Dagger limits

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Structure of the talk

1. Dagger categories
2. Dagger limits
3. Completeness
4. Polar decomposition

Dagger = a functorial way of reversing arrows:

$$A \xrightarrow{f = f^{\dagger\dagger}} B$$

$$A \xleftarrow[f^{\dagger}]{} B$$

Category	Objects	Morphisms	Dagger
Rel	Sets	Relations	inverse
PInj	Sets	Partial injections	inverse
FHilb	F.d. Hilbert spaces	linear maps	adjoint
Groupoid **G**	ob(**G**)	mor(**G**)	inverse

Dictionary

Ordinary notion	Dagger counterpart	Added condition
Isomorphism	Unitary	$f^{-1} = f^\dagger$
Mono	Dagger mono	$f^\dagger f = \mathrm{id}$
Epi	Dagger epi	$ff^\dagger = \mathrm{id}$
	Partial isometry	$f = ff^\dagger f$
Idempotent $p = p^2$	Projection	$p = p^\dagger$
Functor	Dagger Functor	$F(f^\dagger) = F(f)^\dagger$

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Limits

The product of A and B is given by a *terminal cone*

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, dashed] & A \times B \arrow[ur, "p_A"'] \arrow[dr, "p_B"] & \\
 & & B
\end{tikzcd}$$

The notion of a limit generalizes this to an arbitrary diagram $D : \mathbf{J} \to \mathbf{C}$. Equalizers are limits of diagrams of shape $\bullet \rightrightarrows \bullet$

Biproducts

A biproduct is a product + coproduct

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} A \oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B$$

such that

$$p_A i_A = \mathrm{id}_A \qquad p_B i_B = \mathrm{id}_B$$
$$p_B i_A = 0_{A,B} \qquad p_A i_B = 0_{B,A}$$

What should dagger limits be?

- Unique up to unique *unitary*
- Defined (canonically) for arbitrary diagrams
- Definition shouldn't depend on additional structure (e.g. enrichment)
- Generalizes dagger biproducts and dagger equalizers
- Connections to dagger adjunctions etc.

Known examples of dagger limits

- Dagger biproduct of A and B is a biproduct of the form $(A \oplus B, p_A, p_B, p_A^\dagger, p_B^\dagger)$
- Dagger equalizer is an equalizer e that is dagger monic
- Given a diagram from an indiscrete category $\mathbf{J}$ to $\mathbf{C}$: one dagger limit for each choice of $A \in \mathbf{J}$

How to generalize?

1. Maps $A \oplus B \to A, B$ are dagger epic, whereas dagger equalizers $E \to A$ are dagger monic.

2. Requiring the structure maps to be partial isometries generalizes both.

3. Based on equalizers and indiscrete diagrams, one can only require this on a weakly initial set.

4. One also needs to generalize from $A \to A \oplus B \to B = 0_{A,B}$

5. This can be done by saying that the induced projections on the limit commute.

Defining dagger limits

Definition

Let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram and let $\Omega \subseteq J$ be weakly initial. *A dagger limit of D with support Ω* is a limit L of D whose cone $l_A\colon L \to D(A)$ satisfies the following two properties:

normalization: l_A is a partial isometry for every $A \in \Omega$;

independence: the projections on L induced by these partial isometries commute, *i.e.* $l_A^\dagger l_A l_B^\dagger l_B = l_B^\dagger l_B l_A^\dagger l_A$ for all $A, B \in \Omega$.

Uniqueness

Theorem

Let L and M be dagger limits of $D \colon \mathbf{J} \to \mathbf{C}$. Then L and M are unitarily isomorphic as limits if and only if they are both dagger limits with support in the same weakly initial class.

Often Ω is forced on us:

- Products $\color{red}{\bullet} \quad \color{red}{\bullet}$
- Equalizers $\color{red}{\bullet} \rightrightarrows \bullet$
- Pullbacks $\color{red}{\bullet} \to \bullet \leftarrow \color{red}{\bullet}$

But not always: $\bullet \leftrightarrows \color{red}{\bullet}$ or $\color{red}{\bullet} \leftrightarrows \bullet$

Definition

A dagger-shaped dagger limit is the dagger limit of a dagger functor.

E.g. products, limits of projections, unitary representations of groupoids.

Definition

A set $\Omega \subset \mathbf{J}$ is a *basis* when every object B allows a unique $A \in \Omega$ making $\mathbf{J}(A, B)$ non-empty.
(Finitely) based dagger limit: support Ω is a (finite) basis

- Products: $\color{red}{\bullet} \qquad \color{red}{\bullet}$
- Equalizers:$\color{red}{\bullet} \rightrightarrows \bullet$
- Indiscrete categories $\bullet \leftrightarrows \color{red}{\bullet}$
- Nonexample: $\bullet \to \bullet \leftarrow \bullet$

- If $\mathbf{C}$ has zero morphisms, L is a dagger-shaped limit iff
  - each $L \to D(A)$ is a partial isometry
  - $D(A) \to L \to D(B) = 0$ whenever $\hom(A, B)$ is empty.
- If $\mathbf{C}$ is enriched in commutative monoids, then finitely based dagger limits can be equivalently defined by

$$\mathrm{id}_L = \sum_{A \in \Omega} L \to D(A) \to L$$

Theorem
A dagger category has dagger-shaped limits iff it has dagger split infima of projections, dagger stabilizers, and dagger products.

Theorem
A dagger category has all finitely based dagger limits iff it has dagger equalizers, dagger intersections and finite dagger products.

Completeness dagger categories?

- **FHilb** has all finitely based dagger limits but no finite products or dagger pullbacks
- **Rel** has all dagger-shaped limits but not equalizers
- **PInj** has connected dagger-shaped limits, dagger equalizers and pullbacks but no products
- Can one have more dagger limits at once?

Pushing dagger limits too far hurts

Theorem
If a dagger category has dagger equalizers and infinite dagger products, then it must be indiscrete.

Theorem
If a dagger category has dagger equalizers, dagger pullbacks and finite dagger products, then it must be indiscrete.

Both theorems are proved using the following Lemma:

Lemma
If a dagger category has dagger products, then it is uniquely enriched in commutative monoids. If it furthermore has dagger equalizers, then addition is cancellative.

Pushing dagger limits too far hurts

Theorem

If a dagger category has dagger equalizers and infinite dagger products, then it must be indiscrete.

Proof.

Infinite dagger products induce the ability to add infinitely many parallel morphisms. Hence the following computation makes sense for any f:

$$\begin{aligned} 0 + (f + f + \cdots) &= f + f + \cdots \\ &= f + (f + f + \cdots) \end{aligned}$$

It now follows from cancellativity that $f = 0$. □

Polar Decomposition

Definition

Let $f : A \to B$ be a morphism in a dagger category. A *polar decomposition* of f consists of two factorizations of f as $f = pi = jp$,

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, "f"] & A \arrow[d, "p"] \\
B \arrow[r, "j"'] & B
\end{tikzcd}
$$

where p is a partial isometry and i and j are self-adjoint isos. A category *admits polar decomposition* when every morphism has a polar decomposition.

Polar Decomposition

If $E \xrightarrow{e} A \rightrightarrows B$ is an equalizer and

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, "e"] & E \arrow[d, "p"] \\
A \arrow[r, "j"'] & A
\end{tikzcd}$$

is a polar decomposition, then $E \xrightarrow{p} A \rightrightarrows B$ is a dagger equalizer.

Theorem

This works for all $\mathbf{J}$ *with a basis (mod independence)*

Theorem

If one replaces D *with* $D' \cong D$*, this always works (mod independence)*

What dagger limits are

- *Often* unique up to unique unitary
- Defined for arbitrary diagrams, *not always uniquely*
- Definition doesn't need additional structure
- Generalizes dagger biproducts and dagger equalizers
- Connections to dagger adjunctions etc.
- Good enough?