Tensor topology

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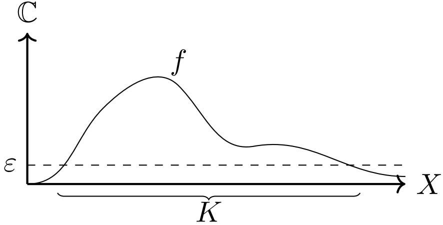
Outline

$\mathbf{Hilb}_{C_0(X)}$

Let X be locally compact Hausdorff space

$$C_0(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \forall \varepsilon > 0\, \exists K \subseteq X \text{ cpt}\colon f(X \setminus K) < \varepsilon\}$$



Hilbert $C_0(X)$-modules $\simeq$ bundles of Hilbert spaces over X

Idempotent subunits

Have $C_0(X)$, want $U \subseteq X$ open

In $\mathbf{Sh}(X)$, $\{\text{subunits}\} \simeq \{U \subseteq X \text{ open}\}$

Define

$$\mathrm{ISub}(\mathbf{C}) = \{ s \colon S \rightarrowtail I \mid \mathrm{id}_S \otimes s \colon S \otimes S \to S \otimes I \text{ iso} \\ \exists\ S \otimes (-) \Rightarrow (-) \otimes S \text{ iso } \} / \simeq$$

In $\mathbf{Hilb}_{C_0(X)}$, subunit $s \colon S \rightarrowtail C_0(X)$ idempotent

$$\Longleftrightarrow$$

$S = C_0(U)$ for open $U \subseteq X$

Idempotent subunits

$\mathrm{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$
'idempotent subunits are open subsets of base space'

$\mathrm{ISub}(\mathbf{Sh}(X)) = \{S \subseteq X \text{ open}\}$
'idempotent subunits are truth values'

$\mathrm{ISub}(Q) = \{x \in Q \mid x^2 = x \leq 1\}$
'idempotent subunits are side-effect-free observations'

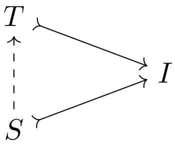
$\mathrm{ISub}(\mathbf{Mod}_R) = \{S \subseteq R \text{ ideal } \mid S = S^2\}$
'idempotent subunits are idempotent ideals'

Idempotent subunits

Define

$$\big(s\colon S \rightarrowtail I\big) \otimes \big(t\colon T \rightarrowtail I\big) = \big(\lambda_I \circ (s \otimes t)\colon S \otimes T \to I\big)$$

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = \mathrm{id}_I$



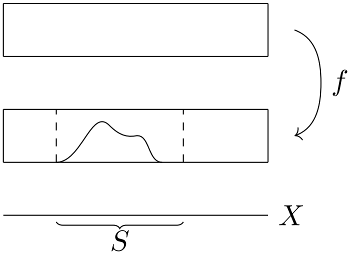
Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes \mathrm{id}_T$ monic, and size issue

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when

$$\begin{array}{ccc} A & \xrightarrow[f]{} & B \\ \downarrow & & \uparrow \simeq \\ B \otimes S & \xrightarrow[\mathrm{id} \otimes s]{} & B \otimes I \end{array}$$

Intuition:



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$$\begin{array}{ccc} f & \longmapsto & \{ s \mid s \text{ supports } f \} \\ \mathbf{C}^{\mathbf{2}} & \xrightarrow{\quad \mathrm{supp} \quad} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \end{array}$$

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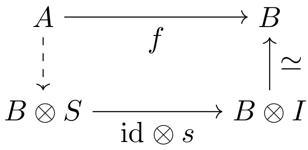

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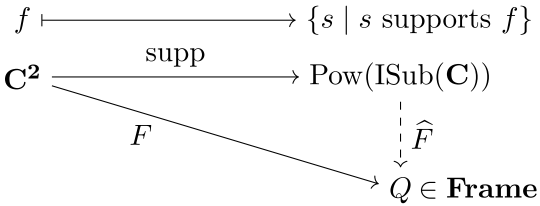

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$$\begin{array}{ccc} f & \longmapsto & \{s \mid s \text{ supports } f\} \\ \mathbf{C}^{\mathbf{2}} & \xrightarrow{\qquad \mathrm{supp} \qquad} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \\ & {\scriptstyle F} \searrow & \downarrow \widehat{F} \\ & & Q \in \mathbf{Frame} \end{array}$$

universal with $F(f) = \bigvee \{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $\mathrm{id}_A \otimes s$ invertible is:

monoidal with tensor unit S

coreflective: $\mathbf{C}\big|_s \underset{\leftarrow - - - -}{\overset{\hookrightarrow}{\perp}} \mathbf{C}$

tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$

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Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{$monocoreflective tensor ideals in $\mathbf{C}\}$

Localisation

A graded monad is a monoidal functor $T\colon \mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$$(\eta\colon A \to T(1),\ \mu\colon T(t) \circ T(s) \to T(s \otimes t))$$

Lemma: Family of restrictions is an ISub($\mathbf{C}$)-graded monad

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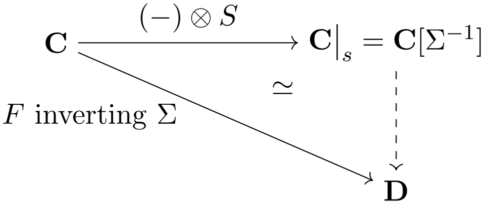
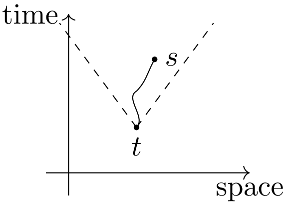
Universal property of localisation for $\Sigma = \{\mathrm{id}_E \otimes s \mid E \in \mathbf{C}\}$



Causal structure

What if X is a spacetime?

Define the causal future of a point as $J^+(t) = \{s \in X \mid t \prec s\}$



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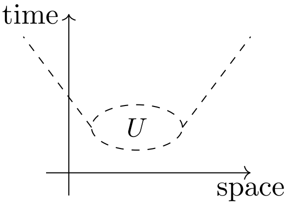

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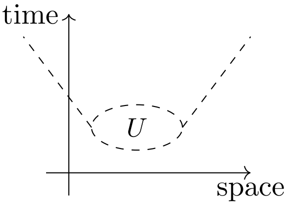
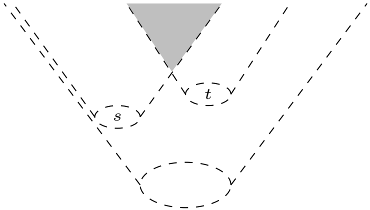


Causal structure on $\mathbf{C}$ is a pair $C^\pm$ of closure operators on $\mathrm{ISub}(\mathbf{C})$

Teleportation



Pair creation at s: $\eta \otimes \mathrm{id}_{C^+(s)}$

Restriction = propagation

Protocol: $(\varepsilon \otimes \mathrm{id}_{C^+(s) \otimes B}) \circ (\mathrm{id}_A \otimes \eta \otimes \mathrm{id}_{C^+(s) \otimes C^+(t)})$

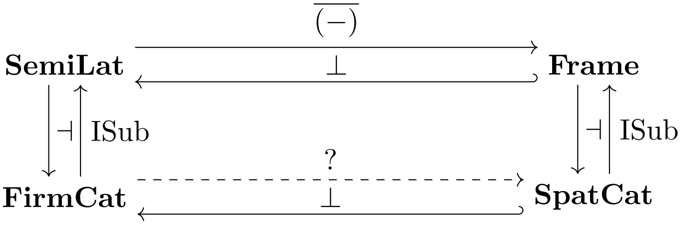
Has support in intersection of Alice's and Bob's causal futures

Spatial categories

Call $\mathbf{C}$ spatial when $\mathrm{ISub}(\mathbf{C})$ is a frame

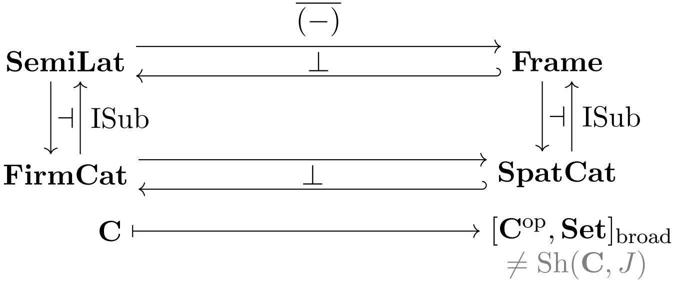
Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$



But $\mathrm{ISub}(\widehat{\mathbf{C}}) \neq \overline{\mathrm{ISub}(\mathbf{C})}$

Spatial categories



$[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{broad}}$ is closed under $\widehat{\otimes}$, includes representables

If $\mathrm{ISub}(\mathbf{C})$ is a complete lattice, any f has *the* support

$$\bigwedge \mathrm{supp}(f) = \bigvee \{t \in \mathrm{ISub}(\mathbf{C}) \mid f \text{ supported on } s \implies t \leq s\}$$

Future directions

Relativistic quantum information protocols

Causality

Graphical calculus

Control flow

Concurrency

Linear logic

Representation theory