Posets of Commutative C*-subalgebras

Combining Viewpoints in Quantum Theory

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Commutative C*-subalgebras

Gelfand duality

Theorem

The category of compact Hausdorff spaces and continuous functions is dual to the category of commutative unital C*-algebras and unital *-homomorphisms via the functor $X \mapsto C(X)$.

Perspectives on Gelfand duality:

- trying to extend it to non-commutative C*-algebras;
- using it to define C*-algebras as 'non-commutative' topological spaces;
- trying to exploit it to study non-commutative C*-algebras.

Definition

Consider a unital C*-algebra A. Then we define

$$\mathcal{C}(A) = \{C \subseteq A : C \text{ is a commutative unital C*-subalgebra of } A\},$$

which we order by inclusion.

Motivation from physics

- C*-algebras can be used to model quantum systems;
- Observables of a classical systems can be represented by continuous functions on a topological space representing its phase space.
- Hence commutative C*-algebras can be used to model classical systems.
- Commutative C*-subalgebras can be used to represent the ‘classical snapshots’ of a quantum system.

Bohr: Can we reconstruct a quantum system if we know all its classical pictures?

Related research

- A. Döring, J. Harding, *Abelian subalgebras and the Jordan structure of von Neumann algebras*, arXiv:1009.4945v1 (2010).
- J. Hamhalter, *Isomorphisms of ordered structures of abelian C*-subalgebras of C*-algebras*, J. Math. Anal. Appl. **383**: 391-399 (2011).
- J. Hamhalter, *Dye's Theorem and Gleason's Theorem for AW*-algebras*, J. of Math. Anal. Appl., **422** 1103–1115 (2015).
- J. Hamhalter, E. Turilova, *Structure of associative subalgebras of Jordan operator algebras*, The Quarterly Journal of Mathematics 64, 397408 (2013).
- C. Heunen, M. L. Reyes, *Active lattices determine AW*-algebras*, Journal of Mathematical Analysis and Applications 416:289-313 (2014).

Related research

J. Hamhalter, *Isomorphisms of ordered structures of abelian C*-subalgebras of C*-algebras*, J. Math. Anal. Appl. **383**: 391-399 (2011).

Theorem

Let A be a commutative C*-algebra and B be a C*-algebra such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then $A \cong B$.

C. Heunen, M. L. Reyes, *Active lattices determine AW*-algebras*, Journal of Mathematical Analysis and Applications 416:289-313 (2014).

Applications

- J. Butterfield, C.J. Isham, *A topos perspective on the Kochen-Specker theorem: I. Quantum states as generalized valuations*, International Journal of Theoretical Physics 37 (11), 2669-2733 (1998).
- A. Döring, C.J. Isham, *'What is a thing?' Topos theory in the foundation of Physics*. In: B. Coecke, ed., *New Structures in Physics*, Lecture Notes in Physics, Springer (2009).
- J. Hamhalter, E. Turilova, *Orthogonal Measures on State Spaces and Context Structure of Quantum Theory*, International Journal of Theoretical Physics, 55,7, 3353-3365 (2016)
- C. Heunen, N.P. Landsman, B. Spitters, *A Topos for Algebraic Quantum Theory*, Commun. Math. Phys. **291**: 63-110 (2009).

Our goals

Theorem

Let A and B be C*-algebras such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then:

(1) $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$;

(2) If A is an AW*-algebra, then so is B. Moreover, given the unique decomposition

$$A = A_{\mathrm{I}} \oplus A_{\mathrm{II}_1} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}},$$

where A_τ is an AW*-algebra of type τ ($\tau = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$), then there exist AW*-algebras B_τ of type τ ($\tau = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$) such that

$$B \cong B_{\mathrm{I}} \oplus B_{\mathrm{II}_1} \oplus B_{\mathrm{II}_\infty} \oplus B_{\mathrm{III}},$$

such that A_{I} and B_{I} are *-isomorphic, and such that A_τ and B_τ are Jordan isomorphic for $\tau = \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$;

(3) If A is a W*-algebra, then so is B.

Projections and orthomodular posets

Projections

Definition

An element p in a C*-algebra A satisfying $p^2 = p = p^*$ is called a *projection*. The set of projections is denoted by $\mathrm{Proj}(A)$.

$\mathrm{Proj}(A)$:

- can be ordered via $p \leq q \iff pq = p$;
- becomes an orthomodular poset if we define $p^\perp = 1_A - p$;
- often encodes much of the structure of A.

Commutativity

Let p and q be elements in an orthomodular poset P:

- p and q are orthogonal ($p \perp q$) if $p \leq q^\perp$;
- p and q commute (pCq) if there are orthogonal e_1, e_2, e_3 such that $p = e_1 \vee e_3$, $q = e_2 \vee e_3$.

The set $C(P)$ of all elements in P that commute with all elements is called the *center* of P.

Lemma

Let $p, q \in \mathrm{Proj}(A)$. Then:

- $p \perp q \iff pq = 0$;
- $pCq \iff pq = qp$.

We have $C(\mathrm{Proj}(A)) = \mathrm{Proj}(Z(A))$.

Posets of Boolean subalgebras

Lemma

Let B be a subset of an orthomodular poset P that contains 0 and 1, and that is closed under joins, meets and the orthocomplementation. Then all elements of B commute if and only if B is a Boolean algebra.

We call such a subset B a *Boolean subalgebra* of P.

Definition

Let P be an orthomodular poset. Then we denote its set of Boolean subalgebras by $\mathcal{B}(P)$, which we order by inclusion.

Commutative AF-algebras

AF-algebras

Definition

A C*-algebra A is called *approximately finite dimensional* (AF) if there is a directed set $\mathcal{D}$ of finite-dimensional C*-subalgebras of A such that $A = \overline{\bigcup \mathcal{D}}$.

Lemma

A commutative C*-algebra A is AF if and only if $A = C^*(\mathrm{Proj}(A))$ if and only if its Gelfand spectrum is a Stone space.

Posets of commutative AF-algebras

Definition

Let A be a C*-algebra. Then $\mathcal{C}_{\mathrm{AF}}(A)$ is defined as the subposet of $\mathcal{C}(A)$ whose elements are AF-algebras.

Lemma (Hamhalter)

Let $C \in \mathcal{C}(A)$ is an atom if and only if it is two dimensional.

Proposition (Heunen-L)

Let $C \in \mathcal{C}(A)$, then $C \in \mathcal{C}_{\mathrm{AF}}(A)$ if and only if C is the supremum of some collection of atoms in $\mathcal{C}(A)$.

Projections

Theorem (Heunen-L)

The map $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{B}(\mathrm{Proj}(A))$, $C \mapsto \mathrm{Proj}(C)$ is an order isomorphism with inverse $B \mapsto C^*(B)$.

Theorem (Harding-Heunen-L-Navara)

Any orthomodular poset P of two or more elements can be reconstructed from $\mathcal{B}(P)$.

Corollary

Let A be a C*-algebra. Then we can reconstruct $\mathrm{Proj}(A)$ from $\mathcal{C}_{\mathrm{AF}}(A)$, hence also from $\mathcal{C}(A)$.

AW*-algebras

Definition of AW*-algebras

Definition

A C*-algebra A is called an *AW*-algebra* if

- $\mathrm{Proj}(A)$ is a complete lattice;
- every maximal commutative C*-subalgebra of A is an AF-algebra.

A C*-subalgebra B of A that is an AW*-algebra such that $\bigvee_{i \in I} p_i \in B$ (as calculated in A) for each $\{p_i\}_{i \in I} \subseteq \mathrm{Proj}(B)$ is called an *AW*-subalgebra*.

Examples

- Any C*-subalgebra of $B(H)$ is precisely a von Neumann algebra if and only if it is an AW*-subalgebra of $B(H)$;
- Any commutative C*-algebra is an AW*-algebra if and only if its Gelfand spectrum is extremally disconnected.

Commutative AW*-subalgebras

Proposition

Let A be an AW*-algebra, and B a C*-algebra such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then B is an AW*-algebra, too.

Definition

Let A be an AW*-algebra. Then we denote the poset of all commutative AW*-subalgebras of A by $\mathcal{A}(A)$.

Two reasons to introduce this poset:

- Generalizing $\mathcal{V}(M)$, the poset of commutative von Neumann subalgebras of a von Neumann algebra M;
- Connecting the C*-algebraic and the von Neumann algebraic frameworks.

The Jordan structure of a C*-algebra

Definition

Let A be a C*-algebra. Then the *Jordan product* on A is given by $a \circ b = \frac{ab+ba}{2}$. A *-preserving linear map $\varphi : A \to B$ between C*-algebras is called a *Jordan homomorphism* if $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$ for each $a, b \in A$.

Theorem

Let A and B be AW*-algebras. Then the following statements are equivalent:

(1) $\mathcal{C}(A) \cong \mathcal{C}(B)$;

(2) $\mathcal{C}_{\mathrm{AF}}(A) \cong \mathcal{C}_{\mathrm{AF}}(B)$;

(3) $\mathcal{A}(A) \cong \mathcal{A}(B)$;

(4) $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$;

(5) There is a Jordan isomorphism $\varphi : A \to B$.

Recognizing von Neumann algebras

Observations:

- Any von Neumann algebra M is an AW*-algebra with a separating family $\{\omega_i\}_{i\in I}$ of states, i.e., $\omega_i : M \to \mathbb{C}$ is bounded, linear, and $\|\omega\| = \omega(1_M) = 1$, and for each non-zero self-adjoint $a \in M$ there is some $i \in I$ such that $\omega(a) \neq 0$;
- Any Jordan isomorphism $M \to B$ is an isometry, so preserves states;
- Hence if M is a von Neumann algebra then B has a separating family of states, too.

Proposition

Let M be a von Neumann algebra and B be a C*-algebra such that $\mathcal{C}(M) \cong \mathcal{C}(B)$. Then B is a von Neumann algebra.

The structure of AW*-algebras

Definition

Let A be an AW*-algebra and $p \in \mathrm{Proj}(A)$. Then p is called:

- *finite* if for each $a \in pAp$ we have $a^*a = p$ if and only if $aa^* = p$.
- *abelian* if pAp is commutative;
- *central* if $p \in Z(A)$ (or equivalently $p \in C(\mathrm{Proj}(A))$).

A central projection q is called the *central cover* of p if it is the least central projection such that $p \leq q$. We call p *faithful* if $C(p) = 1_A$.

Proposition

Let p be a projection in an AW*-algebra A. Then

- p is abelian if and only if $q = p \wedge C(q)$ for each $q \in \mathrm{Proj}(A)$ such that $q \leq p$;
- p is finite if and only if $\downarrow p \subseteq \mathrm{Proj}(A)$ is a modular lattice.

Type classification

Definition

Let A be an AW*-algebra. Then A is of

- *type I* if it has a faithful abelian projection; if A has a collection $\{p_i\}_{i \in J}$ of faithful abelian projections that are mutually orthogonal such that $\bigvee_{i \in J} p_i = 1_A$, then A is called *homogeneous of order* $|J|$.
- *type II* if it has a faithful finite projection and 0 is the only abelian projection. If 1_A is a finite projection, then A is of type II_1; if 0 is the only finite central projection, then A is of type II_∞.
- *type III* if 0 is the only finite projection.

For any AW*-algebra A there is a unique decomposition

$$A = A_{\mathrm{I}} \oplus A_{\mathrm{II}_1} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}},$$

where A_τ is an AW*-algebra of type τ ($\tau = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$).

Preservation of types

Proposition

Let $A = \bigoplus_{i \in I} A_i$ be a direct sum of AW*-algebras A_i, and let B an AW*-algebra such that $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$. Then $B \cong \bigoplus_{i \in I} B_i$, where $\mathrm{Proj}(B_i) \cong \mathrm{Proj}(A_i)$.

Theorem (Kaplansky)

Let A and B be AW*-algebras.

- If A and B are both homogeneous of the same order and $Z(A) \cong Z(B)$, then $A \cong B$;
- If A is of type I, then it is isomorphic to $\bigoplus_{i \in I} A_i$, where each A_i is homogeneous.

Summary

Theorem

Let A and B be C*-algebras such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then:

(1) $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$;

(2) If A is an AW*-algebra, then so is B. Moreover, given the unique decomposition

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where A_τ is an AW*-algebra of type τ ($\tau = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$), then there exist AW*-algebras B_τ of type τ ($\tau = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$) such that

$$B \cong B_{\mathrm{I}} \oplus B_{\mathrm{II}_1} \oplus B_{\mathrm{II}_\infty} \oplus B_{\mathrm{III}},$$

such that A_{I} and B_{I} are *-isomorphic, and such that A_τ and B_τ are Jordan isomorphic for $\tau = \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$;

(3) If A is a W*-algebra, then so is B.

Relevant literature on AW*-algebras

- I. Kaplansky, *Projections in Banach Algebras*, Annals of Mathematics, Second Series, Vol. 53, No. 2, pp. 235-249 (1951).
- I. Kaplansky, *Algebras of Type I*, Annals of Mathematics, Second Series, Vol. 56, No. 3, pp. 460-472 (1952).
- M. Ozawa, *A classification of type I AW* algebras and Boolean valued analysis*, J. Math. Soc. Japan Volume 36, Number 4, 589-608 (1984).
- K. Saito, J.D.M. Wright, *On Defining AW*-algebras and Rickart C*-algebras*, Q. J. Math. 66, no. 3, 979989 (2015).