Two linearities for quantum computing in the lambda calculus

Alejandro Díaz-Caro

Universidad Nacional de Quilmes
& Instituto de Ciencias de la Computacion / CONICET
Buenos Aires, Argentina

Combining Viewpoints in Quantum Theory

19-22 March 2018 – Edinburgh, UK

Joint work with

Gilles Dowek (Paris, France)
Juan Pablo Rinaldi (Rosario, Argentina)

Ongoing works with

Octavio Malherbe (Montevideo, Uruguay)
Ignacio Grimma (Rosario, Argentina)
Pablo E. Martínez López (Quilmes, Argentina)

Motivation

We are interested in the most natural way of

forbidding duplication

in

quantum lambda calculus

(with quantum control)

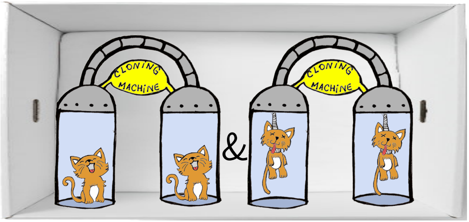
Motivation

Two approaches in the literature to deal with no cloning

Linear-logic approach

e.g. $\lambda x.(x \otimes x)$ is forbidden

Linear-algebra approach



e.g. $f(\alpha\,|0\rangle + \beta\,|1\rangle) \to \alpha f(|0\rangle) + \beta f(|1\rangle)$

Motivation

Measurement

The linear-algebra approach does not make sense here...

...but the linear-logic one, does

---

e.g.

$$(\lambda x.\pi x)\ (\alpha.\ |0\rangle + \beta.\ |1\rangle) \longrightarrow \alpha.(\lambda x.\pi x)\ |0\rangle + \beta.(\lambda x.\pi x)\ |1\rangle$$ **Wrong!**

(Measurement operator)

Key point

We need to distinguish superposed states from basis states using types

Basis states can be cloned
Superposed states cannot

Functions receiving superposed states, cannot clone its argument

Grammars

First version, without tensor

Types

$$\Psi := \mathbb{B} \mid S(\Psi) \qquad \text{Qubit types}$$

$$A := \Psi \mid \Psi \Rightarrow A \mid S(A) \qquad \text{Types}$$

Terms

$$t := \underbrace{x \mid \lambda x^{\Psi}.t \mid |0\rangle \mid |1\rangle}_{\text{basis terms}} \mid tt \mid \pi t \mid ?t{\cdot}t \mid \underbrace{(t+t) \mid \alpha.t \mid \vec{0}_{S(A)}}_{\text{linear combinations}}$$

where $\alpha \in \mathbb{C}$

Intuition

If A is a set of terms, $S(A)$ is its span

e.g. $\mathbb{B} = \{|0\rangle, |1\rangle\} \quad S(\mathbb{B}) = \mathbb{C}^2$

Two kinds of linearity

$$(\lambda x^{\mathbb{B}}.t)\underbrace{b}_{\mathbb{B}} \quad\to\quad t[b/x] \qquad \text{call-by-base}$$

$$\underbrace{(\lambda x^{S(\Psi)}.t)}_{\text{linear abstraction}}\ \underbrace{u}_{S(\Psi)} \quad\to\quad t[u/x] \qquad \text{call-by-name}$$

$$(\lambda x^{\mathbb{B}}.t)\underbrace{(b_1+b_2)}_{S(\mathbb{B})} \quad\to\quad (\lambda x^{\mathbb{B}}.t)\underbrace{b_1}_{\mathbb{B}}+(\lambda x^{\mathbb{B}}.t)\underbrace{b_2}_{\mathbb{B}} \qquad \text{linear distribution}$$

Typing applications

$$\frac{\Gamma \vdash t : \Psi \Rightarrow A \quad \Delta \vdash u : \Psi}{\Gamma, \Delta \vdash tu : A} \Rightarrow_E$$

What about $(\lambda x^{\mathbb{B}}.t)\underbrace{(b_1 + b_2)}_{S(\mathbb{B})}$?

$$\frac{\Gamma \vdash t : \Psi \Rightarrow A \quad \Delta \vdash u : S(\Psi)}{\Gamma, \Delta \vdash tu : S(A)}$$

What about $\underbrace{((\lambda x^{\mathbb{B}}.t) + (\lambda y^{\mathbb{B}}.u))}_{S(\mathbb{B}\Rightarrow A)}\ v$?

$$\frac{\Gamma \vdash t : S(\Psi \Rightarrow A) \quad \Delta \vdash u : S(\Psi)}{\Gamma, \Delta \vdash tu : S(A)} \Rightarrow_{ES}$$

Quantum conditional

$$(?t{\cdot}r)\,|1\rangle \longrightarrow t \qquad\qquad (?t{\cdot}r)\,|0\rangle \longrightarrow r$$

Quantum conditional

$$(?t{\cdot}r)\,|1\rangle \longrightarrow t \qquad\qquad (?t{\cdot}r)\,|0\rangle \longrightarrow r$$

$$(?t{\cdot}r)(\alpha.\,|1\rangle + \beta.\,|0\rangle) \longrightarrow \alpha.(?t{\cdot}r)\,|1\rangle + \beta.(?t{\cdot}r)\,|0\rangle \longrightarrow \alpha.t + \beta.r$$

Quantum conditional

$$(?t\cdot r)\,|1\rangle \longrightarrow t \qquad\qquad (?t\cdot r)\,|0\rangle \longrightarrow r$$

$$(?t\cdot r)(\alpha.\,|1\rangle + \beta.\,|0\rangle) \longrightarrow \alpha.(?t\cdot r)\,|1\rangle + \beta.(?t\cdot r)\,|0\rangle \longrightarrow \alpha.t + \beta.r$$

$$\dfrac{\dfrac{\dfrac{\color{red}{\vdash t : A \quad \vdash r : A}}{\vdash ?t\cdot r : \mathbb{B} \Rightarrow A}\,If}{\vdash ?t\cdot r : S(\mathbb{B} \Rightarrow A)}\,\preceq \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash |1\rangle : \mathbb{B}}\,Ax}{\vdash \alpha.\,|1\rangle : S(\mathbb{B})}\,S_I^{\alpha} \quad \dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{B}}\,Ax}{\vdash \beta.\,|0\rangle : S(\mathbb{B})}\,S_I^{\alpha}}{\vdash \alpha.\,|1\rangle + \beta.\,|0\rangle : S(S(\mathbb{B}))}\,S_I^{+}}{\vdash \alpha.\,|1\rangle + \beta.\,|0\rangle : S(\mathbb{B})}\,\preceq}{\vdash (?t\cdot r)(\alpha.\,|1\rangle + \beta.\,|1\rangle) : \color{red}{S(A)}}\,\Rightarrow_{ES}$$

Quantum conditional

$$(?t{\cdot}r)\,|1\rangle \longrightarrow t \qquad\qquad (?t{\cdot}r)\,|0\rangle \longrightarrow r$$

$$(?t{\cdot}r)(\alpha.\,|1\rangle + \beta.\,|0\rangle) \longrightarrow \alpha.(?t{\cdot}r)\,|1\rangle + \beta.(?t{\cdot}r)\,|0\rangle \longrightarrow \alpha.t + \beta.r$$

$$\dfrac{\dfrac{\dfrac{\color{red}{\vdash t : A \quad \vdash r : A}}{\vdash ?t{\cdot}r : \mathbb{B} \Rightarrow A}\ If}{\vdash ?t{\cdot}r : S(\mathbb{B} \Rightarrow A)}\ \preceq \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash |1\rangle : \mathbb{B}}\ Ax}{\vdash \alpha.\,|1\rangle : S(\mathbb{B})}\ S_I^\alpha \quad \dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{B}}\ Ax}{\vdash \beta.\,|0\rangle : S(\mathbb{B})}\ S_I^\alpha}{\vdash \alpha.\,|1\rangle + \beta.\,|0\rangle : S(S(\mathbb{B}))}\ S_I^+}{\vdash \alpha.\,|1\rangle + \beta.\,|0\rangle : S(\mathbb{B})}\ \preceq}{\vdash (?t{\cdot}r)(\alpha.\,|1\rangle + \beta.\,|1\rangle) : \color{red}{S(A)}}\ \Rightarrow_{ES}$$

$$H = \lambda x^{\mathbb{B}}.(\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.(?{-}|1\rangle{\cdot}|1\rangle)x)$$

$$H\,|0\rangle \longrightarrow (\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle) \qquad H\,|1\rangle \longrightarrow (\frac{1}{\sqrt{2}}.\,|0\rangle - \frac{1}{\sqrt{2}}.\,|1\rangle)$$

Measurement

$$\pi(\alpha_1.b_1 + \alpha_2.b_2) \longrightarrow_{\left(\frac{|\alpha_k|^2}{|\alpha_1|^2+|\alpha_2|^2}\right)} b_k$$

Where $b_i \in \{|0\rangle, |1\rangle\}$.

Example

$$\begin{tikzcd}[column sep=large] & |0\rangle \\ \pi(i.|0\rangle + 2.|1\rangle) \arrow[ur, "\left(\frac{1}{5}\right)"'] \arrow[dr, "\left(\frac{4}{5}\right)"'] & \\ & |1\rangle \end{tikzcd}$$

Adding tensor product

Interpretation of types

$$\begin{aligned}
\llbracket \mathbb{B} \rrbracket &= \{ |0\rangle, |1\rangle \} \subseteq \mathbb{C}^2 \\
\llbracket A \times B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket \\
\llbracket S(A) \rrbracket &= \mathcal{G}\, \llbracket A \rrbracket \\
\mathcal{G}(B_1 \times B_2) &\simeq \mathcal{G}(B_1) \otimes \mathcal{G}(B_2)
\end{aligned}$$

Adding tensor product

Interpretation of types

$$
\begin{aligned}
[\![\mathbb{B}]\!] &= \{|0\rangle, |1\rangle\} \subseteq \mathbb{C}^2 \\
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![S(A)]\!] &= \mathcal{G}\, [\![A]\!] \\
\mathcal{G}(B_1 \times B_2) &\simeq \mathcal{G}(B_1) \otimes \mathcal{G}(B_2)
\end{aligned}
$$

Examples:

$$
\begin{aligned}
\mathcal{G}(\{|0\rangle, |1\rangle\} \times \{|0\rangle, |1\rangle\}) &= \mathcal{G}\{(|0\rangle, |0\rangle), (|0\rangle, |1\rangle), (|1\rangle, |0\rangle), (|1\rangle, |1\rangle)\} \\
&\simeq \mathcal{G}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \\
&= \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \\
&= \mathcal{G}\{|0\rangle, |1\rangle\} \otimes \mathcal{G}\{|0\rangle, |1\rangle\}
\end{aligned}
$$

$$
(\underbrace{|0\rangle}_{\mathbb{B}}, \underbrace{(1/\sqrt{2}.\,|0\rangle + 1/\sqrt{2}.\,|1\rangle)}_{S(\mathbb{B})}) \;\in\; \{|0\rangle, |1\rangle\} \times \mathbb{C}^2
$$

$$
\underbrace{1/\sqrt{2}.(|0\rangle, |0\rangle) \;+\; 1/\sqrt{2}.(|0\rangle, |1\rangle)}_{S(\mathbb{B}\times\mathbb{B})} \;\in\; \mathbb{C}^2 \otimes \mathbb{C}^2
$$

Some information is lost on reduction

Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \leq S(\mathbb{B})$$

$$\mathcal{G}(\mathcal{G}A) = \mathcal{G}A \quad \text{then} \quad S(S(\mathbb{B})) \leq S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

Some information is lost on reduction

Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \leq S(\mathbb{B})$$

$$\mathcal{G}(\mathcal{G}A) = \mathcal{G}A \qquad \text{then} \qquad S(S(\mathbb{B})) \leq S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

$$(|0\rangle, \ |0\rangle + |1\rangle) \quad : \quad \mathbb{B} \times S(\mathbb{B})$$

$$(|0\rangle, |0\rangle) + (|0\rangle, |1\rangle) \quad : \quad S(\mathbb{B} \times \mathbb{B})$$

Some information is lost on reduction

Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \leq S(\mathbb{B})$$

$$\mathcal{G}(\mathcal{G}A) = \mathcal{G}A \qquad \text{then} \qquad S(S(\mathbb{B})) \leq S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

$$(|0\rangle, \ |0\rangle + |1\rangle) \quad : \quad \mathbb{B} \times S(\mathbb{B})$$

$$(|0\rangle, |0\rangle) \ + \ (|0\rangle, |1\rangle) \quad : \quad S(\mathbb{B} \times \mathbb{B})$$

Some information is lost on reduction

Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \leq S(\mathbb{B})$$

$$\mathcal{G}(\mathcal{G}A) = \mathcal{G}A \quad \text{then} \quad S(S(\mathbb{B})) \leq S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

$$(|0\rangle, |0\rangle + |1\rangle) \quad : \quad \mathbb{B} \times S(\mathbb{B})$$

$$(|0\rangle, |0\rangle) + (|0\rangle, |1\rangle) \quad : \quad S(\mathbb{B} \times \mathbb{B})$$

Sure! We are distributing!

$$(X-1)(X-2) \longrightarrow X^2 - 3X + 2$$

we lost the information that it was a product

Some information is lost on reduction

Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \leq S(\mathbb{B})$$
$$\mathcal{G}(\mathcal{G}A) = \mathcal{G}A \quad \text{then} \quad S(S(\mathbb{B})) \leq S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{then} \quad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

$$(|0\rangle, \ |0\rangle + |1\rangle) \quad : \quad \mathbb{B} \times S(\mathbb{B})$$
$$(|0\rangle, |0\rangle) + (|0\rangle, |1\rangle) \quad : \quad S(\mathbb{B} \times \mathbb{B})$$

Sure! We are distributing!

$$(X-1)(X-2) \longrightarrow X^2 - 3X + 2$$

we lost the information that it was a product

Solution: casting

$$(|0\rangle, |0\rangle + |1\rangle) \not\rightarrow (|0\rangle, |0\rangle) + (|0\rangle, |1\rangle)$$
$$\Uparrow_\ell(|0\rangle, |0\rangle + |1\rangle) \rightarrow (|0\rangle, |0\rangle) + (|0\rangle, |1\rangle)$$

Full grammars

Types

$$\Psi := \mathbb{B} \mid S(\Psi) \mid \Psi \times \Psi \qquad \text{Qubit types}$$

$$A := \Psi \mid \Psi \Rightarrow A \mid S(A) \mid A \times A \qquad \text{Types}$$

Terms

$$t := x \mid \lambda x^{\Psi}.t \mid |0\rangle \mid |1\rangle \mid tt \mid \pi_j t \mid ?t{\cdot}t \mid (t+t) \mid \alpha.t \mid \vec{0}_{S(A)}$$
$$\mid t \times t \mid head\ t \mid tail\ t \mid \Uparrow_r t \mid \Uparrow_\ell t$$

where $\alpha \in \mathbb{C}$

Measurement of the first j qubits

$$\pi_j\Big(\sum_{i=1}^{n}[\alpha_i.]\prod_{h=1}^{m} b_{hi}\Big) \longrightarrow_{(p_k)} \left(\prod_{l=1}^{j} b_{lk}\right) \times \sum_{i\in P}\left(\frac{\alpha_i}{\sqrt{\sum\limits_{r\in P}|\alpha_r|^2}}\right) . \prod_{h=j+1}^{m} b_{hi}$$

$$\boxed{k \leq n.}$$

$$\boxed{\begin{array}{c} P \subseteq \mathbb{N}^{\leq n}, \text{ such that} \\ \forall i \in P, \forall h \leq j, \\ b_{hi} = b_{hk}. \end{array}}$$

$$\boxed{p_k = \sum_{i\in P}\left(\frac{|\alpha_i|^2}{\sum\limits_{r=1}^{n}|\alpha_r|^2}\right)}$$

Example

$$\pi_2(\ 2\,|{\bf 01}1\rangle\ +\ |{\bf 01}0\rangle\ +\ 3\,|{\bf 11}1\rangle\ )$$

$$\xrightarrow{(\frac{5}{14})}\ |{\bf 01}\rangle \times \left(\tfrac{2}{\sqrt{5}}\,|1\rangle + \tfrac{1}{\sqrt{5}}\,|0\rangle\right)$$

$$\xrightarrow{(\frac{9}{14})}\ |{\bf 11}\rangle \times (1\,|1\rangle)$$

The full type system

$$\begin{array}{lll} \Psi & := \mathbb{B} \mid S(\Psi) \mid \Psi \times \Psi & \text{Qubit types} \\ A & := \Psi \mid \Psi \Rightarrow A \mid S(A) \mid A \times A & \text{Types} \end{array}$$

$$\frac{}{x : \Psi \vdash x : \Psi}\ Ax \qquad \frac{}{\vdash \vec{0}_{S(A)} : S(A)}\ Ax_{\vec{0}} \qquad \frac{}{\vdash |0\rangle : \mathbb{B}}\ Ax_{|0\rangle} \qquad \frac{}{\vdash |1\rangle : \mathbb{B}}\ Ax_{|1\rangle}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \alpha.t : S(A)}\ S_I^{\alpha} \qquad \frac{\Gamma \vdash t : A \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash (t + u) : S(A)}\ S_I^{+} \qquad \frac{\Gamma \vdash t : S(\mathbb{B}^n)}{\Gamma \vdash \pi_j t : \mathbb{B}^j \times S(\mathbb{B}^{n-1})}\ S_E$$

$$\frac{\Gamma \vdash t : A \ (A \preceq B)}{\Gamma \vdash t : B}\ \preceq \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash ?t{\cdot}u : \mathbb{B} \Rightarrow A}\ If \qquad \frac{\Gamma, x : \Psi \vdash t : A}{\Gamma \vdash \lambda x : \Psi\ t : \Psi \Rightarrow A}\ \Rightarrow_I$$

$$\frac{\Gamma \vdash t : \Psi \Rightarrow A \quad \Delta \vdash u : \Psi}{\Gamma, \Delta \vdash tu : A}\ \Rightarrow_E \qquad \frac{\Gamma \vdash t : S(\Psi \Rightarrow A) \quad \Delta \vdash u : S(\Psi)}{\Gamma, \Delta \vdash tu : S(A)}\ \Rightarrow_{ES}$$

$$\frac{\Gamma \vdash t : A}{\Gamma, x : \mathbb{B}^n \vdash t : A}\ W \qquad \frac{\Gamma, x : \mathbb{B}^n, y : \mathbb{B}^n \vdash t : A}{\Gamma, x : \mathbb{B}^n \vdash (x/y)t : A}\ C$$

$$\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma, \Delta \vdash t \times u : A \times B}\ \times_I \qquad \frac{\Gamma \vdash t : \mathbb{B}^n}{\Gamma \vdash head\ t : \mathbb{B}}\ \times_{Er} \qquad \frac{\Gamma \vdash t : \mathbb{B}^n}{\Gamma \vdash tail\ t : \mathbb{B}^{n-1}}\ \times_{El}$$

$$\frac{\Gamma \vdash t : S(S(A) \times B)}{\Gamma \vdash \Uparrow_r t : S(A \times B)}\ \Uparrow_r \qquad \frac{\Gamma \vdash t : S(A \times S(B))}{\Gamma \vdash \Uparrow_\ell t : S(A \times B)}\ \Uparrow_\ell$$

Rewrite rules

Beta	If b has type $\mathbb{B}^n$ and $b \in \mathcal{B}$, $(\lambda x^{\mathbb{B}^n}.t)b \longrightarrow_{(1)} (b/x)t$	(β_b)
	If u has type $S(\Psi)$, $(\lambda x^{S(\Psi)}.t)u \longrightarrow_{(1)} (u/x)t$	(β_n)
If	$\|1\rangle ?t{\cdot}r \longrightarrow_{(1)} t$	(if_1)
	$\|0\rangle ?t{\cdot}r \longrightarrow_{(1)} r$	(if_0)
Linear distribution	If t has type $\mathbb{B}^n \Rightarrow A$, $t(u+v) \longrightarrow_{(1)} (tu+tv)$	(lin_r^+)
	If t has type $\mathbb{B}^n \Rightarrow A$, $(\alpha.u) \longrightarrow_{(1)} \alpha.tu$	(lin_r^α)
	If t has type $\mathbb{B}^n \Rightarrow A$, $t\vec{0}_{S(\mathbb{B}^n)} \longrightarrow_{(1)} \vec{0}_{S(A)}$	(lin_r^0)
	$(t+u)v \longrightarrow_{(1)} (tv+uv)$	(lin_l^+)
	$(\alpha.t)u \longrightarrow_{(1)} \alpha.tu$	(lin_l^α)
	$\vec{0}_{S(\mathbb{B}^n \Rightarrow A)}t \longrightarrow_{(1)} \vec{0}_{S(A)}$	(lin_l^0)

Rewrite rules

Continuation

Vector space axioms	$(\vec{0}_{S(A)} + t) \longrightarrow_{(1)} t$	(*neutral*)
	$1.t \longrightarrow_{(1)} t$	(*unit*)
	If t has type A, $0.t \longrightarrow_{(1)} \vec{0}_{S(A)}$	($zero_\alpha$)
	$\alpha.\vec{0}_{S(A)} \longrightarrow_{(1)} \vec{0}_{S(A)}$	(*zero*)
	$\alpha.(\beta.t) \longrightarrow_{(1)} (\alpha\beta).t$	(*prod*)
	$\alpha.(t+u) \longrightarrow_{(1)} (\alpha.t + \alpha.u)$	(α*dist*)
	$(\alpha.t + \beta.t) \longrightarrow_{(1)} (\alpha+\beta).t$	(*fact*)
	$(\alpha.t + t) \longrightarrow_{(1)} (\alpha+1).t$	($fact^1$)
	$(t+t) \longrightarrow_{(1)} 2.t$	($fact^2$)
	$\vec{0}_{S(S(A))} \longrightarrow_{(1)} \vec{0}_{S(A)}$	($zero_S$)
$=$	$(t+r) =_{AC} (r+t)$	(*comm*)
	$((t+r)+s) =_{AC} (t+(r+s))$	(*assoc*)
Lists	If $h \neq u \times v$ and $h \in \mathcal{B}$, $head\ h \times t \longrightarrow_{(1)} h$	(*head*)
	If $h \neq u \times v$ and $h \in \mathcal{B}$, $tail\ h \times t \longrightarrow_{(1)} t$	(*tail*)

Rewrite rules

Continuation

Typing casts

	$\Uparrow_r (r+s)\times u \longrightarrow_{(1)} (\Uparrow_r r\times u + \Uparrow_r s\times u)$	$(dist_r^+)$
	$\Uparrow_\ell u\times(r+s) \longrightarrow_{(1)} (\Uparrow_\ell u\times r + \Uparrow_\ell u\times s)$	$(dist_l^+)$
	$\Uparrow_r (\alpha.r)\times u \longrightarrow_{(1)} \alpha.\Uparrow_r r\times u$	$(dist_r^\alpha)$
	$\Uparrow_\ell u\times(\alpha.r) \longrightarrow_{(1)} \alpha.\Uparrow_r u\times r$	$(dist_l^\alpha)$
If u has type B,	$\Uparrow_r \vec{0}_{S(A)}\times u \longrightarrow_{(1)} \vec{0}_{S(A\times B)}$	$(dist_r^0)$
If u has type A,	$\Uparrow_\ell u\times\vec{0}_{S(B)} \longrightarrow_{(1)} \vec{0}_{S(A\times B)}$	$(dist_l^0)$
	$\Uparrow (t+u) \longrightarrow_{(1)} (\Uparrow t + \Uparrow u)$	$(dist_\Uparrow^+)$
	$\Uparrow (\alpha.t) \longrightarrow_{(1)} \alpha.\Uparrow t$	$(dist_\Uparrow^\alpha)$
If $u\in\mathcal{B}$,	$\Uparrow_r u\times v \longrightarrow_{(1)} u\times v$	$(neut_r^\Uparrow)$
If $v\in\mathcal{B}$,	$\Uparrow_\ell u\times v \longrightarrow_{(1)} u\times v$	$(neut_\ell^\Uparrow)$

Projection

$$\pi\Big(\sum_{i=1}^n[\alpha_i.]\prod_{h=1}^m b_{hi}\Big) \longrightarrow_{(p)} \Big(\prod_{h=1}^j b_{hk}\Big)\times\sum_{i\in P}\left(\frac{\alpha_i}{\sqrt{\sum\limits_{r\in P}|\alpha_r|^2}}\right)\prod_{h=j+1}^m b_{hi} \qquad (proj)$$

where $k\leq n$; $P\subseteq\mathbb{N}^{\leq n}$ s.t. $\forall i\in P$, $\forall h\leq j$, $b_{hi}=b_{hk}$; $p=\sum\limits_{i\in P}\frac{|\alpha_i|^2}{\sum_{r=1}^n|\alpha_r|^2}$; $\forall i, b_i=|0\rangle$ or $b_i=|1\rangle$; $\sum_{i=1}^n[\alpha_i.]\prod_{h=1}^m b_{hi}$ is a normal term; and if an α_k is absent, $|\alpha_k|^2=1$.

Categorical interpretation

Work-in-progress with Octavio Malherbe (early ideas)

$$\left[\!\left[\frac{}{x:\Psi\vdash x:\Psi}\right]\!\right] = \Psi \xrightarrow{\mathsf{Id}} \Psi$$

$$\left[\!\left[\frac{\Gamma\vdash t:A\quad \Delta\vdash r:A}{\Gamma,\Delta\vdash t+r:S(A)}\right]\!\right] = \Gamma\times\Delta \xrightarrow{t\times r} A\times A \xrightarrow{+} S(A)$$

$$\left[\!\left[\frac{\Delta\vdash r:\Psi\quad \Gamma\vdash t:\Psi\Rightarrow A}{\Delta,\Gamma\vdash tr:A}\right]\!\right] = \Delta\times\Gamma \xrightarrow{r\times t} \Psi\times[\Psi,A] \xrightarrow{\varepsilon} A$$

$$\left[\!\left[\frac{\Delta\vdash r:S(\Psi)\quad \Gamma\vdash t:S(\Psi\Rightarrow A)}{\Delta,\Gamma\vdash tr:S(A)}\right]\!\right] = \Delta\times\Gamma \xrightarrow{r\times t} S(\Psi)\times S([\Psi,A])$$

$$\xrightarrow{\otimes} S(\Psi)\otimes S([\Psi,A]) \approx S(\Psi\times[\Psi,A])$$

$$\xrightarrow{S(\varepsilon)} S(A)$$

$$\left[\!\left[\frac{\Gamma\vdash t:S(S(A)\times B)}{\Gamma\vdash_{\Uparrow_r} t:S(A\times B)}\right]\!\right] = \Gamma \xrightarrow{t} S(S(A)\times B) \approx S(S(A))\otimes S(B)$$

$$\xrightarrow{\mu\otimes\mathsf{Id}} S(A)\otimes S(B) \approx S(A\times B)$$

Categorical interpretation

Soundness example

$$\dfrac{\dfrac{\vdash t : \mathbb{B} \Rightarrow A}{\vdash t : S(\mathbb{B} \Rightarrow A)} \qquad \dfrac{\vdash r : \mathbb{B} \quad \vdash s : \mathbb{B}}{\vdash r + s : S(\mathbb{B})}}{\vdash t(r+s) : S(A)} \qquad \dfrac{\dfrac{\vdash t : \mathbb{B} \Rightarrow A \quad \vdash r : \mathbb{B}}{\vdash tr : A} \qquad \dfrac{\vdash t : \mathbb{B} \Rightarrow A \quad \vdash s : \mathbb{B}}{\vdash ts : A}}{\vdash tr + ts : S(A)}$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dashed, "\approx"] \arrow[d, dashed, "\approx"] & \mathbf{1} \times \mathbf{1} \times \mathbf{1} \arrow[r, "r \times s \times t"] & \mathbb{B} \times \mathbb{B} \times [\mathbb{B}, A] \arrow[d, "+ \times \eta"] \\
\mathbf{1} \times \mathbf{1} \times \mathbf{1} \times \mathbf{1} \arrow[d, "r \times t \times s \times t"] & & S(\mathbb{B}) \times S([\mathbb{B}, A]) \arrow[d, "\otimes"] \\
\mathbb{B} \times [\mathbb{B}, A] \times \mathbb{B} \times [\mathbb{B}, A] \arrow[d, "\varepsilon \times \varepsilon"] & & S(\mathbb{B}) \otimes S([\mathbb{B}, A]) \arrow[d, dashed, "\approx"] \\
A \times A \arrow[dr, "+"] & & S(\mathbb{B} \times [\mathbb{B}, A]) \arrow[dl, "S(\varepsilon)"'] \\
& S(A) &
\end{tikzcd}$$

Summarizing

- First-order quantum lambda calculus (w/quantum control)
- Algebraic linearity and logical linearity combined to avoid cloning
- Cartesian category, with internal tensor products

Works-in-progress

- Strong normalization (under review) (with J. P. Rinaldi)
- Abstract category model (with O. Malherbe)
- Haskell implementation (with I. Grimma and P. E. Martínez López)

Backup slides

Why first order

$$\mathsf{CM} = \lambda y^{S(\mathbb{B})}.((\lambda x^{\mathbb{B}\Rightarrow S(\mathbb{B})}.(x\,|0\rangle\,, x\,|0\rangle))\ (\lambda z^{\mathbb{B}}.y))$$

$$\begin{aligned}
&\mathsf{CM}\ (\alpha.\,|0\rangle + \beta.\,|1\rangle)\\
&\to (\lambda x^{\mathbb{B}\Rightarrow S(\mathbb{B})}.(x\,|0\rangle\,, x\,|0\rangle))\ (\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\\
&\to ((\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\,|0\rangle\,, (\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\,|0\rangle)\\
&\to^2 (\alpha.\,|0\rangle + \beta.\,|1\rangle\,, \alpha.\,|0\rangle + \beta.\,|1\rangle)
\end{aligned}$$

Deutsch algorithm

Preliminaries

Hadamard

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Deutsch algorithm

Preliminaries

Hadamard

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \qquad H\,|1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$$

$$H = \lambda x^{\mathbb{B}}.{}^{1}\!/\!_{\sqrt{2}}.((|0\rangle + x?{-|1\rangle}{\cdot}|1\rangle))$$

Deutsch algorithm

Preliminaries

Hadamard

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$H = \lambda x^{\mathbb{B}}.{}^{1}\!/\!\sqrt{2}.((|0\rangle + x?{-}|1\rangle{\cdot}|1\rangle))$$

Oracle

A "black box" implementing a function $f : \{0,1\} \to \{0,1\}$

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle$$

Deutsch algorithm

Preliminaries

Hadamard

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$H = \lambda x^{\mathbb{B}}.{}^{1}\!/\!\sqrt{2}.((|0\rangle + x?{-|1\rangle}\cdot|1\rangle))$$

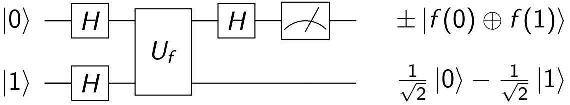
Oracle

A "black box" implementing a function $f : \{0,1\} \to \{0,1\}$

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle$$

$$\begin{aligned} not &= \lambda x^{\mathbb{B}}.x?|0\rangle\cdot|1\rangle \\ U_f &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(head\ x, (tail\ x)?not(f(head\ x))\cdot f(head\ x)) \end{aligned}$$

Deutsch in λ



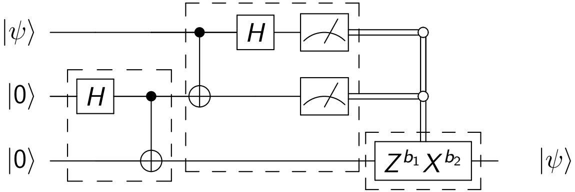
$$\begin{aligned}
\mathit{not} &= \lambda x^{\mathbb{B}}.x?|0\rangle\cdot|1\rangle \\
H &= \lambda x^{\mathbb{B}}.{}^{1}\!/\!\sqrt{2}.((|0\rangle + x?{-|1\rangle}\cdot|1\rangle)) \\
H^{\otimes 2} &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(H(\mathit{head}\ x), H(\mathit{tail}\ x)) \\
U_f &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(\mathit{head}\ x, (\mathit{tail}\ x)?\mathit{not}(f(\mathit{head}\ x))\cdot f(\mathit{head}\ x)) \\
H_1 &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(H(\mathit{head}\ x), \mathit{tail}\ x)
\end{aligned}$$

$$\mathit{Deutsch}_f = \pi_1(\Uparrow_r H_1(U_f \Uparrow_\ell \Uparrow_r H^{\otimes 2}(|0\rangle\,, |1\rangle)$$

$$\vdash \mathit{Deutsch}_f : \mathbb{B} \times S(\mathbb{B})$$

$$\begin{aligned}
\mathit{Deutsch}_{id} &\longrightarrow^{*}_{(1)} \pi_1({}^{1}\!/\!\sqrt{2}.\,|10\rangle - {}^{1}\!/\!\sqrt{2}.\,|11\rangle) \\
&\longrightarrow_{(1)} (|1\rangle\,, {}^{1}\!/\!\sqrt{2}.\,|0\rangle - {}^{1}\!/\!\sqrt{2}.\,|1\rangle)
\end{aligned}$$

Teleportation in λ



$$
\begin{aligned}
&\text{epr} = \lambda x^{\mathbb{B}\times\mathbb{B}}.\text{cnot}(H_1\ x)\\
&\text{alice} = \lambda x^{S(\mathbb{B})\times S(\mathbb{B}\times\mathbb{B})}.\pi_2(\Uparrow_r\ H_1^3(cnot_{12}^3\ \Uparrow_\ell\Uparrow_r\ x))\\
&U^b = (\lambda b^{\mathbb{B}}.\lambda x^{\mathbb{B}}.b?Ux\cdot x)\ b\\
&\text{bob} = \lambda x^{\mathbb{B}\times\mathbb{B}\times\mathbb{B}}.Z^{head\ x}not^{head\ (tail\ x)}.(tail\ (tail\ x))\\
&Teleportation = \lambda q^{S(\mathbb{B})}.\text{bob}\ (\Uparrow_\ell\ \text{alice}\ (q,\text{epr}\ |00\rangle))
\end{aligned}
$$

$\vdash Teleportation : S(\mathbb{B}) \Rightarrow S(\mathbb{B})$

$Teleportation\ q \longrightarrow_{(1)} q$