A statistical interpretation of Grothendieck's inequality and its relation to the size of non-locality of quantum mechanics

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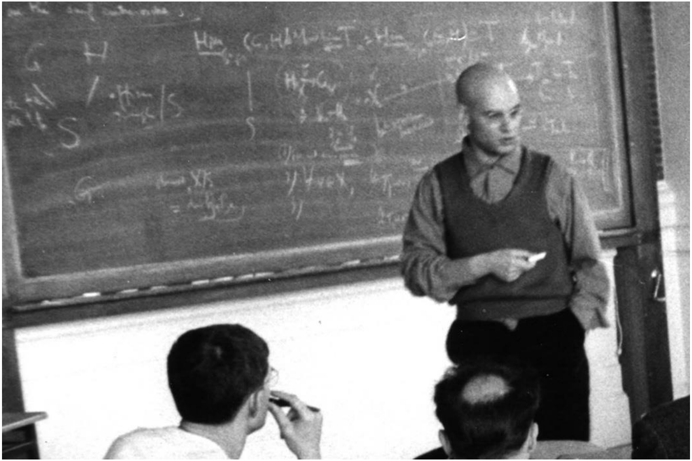
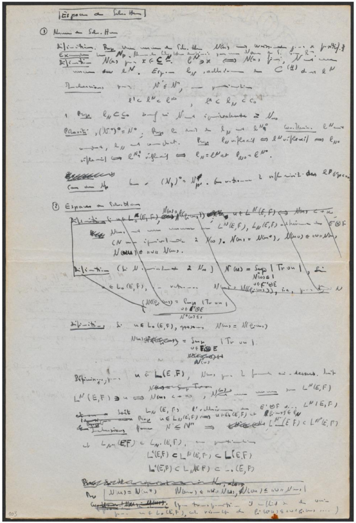
5. Towards a determination of Grothendieck's constant $K_G^{\mathbb{R}}$

A portrait of A. Grothendieck

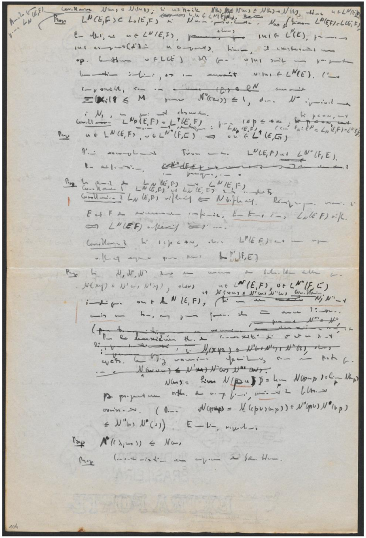
A. Grothendieck lecturing at IHES (1958-1970)



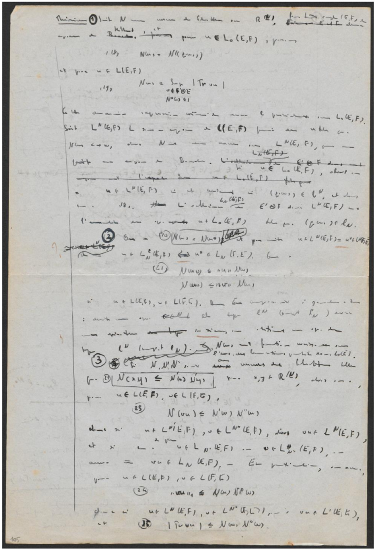
Excerpt from A. Grothendieck's handwritten lecture notes I



Excerpt from A. Grothendieck's handwritten lecture notes II



Excerpt from A. Grothendieck's handwritten lecture notes III



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Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

*Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Then there exists a universal constant $K > 0$ - not depending on m and n - such that for all matrices $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{F})$, for all $\mathbb{F}$-Hilbert spaces H, for all **unit** vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in S_H$ the following inequality is satisfied:*

$$\Big|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle u_i, v_j\rangle_H\Big| \leq K \max\Big\{\Big|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j\Big| : p_i, q_j \in \{-1,1\}\Big\}.$$

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant.

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant. Computing the exact numerical value of this constant is an open problem (unsolved since 1953)!

Grothendieck's inequality in matrix form II

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and H be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$. Let $K_{GH}^{\mathbb{F}}$ denote the Grothendieck constant, derived from Grothendieck's inequality "restricted" to the set of all positive semidefinite $n \times n$-matrices over $\mathbb{F}$. Then

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$$K_{GH}^{\mathbb{R}} = \frac{\pi}{2}$$

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From now on are going to consider the real case (i. e., $\mathbb{F} = \mathbb{R}$) only. Nevertheless, we allow an unrestricted use of all matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ for any $m, n \in \mathbb{N}$ in GT.

Grothendieck's inequality in matrix form III

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

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$$1,676 < K_G^{\mathbb{R}} \overset{(!)}{<} \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1,782\,.$$

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Conjecture

Is $K_G^{\mathbb{R}} = \sqrt{\pi} \approx 1,772$*?*

Grothendieck's inequality rewritten I

By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear mappings between correlation matrices.

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We will sketch this approach which might lead to a constructive improvement of Krivine's upper bound $\frac{\pi}{2\ln(1+\sqrt{2})}$. At least it also can be reproduced in this approach.

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

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Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H =$$

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$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \mathrm{tr}\big(A^\top \, \Gamma_H(u, v)\big) =$$

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is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where

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$$\Gamma_H(u, v) := \begin{pmatrix} \langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \ldots & \langle u_1, v_n \rangle_H \\ \langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \ldots & \langle u_2, v_n \rangle_H \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \ldots & \langle u_m, v_n \rangle_H \end{pmatrix}.$$

Grothendieck's inequality rewritten III

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), p := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \operatorname{tr}\left(A^\top \Gamma_{\mathbb{R}}(p, q)\right) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,$$

where now

$$\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix} \pm 1 & \mp 1 & \ldots & \pm 1 \\ \mp 1 & \mp 1 & \ldots & \mp 1 \\ \vdots & \vdots & \vdots & \vdots \\ \pm 1 & \mp 1 & \ldots & \pm 1 \end{pmatrix}.$$

Full matrix representation of the Hilbert space vectors

Pick all $m+n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

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Does this matrix look familiar to you?

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It is a part of something larger...

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Namely:

Block matrix representation I

$$
\begin{pmatrix}
 & \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix} \\
\begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}^{\mathbf{T}} &
\end{pmatrix}
$$

Block matrix representation I

$$
\left(\begin{array}{cccccccc}
 & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
 & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle u_1, v_1\rangle & \langle u_2, v_1\rangle & \dots & \langle u_m, v_1\rangle & & & & \\
\langle u_1, v_2\rangle & \langle u_2, v_2\rangle & \dots & \langle u_m, v_2\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle u_1, v_n\rangle & \langle u_2, v_n\rangle & \dots & \langle u_m, v_n\rangle & & & &
\end{array}\right)
$$

Block matrix representation I

$$\begin{pmatrix} & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\ \langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & & & & \\ \langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ \langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & & & & \end{pmatrix}$$

Block matrix representation II

$$
\begin{pmatrix}
\langle u_1, u_1\rangle & \langle u_1, u_2\rangle & \dots & \langle u_1, u_m\rangle & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
\langle u_2, u_1\rangle & \langle u_2, u_2\rangle & \dots & \langle u_2, u_m\rangle & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1\rangle & \langle u_m, u_2\rangle & \dots & \langle u_m, u_m\rangle & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & \langle v_n, v_1\rangle & \langle v_m, v_2\rangle & \dots & \langle v_n, v_n\rangle
\end{pmatrix}
$$

Block matrix representation III

$$\begin{pmatrix}
1 & \langle u_1, u_2\rangle & \dots & \langle u_1, u_m\rangle & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
\langle u_2, u_1\rangle & 1 & \dots & \langle u_2, u_m\rangle & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1\rangle & \langle u_m, u_2\rangle & \dots & 1 & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & \langle v_n, v_1\rangle & \langle v_m, v_2\rangle & \dots & 1
\end{pmatrix}$$

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$PSD(n;\mathbb{R}) := \{S : S \in \mathbb{M}(n \times n;\mathbb{R})$ and S is positive semidefinite$\}$.

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Recall that $PSD(n;\mathbb{R})$ is a closed convex cone which (by definition!) consists of symmetric matrices only.

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Recall that $PSD(n;\mathbb{R})$ is a closed convex cone which (by definition!) consists of symmetric matrices only.

Moreover, we consider the set

$$C(n;\mathbb{R}) := \Big\{S \in PSD(n;\mathbb{R}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\Big\}.$$

A refresher of a few definitions II

Let $d, k \in \mathbb{N}$ and $(H, \langle\cdot,\cdot\rangle)$ be an arbitrary d-dimensional Hilbert space (i. e, $H = l_2^d$). Let $w_1, w_2, \ldots, w_k \in H$. Put $w := (w_1, \ldots, w_k)^\top \in H^k$ and $S := \left(w_1 \mid w_2 \mid \ldots \mid w_k\right) \in \mathbb{M}(d \times k; \mathbb{R})$. The matrix $\Gamma_H(w,w) \in PSD(k;\mathbb{R})$, defined as

$$\Gamma_H(w,w)_{ij} := \langle w_i, w_j \rangle = \left(S^\top S\right)_{ij} \qquad \left(i, j \in [k] := \{1, 2, \ldots, k\}\right)$$

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Observe that

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is not a Gram matrix!

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Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \dots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \dots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

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Recall that ξ is an n-dimensional Gaussian random vector with respect to the "parameters" μ and C (short: $\xi \sim N_n(\mu, C)$) if and only if for all $a \in \mathbb{R}^n$ there exists $\eta_a \sim N_1(0, 1)$ such that

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Note that we don’t require here that C is invertible! Following Feller, the matrix $\mathbb{V}(\xi)$ defined as

$$\mathbb{V}(\xi)_{ij} := \mathbb{E}[\xi_i \xi_j] - \mathbb{E}[\xi_i]\mathbb{E}[\xi_j] \overset{(!)}{=} C_{ij} \qquad (i, j \in [n])$$

is known as the *variance matrix of the Gaussian random vector* ξ.

Structure of correlation matrices I

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In particular, condition (i) implies that $\sigma_{ij} \in [-1, 1]$ *for all* $i, j \in [n]$.

Structure of correlation matrices II

Observation

Let $k \in \mathbb{N}$. Then the sets $\left\{S : S = xx^\top \text{ for some } x \in \{-1, 1\}^k\right\}$ and $\left\{\Theta : \Theta \in C(k; \mathbb{R}) \text{ and } rk(\Theta) = 1\right\}$ coincide.

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Proposition (K. R. Parthasarathy (2002))

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Let $k \in \mathbb{N}$. Put

$$C_1(k; \mathbb{R}) := \left\{\Theta : \Theta \in C(k; \mathbb{R}) \text{ and } \mathrm{rk}(\Theta) = 1\right\}.$$

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Definition

Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{R})$ arbitrary. Put

$$\widehat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

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Observe that $\widehat{A}$ is symmetric, implying that $\widehat{A} = \widehat{A}^\top$.

A further equivalent rewriting of GT I

Proposition

Let H be an arbitrary Hilbert space over $\mathbb{R}$. Let $m, n \in \mathbb{N}$ and $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

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We don't know whether condition (ii) holds for all matrices in $\mathbb{M}((m+n) \times (m+n); \mathbb{R})$.

GT versus NP-hard optimisation

Observation

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

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Thus, Grothendieck's constant $K_G^{\mathbb{R}}$ is precisely the "integrality gap"; i. e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!

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(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a (Kolmogorovian) probability space. Let $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. $A \in C_{\text{loc}}(m \times n; \mathbb{R})$ iff $a_{ij} = \mathbb{E}_{\mathbb{P}}\left[X_i Y_j\right]$, where $X_i, Y_j : \Omega \longrightarrow [-1, 1]$ are random variables - all defined on the same given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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(ii) Let $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{R})$. $A \in QC(m \times n; \mathbb{R})$ iff there are $k, l \in \mathbb{N}$, a density matrix ρ on $\mathcal{B}\big(H_{k,l}\big)$, where $H_{k,l} := \mathbb{C}^k \otimes \mathbb{C}^l$, and linear operators $A_i \in \mathcal{B}\big(\mathbb{C}^k\big)$, $B_j \in \mathcal{B}\big(\mathbb{C}^l\big)$ such that $\|A_i\| \leq 1$, $\|B_j\| \leq 1$ and

$$a_{ij} = \langle \rho, A_i \otimes B_j \rangle = \operatorname{tr}\big(\rho(A_i \otimes B_j)\big) = \operatorname{tr}\big(\rho(A_i \otimes Id^{(l)})(Id^{(k)} \otimes B_j)\big)$$

for all $(i, j) \in [m] \times [n]$.

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Theorem (Tsirelson (1987, 1993))

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(ii) $A = \Gamma_{l_2^k}(u, v)$ *for some* $k \in \mathbb{N}$ *and some* $u \in \left(S^{k-1}\right)^m$ *and* $v \in \left(S^{k-1}\right)^n$.

Modelling quantum correlation III

$$\Gamma_{l_2^k}(u,v) = \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}$$

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$\Gamma_{l_2^k}(u,v) = UV$ is the product of the matrices $U : l_2^k \longrightarrow l_\infty^m$ and $V : l_1^n \longrightarrow l_2^k$, where

$$V := \left(v_1 \mid v_2 \mid \dots \mid v_n \right) \text{ and } U := \begin{pmatrix} u_1^\top \\ u_2^\top \\ \vdots \\ u_m^\top \end{pmatrix} .$$

Modelling quantum correlation IV

Hence, we see that if $u \in S_H^m$ and $v \in S_H^n$ one can canonically associate a linear operator to the $(m \times n)$-matrix $\Gamma_H(u,v)$ which factors through the Hilbert space $H := l_2^k$ such that $\Gamma_H(u,v) = UV$ for some $(m \times k)$-matrix U and some $(k \times n)$-matrix V, satisfying

$$\gamma_2\big(\Gamma_H(u,v)\big) \leq \|U\|_{2,\infty} \cdot \|V\|_{1,2} \leq 1 \ :$$

$$\begin{tikzcd}
l_1^n \arrow[rr, "\Gamma_H(u,v)"] \arrow[dr, "V"'] & & l_\infty^m \\
& H \arrow[ur, "U"'] &
\end{tikzcd}$$

Modelling quantum correlation V

Theorem (Grothendieck (1953), Tsirelson (1987), Pisier (2001))

Let H be a separable Hilbert space and $m, n \in \mathbb{N}$. Let $u := (u_1, \dots, u_m)^\top \in S_H^m$ and $v := (v_1, \dots, v_n)^\top \in S_H^n$. Then

$$\begin{aligned} \Gamma_H(u, v) &\in K_G^{\mathbb{R}}\, \mathit{cx}\left(\left\{pq^\top : p \in \{-1, 1\}^m, q \in \{-1, 1\}^n\right\}\right) \\ &= K_G^{\mathbb{R}}\, C_{\mathit{loc}}(m \times n; \mathbb{R}) \,. \end{aligned}$$

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Corollary (Tsirelson (1987, 1993))

Let $m, n \in \mathbb{N}$. Then

$$QC(m \times n; \mathbb{R}) \subseteq K_G^{\mathbb{R}} \, C_{\mathit{loc}}(m \times n; \mathbb{R}) \, .$$

Moreover, $C_{\mathit{loc}}(m \times n; \mathbb{R}) \subseteq QC(m \times n; \mathbb{R})$. The latter set inclusion is strict.

Bell's inequalities and GT I

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Purely in terms of a very elementary application of classical Kolmogorovian probability theory and a bit of elementary algebra - and completely independent of any modelling assumptions in physics - Bell's inequalities can be represented in form of an inequality originating from *J. F. Clauser, M. A. Horne, A. Shimony* and *R. A. Holt* in 1969.

Bell's inequalities and GT I

It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as *Bell's inequalities*.

Purely in terms of of a very elementary application of classical Kolmogorovian probability theory and a bit of elementary algebra - and "without the annoying adherence to physics" (as we have learnt from Niel ☺) - Bell's inequalities can be represented in form of an inequality originating from *J. F. Clauser, M. A. Horne, A. Shimony* and *R. A. Holt* in 1969.

Bell's inequalities and GT II

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space. Let* X_1, X_2, X_3 *and* X_4 *be arbitrary random variables with values in* $[-1,1]$ $\mathbb{P}$*-a.s., all defined on* Ω*. Then*

$$|\mathbb{E}_{\mathbb{P}}[X_i X_2] - \mathbb{E}_{\mathbb{P}}[X_i X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2 X_3] \text{ for all } i \in \{1, 4\}$$

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In other words:

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Observation (BCHSH Inequality in matrix form)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space (in the sense of Kolmogorov).*

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Then

$$|\langle A^{\mathit{Had}}, \Gamma \rangle| = |\mathit{tr}\big(A^{\mathit{Had}}\,\Gamma\big)| \leq 2 \;\; \textit{for all} \;\; \Gamma \in C_{\mathit{loc}}(2 \times 2; \mathbb{R}) \,.$$

Bell's inequalities and GT IV

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Theorem (Tsirelson (1980))

Let H be an arbitrary Hilbert space, $u \in S_H^2$ and $v \in S_H^2$. Then

$$|\langle A^{Had}, \Gamma_H(u,v)\rangle| = |tr\big(A^{Had}\,\Gamma_H(u,v)\big)| \leq 2\sqrt{2}$$

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To this end, we recall the main ideas underlying the EPR/Bell-CHSH experiment.

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Bear also Rui's talk in mind!

Bell's inequalities and GT V

A source emits in opposite directions two spin $\frac{1}{2}$ particles created from one particle of spin 0. By rotating magnets perpendicular to the directions of the two spin $\frac{1}{2}$ particles, both, Alice and Bob measure the spin in 2 different directions, leading to angles $-\frac{\pi}{2} \leq \alpha_1, \alpha_2 < \frac{\pi}{2}$ for Alice and $-\frac{\pi}{2} \leq \beta_1, \beta_2 < \frac{\pi}{2}$ for Bob. Only one angle per measurement can be chosen on both sides. The outcome of this experiment is a “random” pair of observables belonging to the set

$$\{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2)\}\,.$$

Any of these observables takes its values in $\{-1, +1\}$.

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Describing this experiment purely in terms of mathematics we immediately recognise that the Bell-Tsirelson constant $2\sqrt{2}$ is attained by the Hadamard matrix, since:

Bell's inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let* $H \ni x := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$ *("entangled Bell state").*

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Put

$$\Gamma^{EPR} := \begin{pmatrix} \langle x, (A_1 \otimes B_1)x\rangle_H & \langle x, (A_1 \otimes B_2)x\rangle_H \\ \langle x, (A_2 \otimes B_1)x\rangle_H & \langle x, (A_2 \otimes B_2)x\rangle_H \end{pmatrix},$$

where $A_i := R(\alpha_i)$, $B_j := R(\beta_j)$ *and*

$$O(2;\mathbb{R}) \ni R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix} \text{ ("rotary reflections")}\,.$$

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Then $\Gamma^{EPR} \in QC(2 \times 2;\mathbb{R})$ *and*

$$|\langle A^{Had}, \Gamma^{EPR}\rangle| = |tr\big(A^{Had}\,\Gamma^{EPR}\big)| = 2\sqrt{2} > 2.$$

1. A very short glimpse at A. Grothendieck's work in functional analysis
2. Grothendieck's inequality in matrix formulation
3. Grothendieck's inequality rewritten
4. Grothendieck's inequality and its relation to non-locality in quantum mechanics
5. Towards a determination of Grothendieck's constant $K_G^{\mathbb{R}}$

Schur product and the matrix $f[A]$

Definition

Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i, j) \in [m] \times [n]$.

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The Schur product (or Hadamard product)

$$(a_{ij}) * (b_{ij}) := (a_{ij}b_{ij})$$

of matrices (a_{ij}) and (b_{ij}) leads to $f[A]$, where $f(x) := x^2$.

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Remark

The notation "$f[A]$" is used to highlight the difference between the matrix $f(A)$ originating from the spectral representation of A (for normal matrices A) and the matrix $f[A]$, defined as above !

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

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Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Sigma_\rho)$, *where*

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Corollary

Let $2 \leq k \in \mathbb{N}$. *Let* $\Sigma \in C(k;\mathbb{R})$ *an arbitrarily given correlation matrix. Then there exists a Gaussian random vector* $\xi \sim N_k(0,\Sigma)$ *such that*

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$$\max_{\substack{\Theta \in C(k;\mathbb{R}) \\ rank(\Theta)=1}} |\langle \widehat{A}, \Theta\rangle| \geq \mathbb{E}\big[|\langle \widehat{A}, \Theta(\xi)\rangle|\big] \geq |\langle \widehat{A}, \mathbb{E}\big[\Theta(\xi)\big]\rangle| = \frac{2}{\pi}|\langle \widehat{A}, \arcsin[\Sigma]\rangle|\,.$$

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More generally, we may list the following two "Schoenberg-type" results (applied to non-linear correlation matrix transforms) which are implied by the Schur product theorem:

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Let $0 \neq f : [-1,1] \longrightarrow \mathbb{R}$ be a function that admits a power series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for some sequence (a_n) of non-negative numbers on $[-1,1]$.

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Conversely, we have:

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Theorem (Schoenberg (1942), Rudin (1959), Christensen/Ressel (1978))

Let $0 \neq g : [-1, 1] \longrightarrow \mathbb{R}$ *be a function such that* $g\big[\Sigma\big]$ *is a* $(k \times k)$*-correlation matrix for all* $(k \times k)$*-correlation matrices* Σ *and all* $k \in \mathbb{N}$*.*

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Moreover, $g : [-1,1] \longrightarrow [-1,1]$ *has to be a function that admits a power series representation* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *for some sequence* (b_n) *of non-negative numbers on* $[-1,1]$*.*

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(iii) A reiteration of the steps (i) and (ii) could lead to an iterative algorithm which might converge to a "suitable" - upper - bound of $K_G^{\mathbb{R}}$.

A phrase of G. H. Hardy

"... at present I will say only that if a chess problem is, in the crude sense, 'useless', then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects..."

– *A Mathematician's Apology* (1940)

Only a - very - few references

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Are there any questions, comments or remarks?