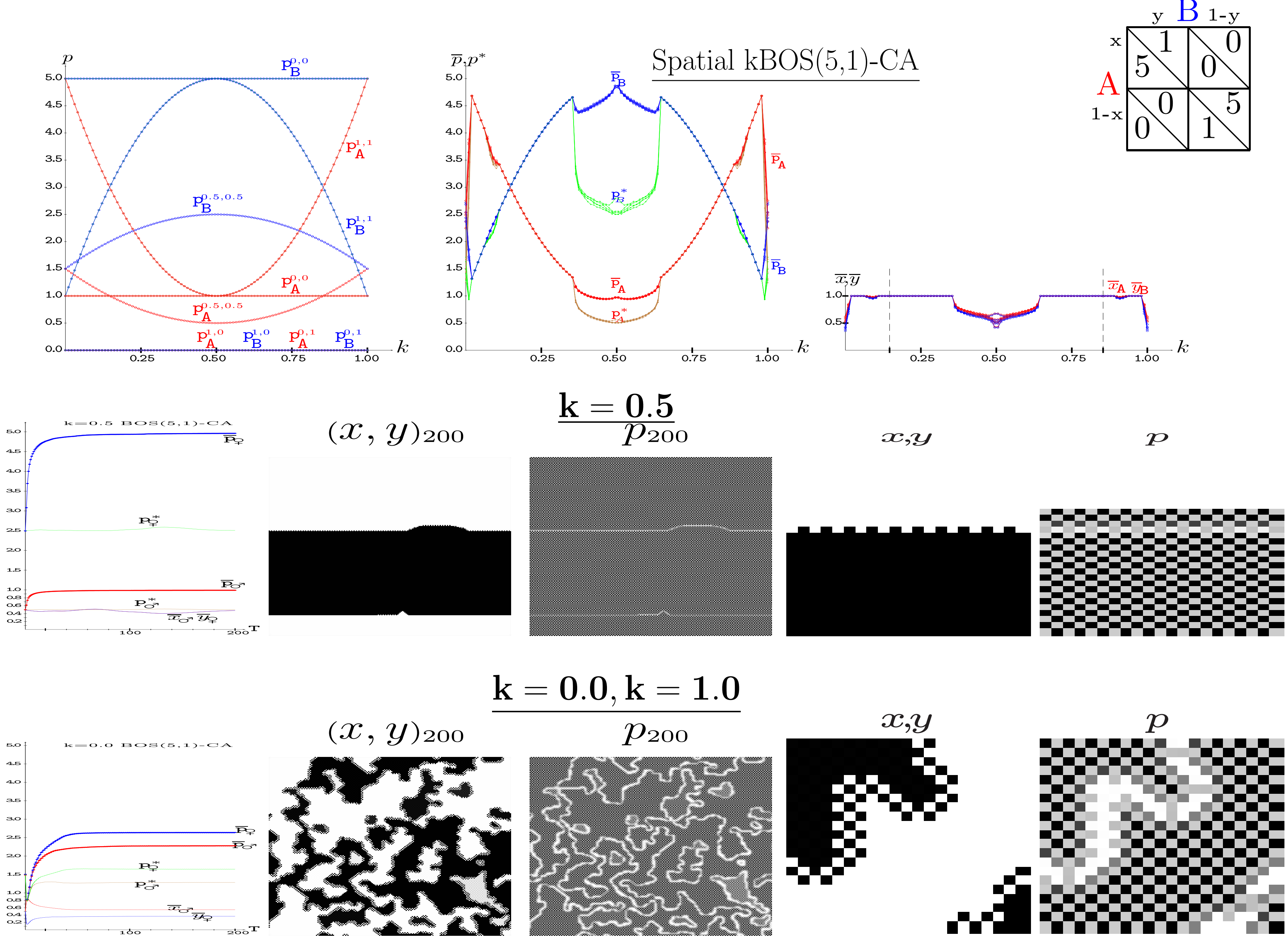
$$\bullet\ x^{*} = \frac{R}{R+r}, y^{*} = \frac{r}{R+r} \rightarrow p_{♂,♀} = \frac{Rr}{R+r} < r$$

CORRELATED games

$$p_{♂} = \pi_{11}R + \pi_{22}r$$
$$p_{♀} = \pi_{11}r + \pi_{22}R$$

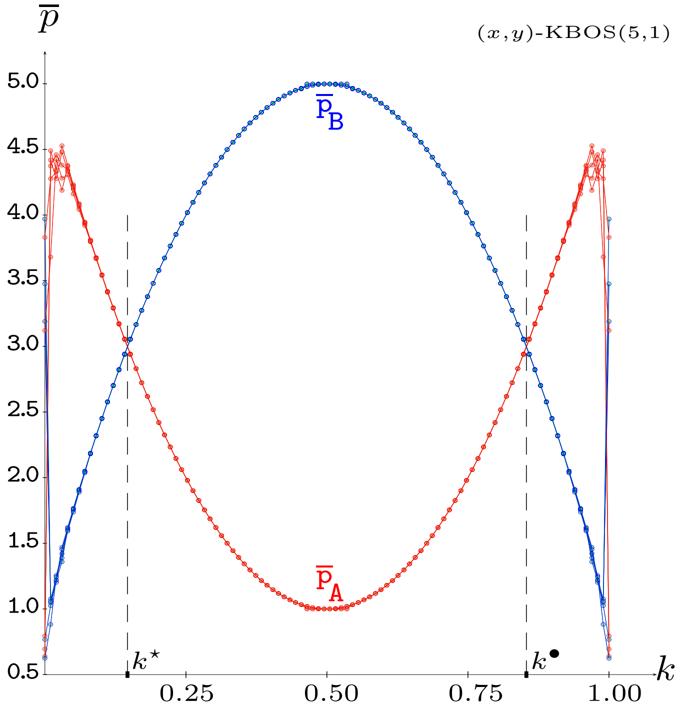
$\pi_{11} = \pi_{22} = \pi \rightarrow p_{♂} = p_{♀} = (R+r)\pi \qquad \pi = 1/2 \rightarrow p^{=}_{max} = (R+r)/2$

SWS: $(1,1) \equiv (F,F)$, $(0,0) \equiv (B,B)$, $\pi_{11} + \pi_{22} = 1$, e.g., $\pi_{11} = 1, \pi_{22} = 1$ $\rightarrow p_{A} + p_{B} = R + r$

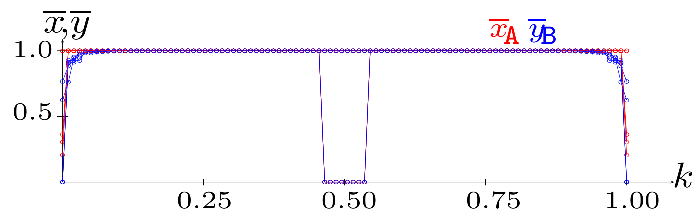


Spatial kBOS(5,1)-CA
k = 0.5
(x, y)200
p200
x,y
p
k=0.5 BOS(5,1)-CA
k = 0.0, k = 1.0
k=0.0 BOS(5,1)-CA

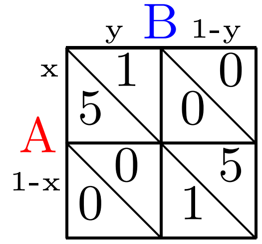
KBOS(5,1)-NW



(x,y)-KBOS(5,1) T=200 N=40×10^3
$\overline{p}$
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
$\overline{\mathrm{p}}_{\mathrm{B}}$
$\overline{\mathrm{p}}_{\mathrm{A}}$
$k^\star$
$k^\bullet$
0.25
0.50
0.75
1.00
k

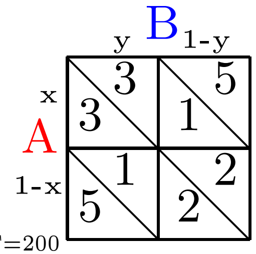


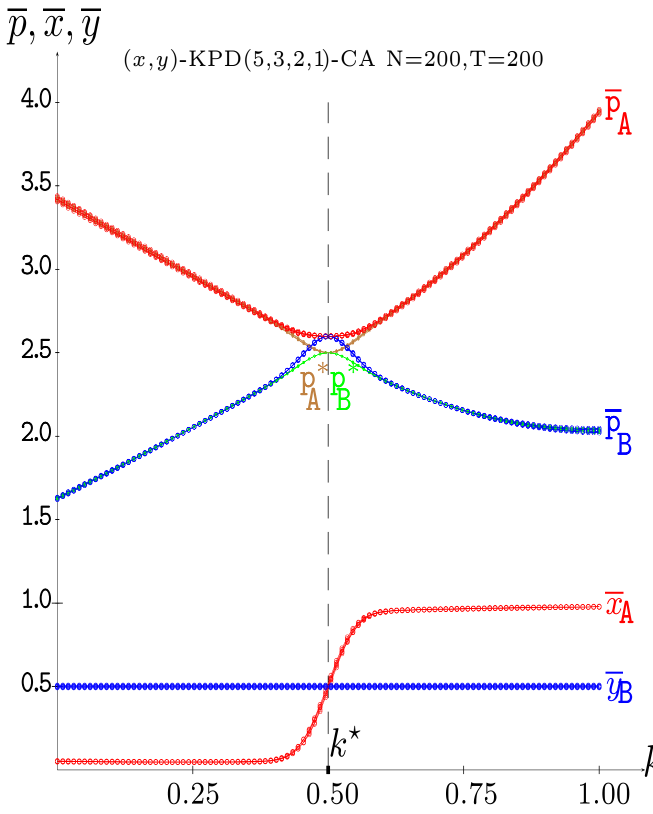
$\overline{x}$,$\overline{y}$
1.0
0.5
$\overline{x}_{\mathrm{A}}$ $\overline{y}_{\mathrm{B}}$
0.25
0.50
0.75
1.00
k

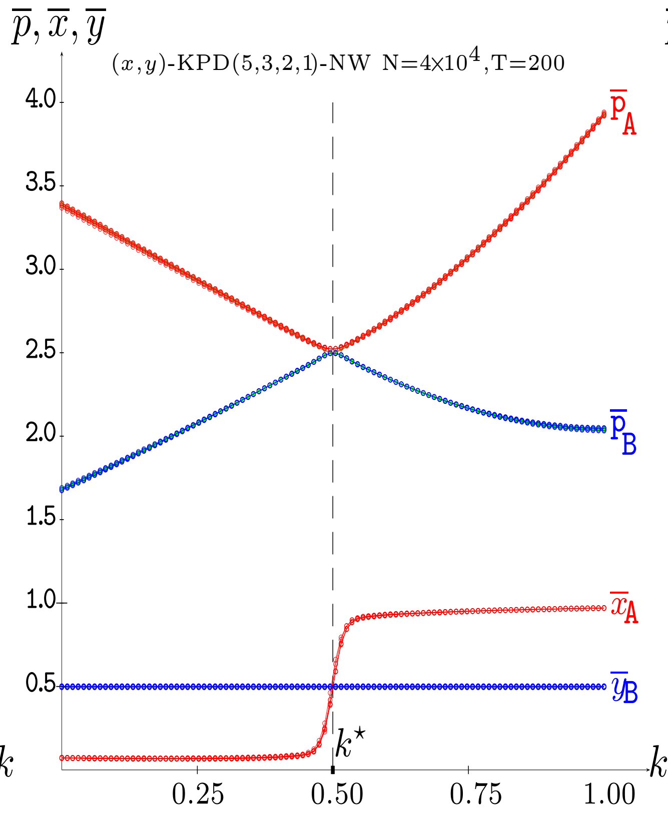


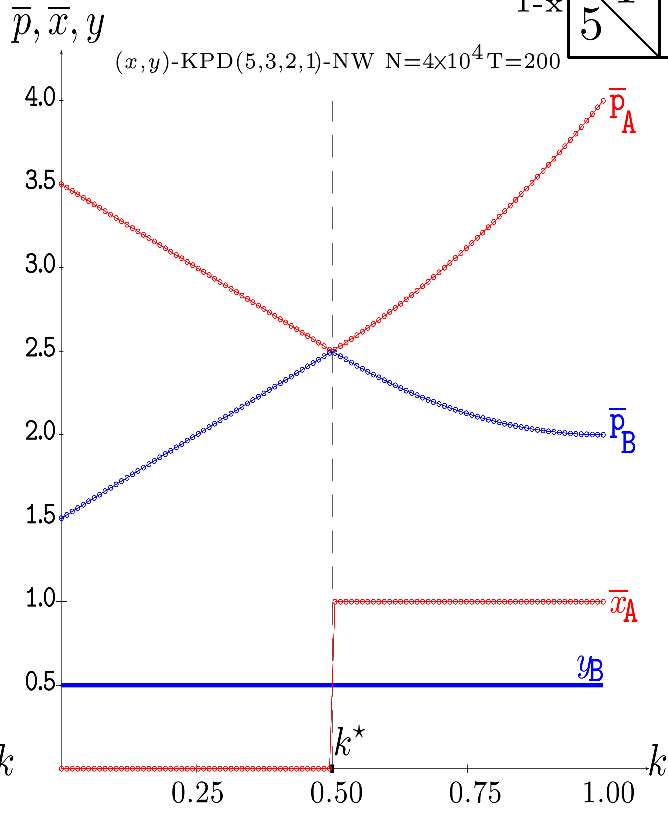
y
B
1-y
x
1-x
A
5
1
0
0
0
0
1
5

ONLY A UPDATES STRATEGY KPD(5,3,2,1)









$$\mathbf{\Pi}(y = \frac{1}{2}) = \frac{1}{2}\begin{pmatrix} (2k-1)^2 x & k-(2k-1)x \\ 1-k+(2k-1)x & 1-(2k-1)^2 x \end{pmatrix}$$

$$p_A = \frac{1}{2}\left(3(2k-1)^2 x + (1-2k)x + k + 5((1-k)+(2k-1)x) + 2(1+((4k(1-k)-1)x)\right) = \frac{1}{2}\left((12k^2-12k+3)+(1-2k)+(10k-5)+(8k-8k^2-2))x + 7 - 4k\right) = \underline{\frac{1}{2}(4k^2+4k-3))x + 7 - 4k)} \quad 4k^2+4k-3=0 \rightarrow \quad k^\star = \frac{1}{2}$$

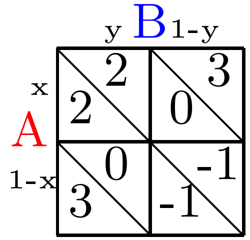
$$\underline{p_A^{(x=0,y=1/2)} = \frac{1}{2}(7-4k)} \quad p_A^{(x=0,y=1/2)}(k=0) = \tfrac{7}{2} = 3.5 \quad p_A^{(x=0,y=1/2)}(k=\tfrac{1}{2}) = \tfrac{5}{2} = 2.5 \qquad \underline{p_A^{(x=1,y=1/2)} = \frac{1}{2}(4k^2+4) = 2k^2+2} \quad p_A^{(x=1,y=1/2)}(k=\tfrac{1}{2}) = \tfrac{5}{2} = 2.5 \quad p_A^{(x=1,y=1/2)}(k=1) = 4.0$$

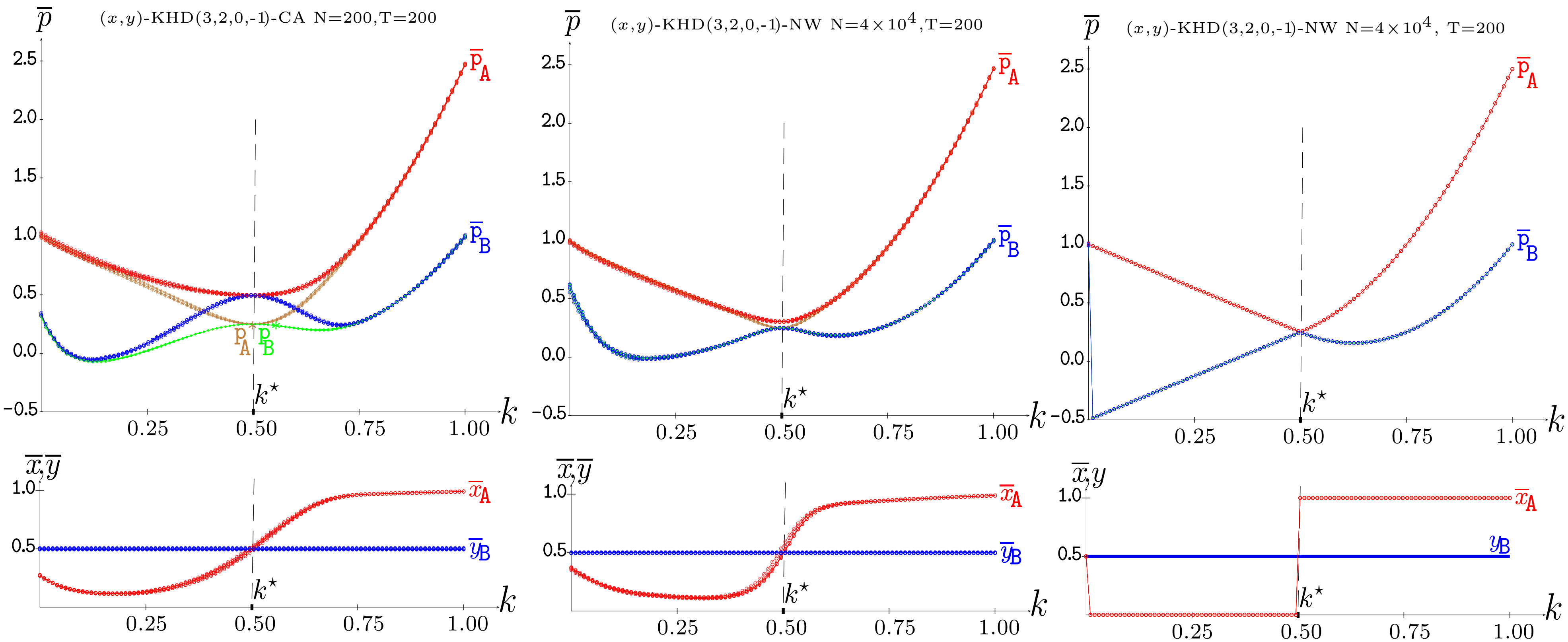
$$p_B = \frac{1}{2}\left(3(2k-1)^2 x + 5((1-2k)x+k) + (1-k) + (2k-1)x + 2(1+((4k(1-k)-1)x)\right) = \frac{1}{2}\left(12k^2-12k+3+5(1-2k)+(2k-1)+2((4k(1-k)-1)x+5k+1-k+2\right) =$$

$$\frac{1}{2}\left(12k^2-12k+3-8k+4+8k-8k^2-2)x+4k+3\right) = \underline{\frac{1}{2}\left(4k^2-12k+5)x+4k+3\right)}$$

$$\underline{p_B^{(x=0,y=1/2)} = 2k + \frac{3}{2}} \quad p_B^{(x=0,y=1/2)}(k=0) = \frac{1}{2}3 = 1.5 \quad p_B^{(x=0,y=1/2)}(k=1/2) = \frac{5}{2} = 2.5 \qquad \underline{p_B^{(x=1,y=1/2)} = 2k^2-4k+4} = \frac{1}{2}(4k^2-8k+8) \quad p_B^{(x=1,y=1/2)}(k=1/2) = \frac{5}{2} \quad p_B^{(x=1,y=1/2)}(k=1) = 2$$

ONLY A UPDATES STRATEGY KHD(3,2,0,-1)





$y = 1/2$

$$p_A = \frac{1}{2}\left(2(2k-1)^2x+3((1-k)+(2k-1)x)-(1-(4k^2-4k+1)x)\right) = \frac{1}{2}\left((8k^2-8k+2)+6k-3+4k^2-4k+1)x+2-3k\right) = \frac{1}{2}\left(12k^2-6k)x+2-3k\right) = \underline{3(2k-1)kx+1-\frac{3}{2}k} \to k^\star = \frac{1}{2}$$

$$\underline{p_A^{(x=0,y=1/2)} = 1-\frac{3}{2}k} \quad p_A^{(x=0,y=1/2)}(k=0) = 1 \quad p_A^{(x=0,y=1/2)}(k=1/2) = 1\frac{3}{4} = \frac{1}{4}$$

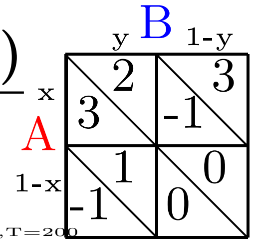
$$\underline{p_A^{(x=1,y=1/2)} = 6k^2-\frac{9}{2}k+1} = 3(2k-1)k+(1-\frac{3}{2}k) = 6k^2-3k-\frac{3}{2}k+1 \quad p_A^{(x=1,y=1/2)}(k=1/2) = \frac{1}{4} \quad p_A^{(x=1,y=1/2)}(k=1) = \frac{5}{2}$$

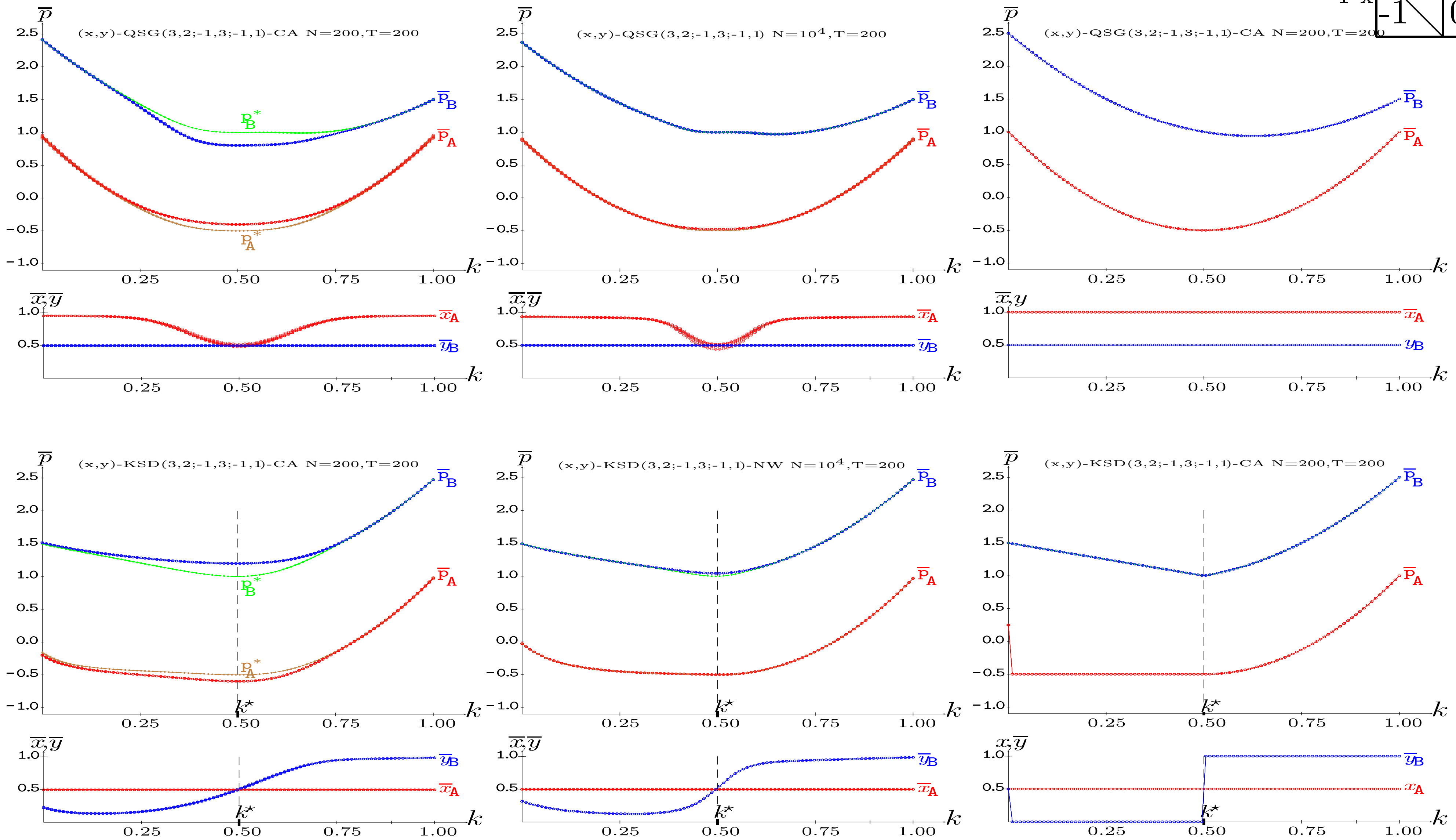
$$p_B = \frac{1}{2}\left(2(2k-1)^2x+3(k+(1-2k)x)-(1-(4k^2-4k+1)x)\right) = \frac{1}{2}\left((8k^2-8k+2+3-6k+4k^2-4k+1))x)+3k-1\right) = \frac{1}{2}\left((12k^2-18k+6)x+3k-1\right) = \underline{(6k^2-9k+3)x+\frac{1}{2}(3k-1)}$$

$$\underline{p_B^{(x=0,y=1/2)} = \frac{1}{2}(3k-1)} \quad p_B^{(x=0,y=1/2)}(k=0) = -\tfrac{1}{2} \quad p_B^{(x=0,y=1/2)}(k=1/2) = \tfrac{1}{4}$$

$$\underline{p_B^{(x=1,y=1/2)} = 6k^2-\frac{15}{2}k+\frac{5}{2}} = 6k^2-9k+3+\frac{1}{2}(3k-1) \quad p_B^{(x=1,y=1/2)}(k=1/2) = \frac{1}{4} \quad p_B^{(x=1,y=1/2)}(k=1) = 1.$$

ONLY ONE PLAYER UPDATES STRATEGY KSD(3,2,1,-1)





y=1/2

$$p_A = \frac{1}{2}(3(2k-1)^2x - (k-(2k-1)x) - ((1-k)+(2k-1)x)) = \frac{1}{2}(12k^2 - 12k + 3 + (2k-1) - (2k-1))x - k - (1-k)) = \frac{1}{2}((12k^2-12k+3)x - 1) = \underline{(6k^2 - 6k + \frac{3}{2})x - \frac{1}{2}} \quad (6k^2 - 6k + \frac{3}{2}) \geq 0 \rightarrow x = 1 \quad \underline{p_A = 6k^2 - 6k + 1}$$

$$p_B = \frac{1}{2}(2(2k-1)^2x + 3(k-(2k-1)x)) + ((1-k)+(2k-1))x) = \frac{1}{2}((8k^2 - 8k + 2 - 3(2k-1) + (2k-1))x + 3k + 1 - k) = \underline{(4k^2 - 6k + 2)x + k + \frac{1}{2}} \rightarrow \underline{p_B = 4k^2 - 5k + \frac{5}{2}}$$

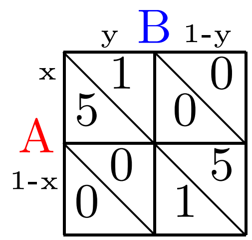
x=1/2

$$p_B = \frac{1}{2}(2(2k-1)^2y + 3((1-k)+(2k-1)y) + (k-(2k-1)y) + y)) = \frac{1}{2}(8k^2 - 8k + 2 + 6k - 3 - 2k + 1)y + 3 - 3k + k) = \frac{1}{2}(8k^2 - 4k)y + 3 - 2k) = (4k^2 - 2k)y + \frac{3}{2} - k) \rightarrow k^\star = 1/2$$

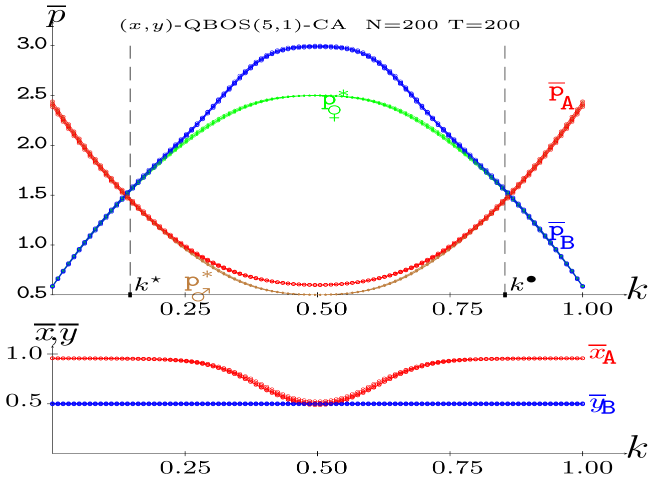
$$\underline{p_B^{(x=1/2,y=0)} = \tfrac{3}{2} - k} \quad \tfrac{3}{2} \quad 1 \qquad \underline{p_B^{(x=1/2,y=1)} = 4k^2 - 3k + \tfrac{3}{2}} \quad 1 \quad \tfrac{5}{2}$$

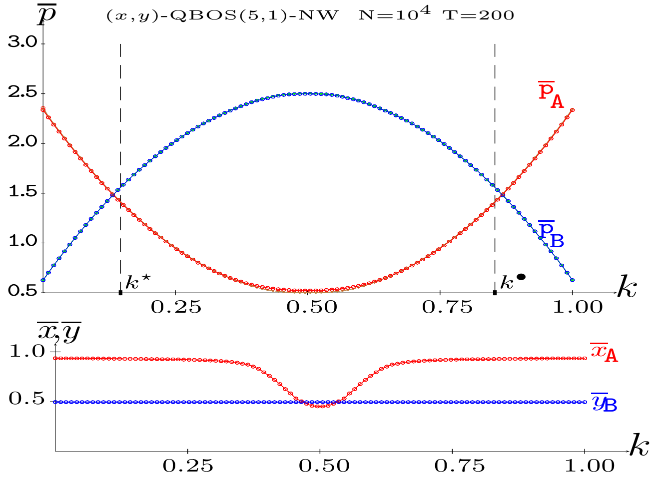
$$p_A = \frac{1}{2}(3(2k-1)^2y - ((1-k)+(2k-1)y) - (k-(2k-1)y+y)) = \frac{1}{2}(12k^2 - 12k + 3 - 2k + 1 + 2k - 1)y - 1 + k - k) = \frac{1}{2}(12k^2 - 12k + 3)y - 1) = (6k^2 - 6k + \frac{3}{2})y - \frac{1}{2} \rightarrow k^\star = 1/2$$

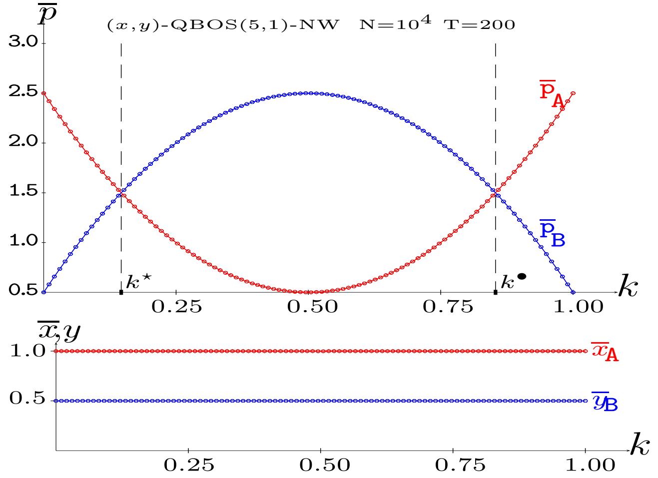
ONLY ONE PLAYER UPDATES STRATEGY KBOS(5,1)

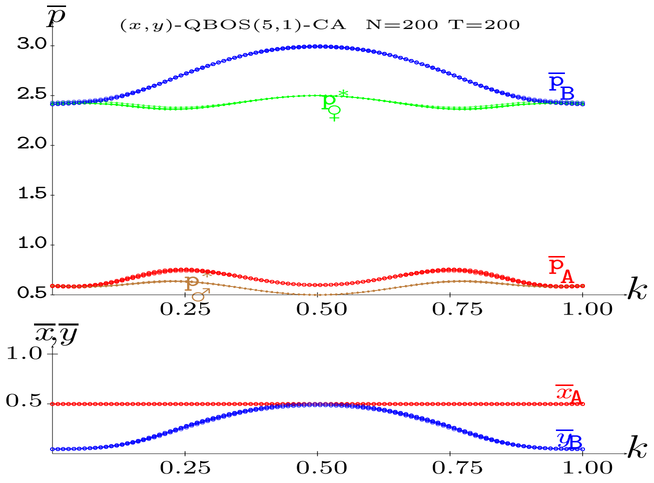


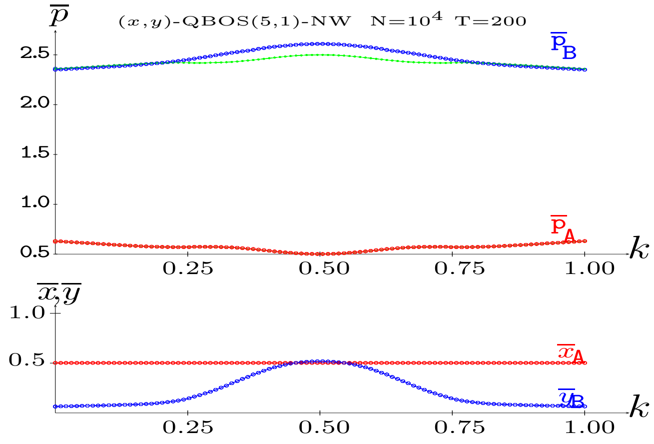
$$x = y = k = \frac{1}{2} \rightarrow \mathbf{\Pi} = \begin{pmatrix} 0 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$$

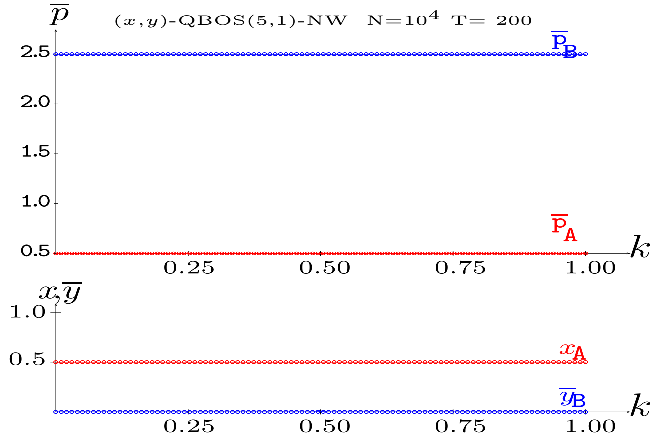












$$\mathbf{\Pi}(y = \frac{1}{2}) = \frac{1}{2} \begin{pmatrix} (2k-1)^2 x & k-(2k-1)x \\ 1-k+(2k-1)x & 1-(2k-1)^2 x \end{pmatrix} \qquad \mathbf{\Pi}(x = \frac{1}{2}) = \frac{1}{2} \begin{pmatrix} (2k-1)^2 y & 1-k+(2k-1)y \\ k-(2k-1)y & 1-(2k-1)^2 y \end{pmatrix}$$

y=1/2

$$p_A = \frac{1}{2}\left(5(2k-1)^2 x + 1 + ((4k(1-k)-1)x\right) = \frac{1}{2}((20k^2-20k+5)x + 1 + (4k-4k^2-1)x) = \underline{(8k^2-8k+2)x + \frac{1}{2}} \qquad 8k^2-8k+2 \geq 0 \rightarrow x = 1 \rightarrow \underline{p_A = 8k^2-8k+\frac{5}{2}}$$

$$p_B = \frac{1}{2}\left((2k-1)^2 x + 5(1 + ((4k(1-k)-1)x)\right) = \frac{1}{2}\left(4k^2-4k+1)^2 x + 5 + (20k-20k^2)-5)x)\right) = \underline{(-8k^2-8k-2)x + \frac{5}{2}} \rightarrow \underline{p_B = -8k^2+8k+\frac{1}{2}}$$

x=1/2

$$p_B = (-8k^2+8k-2)y + \frac{5}{2}) \qquad -8k^2+8k-2 \leq 0 \rightarrow y = 0 \rightarrow \underline{p_B = \frac{5}{2}}, \underline{p_A = \frac{1}{2}}$$

CONCLUSSIONS

- The collective simulation of iterated games on spatial lattices and networks is a very useful tool for the study of two-person games, providing interesting additional information on their features.

- High correlation enables the emergence of new Nash equilibria.

- In the Prisoner's Dilemma, Hawk and Dove, and Samaritan's Dilemma games the new Nash equilibria achieved with high correlation maximize the sum of the payoffs of both players, i.e., provide their (unique) so called social welfare solution.

- The Battle of the Sexes appears as the most challenging game because it has two social welfare solutions and the correlation mechanism adopted in this study tends to favor to one of the players.