A statistical interpretation of Grothendieck’s inequality and its relation to the size of non-locality of quantum mechanics

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Grothendieck’s inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Then there exists a universal constant $K > 0$ - not depending on $m$ and $n$ - such that for all matrices $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$, for all $\mathbb{F}$-Hilbert spaces $H$, for all unit vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in S_H$ the following inequality is satisfied:

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max \left\{ \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right| : p_i, q_j \in \{-1, 1\} \right\}.$$
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$$\left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max \left\{ \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right| : p_i, q_j \in \{-1, 1\} \right\}.$$

The smallest possible value of the corresponding constant $K$ is denoted by $K^F_G$. It is called Grothendieck’s constant.
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The smallest possible value of the corresponding constant $K$ is denoted by $K^\mathbb{F}_G$. It is called Grothendieck’s constant. Computing the exact numerical value of this constant is an open problem (unsolved since 1953)!
Grothendieck’s inequality in matrix form II


Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $H$ be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$. Let $K_{GH}^F$ denote the Grothendieck constant, derived from Grothendieck’s inequality “restricted” to the set of all positive semidefinite $n \times n$-matrices over $\mathbb{F}$. Then

$$K_{GH}^\mathbb{R} = \frac{\pi}{2} \quad \text{and} \quad K_{GH}^\mathbb{C} = 4\pi.$$
Grothendieck’s inequality in matrix form II


Let $F \in \{ \mathbb{R}, \mathbb{C} \}$ and $H$ be an arbitrary Hilbert space over $F$. Let $n \in \mathbb{N}$. Let $K_{GH}^F$ denote the Grothendieck constant, derived from Grothendieck’s inequality “restricted” to the set of all positive semidefinite $n \times n$-matrices over $F$. Then

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\[
K_{\text{GH}}^\mathbb{R} = \frac{\pi}{2} \quad \text{and} \quad K_{\text{GH}}^\mathbb{C} = \frac{4}{\pi}.
\]

Let $F \in \{\mathbb{R}, \mathbb{C}\}$ and $H$ be an arbitrary Hilbert space over $F$. Let $n \in \mathbb{N}$. Let $K^{F}_{GH}$ denote the Grothendieck constant, derived from Grothendieck’s inequality “restricted” to the set of all positive semidefinite $n \times n$-matrices over $F$. Then

$$K^{\mathbb{R}}_{GH} = \frac{\pi}{2} \quad \text{and} \quad K^{\mathbb{C}}_{GH} = \frac{4}{\pi}.$$ 

From now on are going to consider the real case (i.e., $F = \mathbb{R}$) only. Nevertheless, we allow an unrestricted use of all matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ for any $m, n \in \mathbb{N}$ in GT.
Grothendieck’s inequality in matrix form III

Until present the following encapsulation of $K^R_G$ holds, primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv):
Grothendieck’s inequality in matrix form III

Until present the following encapsulation of $K_{RG}^G$ holds, primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv):

\[
1.676 < K_{RG}^G \left(\frac{\pi}{2 \ln(1 + \sqrt{2})}\right) \approx 1.782.
\]
Grothendieck’s inequality in matrix form III

Until present the following encapsulation of $K^{\mathbb{R}}_G$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

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Screening these numbers we might be tempted to guess the following
Until present the following encapsulation of $K_{G}^{\mathbb{R}}$ holds, primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv):

$$1,676 < K_{G}^{\mathbb{R}} < \frac{\pi}{2 \ln(1 + \sqrt{2})} \approx 1,782.$$ 

Screening these numbers we might be tempted to guess the following

**Conjecture**

Is $K_{G}^{\mathbb{R}} = \sqrt{\pi} \approx 1,772$?
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By transforming Grothendieck’s inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear mappings between correlation matrices.
By transforming Grothendieck’s inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear mappings between correlation matrices.

We will sketch this approach which might lead to a constructive improvement of Krivine’s upper bound \( \frac{\pi}{2 \ln(1 + \sqrt{2})} \). At least it also can be reproduced in this approach.
Grothendieck’s inequality rewritten II

Let $m, n \in \mathbb{N}$, $A \in \mathbb{M}(m \times n; \mathbb{R})$, $u := (u_1, \ldots, u_m)^\top \in S^m_H$ and $v := (v_1, \ldots, v_n)^\top \in S^n_H$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in $H$. 

Firstly, note that

$$m \sum_{i=1}^{m} u_i^\top v_i = \text{tr}(A^\top \Gamma_H(u, v)) = \langle A, \Gamma_H(u, v) \rangle,$$

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where $\Gamma_H(u, v) := \begin{pmatrix} \langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \cdots & \langle u_1, v_n \rangle_H \\ \langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \cdots & \langle u_2, v_n \rangle_H \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \cdots & \langle u_m, v_n \rangle_H \end{pmatrix}$. 

Grothendieck’s inequality rewritten II

Let \( m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S^m_H \) and \( v := (v_1, \ldots, v_n)^\top \in S^n_H \) be given, where \( S_H := \{ w \in H : \|w\| = 1 \} \) denotes the unit sphere in \( H \).

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\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H =
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Let \( m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)\top \in S_H^m \) and \( v := (v_1, \ldots, v_n)\top \in S_H^n \) be given, where \( S_H := \{w \in H : \|w\| = 1\} \) denotes the unit sphere in \( H \).

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\]

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices \( A \in \mathbb{M}(m \times n; \mathbb{R}) \) and \( \Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R}) \), where \( \Gamma_H(u, v) := \begin{pmatrix} \langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \cdots & \langle u_1, v_n \rangle_H \\ \langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \cdots & \langle u_2, v_n \rangle_H \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \cdots & \langle u_m, v_n \rangle_H \end{pmatrix} \).
Grothendieck’s inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in $H$.

Firstly, note that

$$
\sum_{i=1}^m \sum_{j=1}^n a_{ij} \langle u_i, v_j \rangle_H = \text{tr}(A^\top \Gamma_H(u, v)) = \langle A, \Gamma_H(u, v) \rangle,
$$

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where
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Let \( m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S^m_H \) and \( v := (v_1, \ldots, v_n)^\top \in S^n_H \) be given, where \( S_H := \{w \in H : \|w\| = 1\} \) denotes the unit sphere in \( H \).

Firstly, note that

\[
\sum_{i=1}^m \sum_{j=1}^n a_{ij}\langle u_i, v_j \rangle_H = \text{tr}(A^\top \Gamma_H(u, v)) = \langle A, \Gamma_H(u, v) \rangle,
\]

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices \( A \in \mathbb{M}(m \times n; \mathbb{R}) \) and \( \Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R}) \), where

\[
\Gamma_H(u, v) := \begin{pmatrix}
\langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \ldots & \langle u_1, v_n \rangle_H \\
\langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \ldots & \langle u_2, v_n \rangle_H \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \ldots & \langle u_m, v_n \rangle_H
\end{pmatrix}.
\]
Grothendieck’s inequality rewritten III

Let \( m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), p := (p_1, \ldots, p_m)^\top \in (S^0)^m \) and \( q := (q_1, \ldots, q_n)^\top \in (S^0)^n \) be given, where \( S^0 := \{-1, 1\} \) denotes the unit “sphere” in \( \mathbb{R} = \mathbb{R}^{0+1} \).

Similarly as before, we obtain

\[
\sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j = \text{tr}(A^\top \Gamma_{\mathbb{R}}(p, q)) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,
\]

where now

\[
\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix}
\pm 1 & \mp 1 & \ldots & \pm 1 \\
\mp 1 & \mp 1 & \ldots & \mp 1 \\
\vdots & \vdots & \ddots & \vdots \\
\pm 1 & \mp 1 & \ldots & \pm 1
\end{pmatrix}.
\]
Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as
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$$
\begin{pmatrix}
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle
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\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle 
\end{pmatrix}
$$

Does this matrix look familiar to you?
Full matrix representation of the Hilbert space vectors

Pick all \( m + n \) Hilbert space unit vectors \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H \) and represent them as

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\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle
\end{pmatrix}
\]

Does this matrix look familiar to you?
It is a part of something larger...
Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

$$
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\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle 
\end{pmatrix}
$$

Does this matrix look familiar to you?
It is a part of something larger...
Namely:
Block matrix representation I

\[
\begin{pmatrix}
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \\
\end{pmatrix}^T
\]

\[
\begin{pmatrix}
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \\
\end{pmatrix}
\]
**Block matrix representation I**

\[
\begin{pmatrix}
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\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \\
\langle u_1, v_n \rangle & \langle u_2, v_n \rangle & \cdots & \langle u_m, v_n \rangle 
\end{pmatrix}
\]
Block matrix representation I

\[
\begin{pmatrix}
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \\
\langle v_1, u_1 \rangle & \langle v_1, u_2 \rangle & \cdots & \langle v_1, u_m \rangle \\
\langle v_2, u_1 \rangle & \langle v_2, u_2 \rangle & \cdots & \langle v_2, u_m \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1 \rangle & \langle v_n, u_2 \rangle & \cdots & \langle v_n, u_m \rangle 
\end{pmatrix}
\]
Block matrix representation II

\[
\begin{pmatrix}
\langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \ldots & \langle u_1, u_m \rangle & \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\
\langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \ldots & \langle u_2, u_m \rangle & \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1 \rangle & \langle u_m, u_2 \rangle & \ldots & \langle u_m, u_m \rangle & \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle \\
\langle v_1, u_1 \rangle & \langle v_1, u_2 \rangle & \ldots & \langle v_1, u_m \rangle & \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \ldots & \langle v_1, v_n \rangle \\
\langle v_2, u_1 \rangle & \langle v_2, u_2 \rangle & \ldots & \langle v_2, u_m \rangle & \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \ldots & \langle v_2, v_n \rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1 \rangle & \langle v_n, u_2 \rangle & \ldots & \langle v_n, u_m \rangle & \langle v_n, v_1 \rangle & \langle v_m, v_2 \rangle & \ldots & \langle v_n, v_n \rangle 
\end{pmatrix}
\]
Block matrix representation III

$$
\begin{bmatrix}
1 & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_m \rangle & \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_n \rangle \\
\langle u_2, u_1 \rangle & 1 & \cdots & \langle u_2, u_m \rangle & \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1 \rangle & \langle u_m, u_2 \rangle & \cdots & 1 & \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \\
\langle v_1, u_1 \rangle & \langle v_1, u_2 \rangle & \cdots & \langle v_1, u_m \rangle & 1 & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\
\langle v_2, u_1 \rangle & \langle v_2, u_2 \rangle & \cdots & \langle v_2, u_m \rangle & \langle v_2, v_1 \rangle & 1 & \cdots & \langle v_2, v_n \rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1 \rangle & \langle v_n, u_2 \rangle & \cdots & \langle v_n, u_m \rangle & \langle v_n, v_1 \rangle & \langle v_m, v_2 \rangle & \cdots & 1 \\
\end{bmatrix}
$$
A refresher of a few definitions I

Let \( n \in \mathbb{N} \). We put

\[
PSD(n; \mathbb{R}) := \{ S : S \in \mathbb{M}(n \times n; \mathbb{R}) \text{ and } S \text{ is positive semidefinite} \}.
\]
Let $n \in \mathbb{N}$. We put

$$PSD(n; \mathbb{R}) := \{S : S \in \mathbb{M}(n \times n; \mathbb{R}) \text{ and } S \text{ is positive semidefinite}\}.$$ 

Recall that $PSD(n; \mathbb{R})$ is a closed convex cone which (by definition!) consists of symmetric matrices only.
A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$\text{PSD}(n; \mathbb{R}) := \{S : S \in M(n \times n; \mathbb{R}) \text{ and } S \text{ is positive semidefinite}\}.$$ 

Recall that $\text{PSD}(n; \mathbb{R})$ is a closed convex cone which (by definition!) consists of symmetric matrices only.

Moreover, we consider the set

$$C(n; \mathbb{R}) := \{S \in \text{PSD}(n; \mathbb{R}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\}.$$
A refresher of a few definitions II

Let $d, k \in \mathbb{N}$ and $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary $d$-dimensional Hilbert space (i.e., $H = l^d_2$). Let $w_1, w_2, \ldots, w_k \in H$. Put $w := (w_1, \ldots, w_k)^\top \in H^k$ and $S := (w_1 \mid w_2 \mid \ldots \mid w_k) \in \mathbb{M}(d \times k; \mathbb{R})$. The matrix $\Gamma_H(w, w) \in PSD(k; \mathbb{R})$, defined as

$$\Gamma_H(w, w)_{ij} := \langle w_i, w_j \rangle = (S^\top S)_{ij} \quad (i, j \in [k] := \{1, 2, \ldots, k\})$$

is called Gram matrix of the vectors $w_1, \ldots, w_k \in H$. 
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is called Gram matrix of the vectors $w_1, \ldots, w_k \in H$.

Observe that

$$\begin{pmatrix}
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle
\end{pmatrix}$$

is not a Gram matrix!
A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let

$\xi := (\xi_1, \xi_2, \ldots, \xi_n)^{\top}: \Omega \rightarrow \mathbb{R}^n$ be a random vector. Let

$\mu := (\mu_1, \mu_2, \ldots, \mu_n)^{\top} \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$. 

Note that we don't require here that $C$ is invertible!
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Let \( n \in \mathbb{N} \). Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n \) be a random vector. Let \( \mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n \) and \( C \in PSD(n; \mathbb{R}) \).

Recall that \( \xi \) is an \( n \)-dimensional Gaussian random vector with respect to the “parameters” \( \mu \) and \( C \) (short: \( \xi \sim N_n(\mu, C) \)) if and only if for all \( a \in \mathbb{R}^n \) there exists \( \eta_a \sim N_1(0, 1) \) such that

\[
\langle a, \xi \rangle = \sum_{i=1}^{n} a_i \xi_i = \langle a, \mu \rangle + \sqrt{\langle a, Ca \rangle} \eta_a 
\]
A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^{\top} : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^{\top} \in \mathbb{R}^n$ and $C \in \text{PSD}(n; \mathbb{R})$.

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\]

Note that we don’t require here that $C$ is invertible! Following Feller, the matrix $\mathbb{V}(\xi)$ defined as
\[
\mathbb{V}(\xi)_{ij} := \mathbb{E}[\xi_i \xi_j] - \mathbb{E}[\xi_i] \mathbb{E}[\xi_j] \overset{(!)}{=} C_{ij} \quad (i, j \in [n])
\]
is known as the variance matrix of the Gaussian random vector $\xi$. 

Corollary

Let $n \in \mathbb{N}$ and $\Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R})$. TFAE:
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Let $n \in \mathbb{N}$ and $\Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R})$. TFAE:

(i) $\Sigma \in C(n; \mathbb{R})$. 

Thereby, $\phi_{ii} = 0$ for all $i \in [n]$. 

In particular, condition (i) implies that $\sigma_{ij} \in [-1, 1]$ for all $i, j \in [n]$. 

Structure of correlation matrices I
Structure of correlation matrices I

Corollary

Let \( n \in \mathbb{N} \) and \( \Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R}) \). TFAE:

(i) \( \Sigma \in C(n; \mathbb{R}) \).

(ii) \( \Sigma = \Gamma_{\chi_2}(x, x) \) for some \( x = (x_1, \ldots, x_n)^\top \in (S^{n-1})^n \).
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Let \( n \in \mathbb{N} \) and \( \Sigma = (\sigma_{ij}) \in M(n \times n; \mathbb{R}) \). TFAE:

(i) \( \Sigma \in C(n; \mathbb{R}) \).

(ii) \( \Sigma = \Gamma_{\|x\|^2}(x, x) \) for some \( x = (x_1, \ldots, x_n)^\top \in (S^{n-1})^n \).

(iii) \( \sigma_{ij} = \cos(\varphi_{ij}) \) for some \( \varphi_{ij} \in [0, \pi] \) for all \( i, j \in [n] \). Thereby, \( \varphi_{ii} = 0 \) for all \( i \in [n] \).
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Let $n \in \mathbb{N}$ and $\Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R})$. TFAE:

(i) $\Sigma \in C(n; \mathbb{R})$.

(ii) $\Sigma = \Gamma_{\frac{1}{2}}(x, x)$ for some $x = (x_1, \ldots, x_n)^\top \in (S^{n-1})^n$.

(iii) $\sigma_{ij} = \cos(\varphi_{ij})$ for some $\varphi_{ij} \in [0, \pi]$ for all $i, j \in [n]$. Thereby, $\varphi_{ii} = 0$ for all $i \in [n]$.

(iv) $\Sigma = \mathcal{V}(\xi)$ is a correlation matrix, induced by some $n$-dimensional Gaussian random vector $\xi \sim \mathcal{N}_n(0, \Sigma)$. 
Corollary

Let \( n \in \mathbb{N} \) and \( \Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R}) \). TFAE:

(i) \( \Sigma \in C(n; \mathbb{R}) \).

(ii) \( \Sigma = \Gamma_{\mathcal{W}_2}(x, x) \) for some \( x = (x_1, \ldots, x_n)^\top \in (S^{n-1})^n \).

(iii) \( \sigma_{ij} = \cos(\varphi_{ij}) \) for some \( \varphi_{ij} \in [0, \pi] \) for all \( i, j \in [n] \). Thereby, \( \varphi_{ii} = 0 \) for all \( i \in [n] \).

(iv) \( \Sigma = \nabla(\xi) \) is a correlation matrix, induced by some \( n \)-dimensional Gaussian random vector \( \xi \sim \mathcal{N}_n(0, \Sigma) \).

In particular, condition (i) implies that \( \sigma_{ij} \in [-1, 1] \) for all \( i, j \in [n] \).
Observation

Let $k \in \mathbb{N}$. Then the sets \( \{ S : S = xx^\top \text{ for some } x \in \{-1, 1\}^k \} \) and \( \{ \Theta : \Theta \in C(k; \mathbb{R}) \text{ and } \text{rk}(\Theta) = 1 \} \) coincide.
Structure of correlation matrices II

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Let $k \in \mathbb{N}$. Then the sets $\{ S : S = xx^\top \text{ for some } x \in \{-1, 1\}^k \}$ and $\{ \Theta : \Theta \in C(k; \mathbb{R}) \text{ and } rk(\Theta) = 1 \}$ coincide.

Proposition (K. R. Parthasarathy (2002))
Let $k \in \mathbb{N}$. $C(k; \mathbb{R})$ is a compact and convex subset of the $k^2$-dimensional vector space $\mathbb{M}(k \times k; \mathbb{R})$. Any $k \times k$-correlation matrix of rank 1 is an extremal point of the set $C(k; \mathbb{R})$. 
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In particular, the (finite) set of all \( k \times k \)-correlation matrices of rank 1 is not convex.
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In particular, the (finite) set of all \( k \times k \)-correlation matrices of rank 1 is not convex.
Let \( k \in \mathbb{N} \). Put

\[
C_1(k; \mathbb{R}) := \{ \Theta : \Theta \in C(k; \mathbb{R}) \text{ and } \text{rk}(\Theta) = 1 \}.
\]
Canonical block injection of $A$

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Having gained - important - additional structure by “enlarging”
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Definition
Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{R})$ arbitrary. Put

$$\hat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

Let us call $\mathbb{M}((m + n) \times (m + n); \mathbb{R}) \ni \hat{A}$ the canonical block injection of $A$.
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Let us call $\mathcal{M}((m + n) \times (m + n); \mathbb{R}) \ni \hat{A}$ the canonical block
injection of $A$.

Observe that $\hat{A}$ is symmetric, implying that $\hat{A} = \hat{A}^\top$. 
A further equivalent rewriting of GT I

Proposition

Let $H$ be an arbitrary Hilbert space over $\mathbb{R}$. Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:
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Let $H$ be an arbitrary Hilbert space over $\mathbb{R}$. Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in M(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

(i) 

$$
\sup_{(u,v)\in S_H^m \times S_H^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max_{(p,q)\in \{-1,1\}^m \times \{-1,1\}^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right|.
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(ii)

$$\sup_{\Sigma \in C(m+n;\mathbb{R})} |\langle \hat{A}, \Sigma \rangle| \leq K \max_{\Theta \in C(m+n;\mathbb{R}) \text{ and } \text{rk}(\Theta)=1} |\langle \hat{A}, \Theta \rangle|.$$
A further equivalent rewriting of GT II

Proposition

Let $H$ be an arbitrary Hilbert space over $\mathbb{R}$. Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

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(ii)

$$\max_{\Sigma \in \mathcal{C}(m+n; \mathbb{R})} |\langle \hat{A}, \Sigma \rangle| \leq K \max_{\Theta \in \mathcal{C}_1(m+n; \mathbb{R})} |\langle \hat{A}, \Theta \rangle| .$$
A further equivalent rewriting of GT II

Proposition

Let $H$ be an arbitrary Hilbert space over $\mathbb{R}$. Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

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$$\max_{\Sigma \in C(m+n;\mathbb{R})} |\langle \hat{A}, \Sigma \rangle| \leq K \max_{\Theta \in C_1(m+n;\mathbb{R})} |\langle \hat{A}, \Theta \rangle|.$$  

We don't know whether condition (ii) holds for all matrices in $\mathbb{M}((m + n) \times (m + n); \mathbb{R})$. 
GT versus NP-hard optimisation

Observation
On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P):

\[
\max_{\Sigma \in C(m+n; \mathbb{R})} |\langle \hat{A}, \Sigma \rangle|
\]

On the right side: an NP-hard, non-convex combinatorial (Boolean) optimisation problem:

\[
\max_{\Theta \in C(m+n; \mathbb{R})} \text{rk}(\Theta) = 1 |\langle \hat{A}, \Theta \rangle|
\]

Thus, Grothendieck's constant \( K_{RG} \) is precisely the "integrality gap"; i.e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!
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Thus, Grothendieck’s constant $K_G^\mathbb{R}$ is precisely the “integrality gap”; i.e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!
1 A very short glimpse at A. Grothendieck’s work in functional analysis

2 Grothendieck’s inequality in matrix formulation

3 Grothendieck’s inequality rewritten

4 Grothendieck’s inequality and its relation to non-locality in quantum mechanics

5 Towards a determination of Grothendieck’s constant $K_{RG}^G$
Modelling quantum correlation I

Following Tsirelson’s approach we consider two sets, the set of all “classical” (local) \((m \times n)\)-cross-correlation matrices and the set of all \((m \times n)\)-quantum correlation matrices:

(i) Let \((\Omega, F, P)\) be a (Kolmogorovian) probability space. Let \(A = (a_{ij}) \in \mathcal{M}(m \times n; \mathbb{R})\). \(A \in \mathcal{C}_{\text{loc}}(m \times n; \mathbb{R})\) iff \(a_{ij} = E_P[X_i Y_j]\), where \(X_i, Y_j : \Omega \rightarrow [-1, 1]\) are random variables - all defined on the same given probability space \((\Omega, F, P)\).

(ii) Let \(A = (a_{ij}) \in \mathcal{M}(m \times n; \mathbb{R})\). \(A \in \mathcal{QC}(m \times n; \mathbb{R})\) iff there are \(k, l \in \mathbb{N}\), a density matrix \(\rho\) on \(B(H_k, l)\), where \(H_k, l := \mathbb{C}^k \otimes \mathbb{C}^l\), and linear operators \(A_i \in B(\mathbb{C}^k)\), \(B_j \in B(\mathbb{C}^l)\) such that \(\|A_i\| \leq 1\), \(\|B_j\| \leq 1\) and \(a_{ij} = \langle \rho, A_i \otimes B_j \rangle = \text{tr}(\rho (A_i \otimes \text{Id}(l) \otimes B_j (\text{Id}(k) \otimes B_j)))\) for all \((i, j) \in [m] \times [n]\).
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(i) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a (Kolmogorovian) probability space. Let \(A = (a_{ij}) \in M(m \times n; \mathbb{R})\). \(A \in C_{loc}(m \times n; \mathbb{R})\) iff \(a_{ij} = \mathbb{E}_{\mathbb{P}}[X_i Y_j]\), where \(X_i, Y_j : \Omega \rightarrow [-1,1]\) are random variables - all defined on the same given probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
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(ii) Let \(A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R}). \) \(A \in QC(m \times n; \mathbb{R})\) iff there are \(k, l \in \mathbb{N}\), a density matrix \(\rho\) on \(\mathcal{B}(H_{k,l})\), where 
\[ H_{k,l} := \mathbb{C}^k \otimes \mathbb{C}^l, \] and linear operators \(A_i \in \mathcal{B}(\mathbb{C}^k), B_j \in \mathcal{B}(\mathbb{C}^l)\) such that \(\|A_i\| \leq 1, \|B_j\| \leq 1\) and 
\[ a_{ij} = \langle \rho, A_i \otimes B_j \rangle = \text{tr}(\rho(A_i \otimes B_j)) = \text{tr}(\rho(A_i \otimes \text{Id}^l)(\text{Id}^k \otimes B_j)) \]
for all \((i,j) \in [m] \times [n]\).
Modelling quantum correlation II

Is there a link between $QC(m \times n; \mathbb{R})$ and the left side of GT?
Is there a link between $QC(m \times n; \mathbb{R})$ and the left side of GT?

Theorem (Tsirelson (1987, 1993))

Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. TFAE:

(i) $A \in QC(m \times n; \mathbb{R})$.

(ii) $A = \Gamma_{l_2^k}(u, v)$ for some $k \in \mathbb{N}$ and some $u \in (S^{k-1})^m$ and $v \in (S^{k-1})^n$. 
Modelling quantum correlation III

\[ \Gamma_{l_2}^{k}(u, v) = \begin{pmatrix} 
\langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\
\langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle 
\end{pmatrix} \]
Modelling quantum correlation III

\[ \Gamma_{l_2^k}(u, v) = \begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_n \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \cdots & \langle u_m, v_n \rangle \end{pmatrix} \]

\[ \Gamma_{l_2^k}(u, v) = UV \] is the product of the matrices \( U : l_2^k \rightarrow l_\infty^m \) and \( V : l_1^n \rightarrow l_2^k \), where

\[ V := (v_1 | v_2 | \ldots | v_n) \] and \( U := \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_m^T \end{pmatrix} \).


Hence, we see that if $u \in S^m_H$ and $v \in S^n_H$ one can canonically associate a linear operator to the $(m \times n)$-matrix $\Gamma_H(u, v)$ which factors through the Hilbert space $H := \ell^k_2$ such that $\Gamma_H(u, v) = UV$ for some $(m \times k)$-matrix $U$ and some $(k \times n)$-matrix $V$, satisfying

$$\gamma_2(\Gamma_H(u, v)) \leq \|U\|_{2,\infty} \cdot \|V\|_{1,2} \leq 1 :$$
Modelling quantum correlation V

Theorem (Grothendieck (1953), Tsirelson (1987), Pisier (2001))

Let $H$ be a separable Hilbert space and $m, n \in \mathbb{N}$. Let $u := (u_1, \ldots, u_m) \top \in S_H^m$ and $v := (v_1, \ldots, v_n) \top \in S_H^n$. Then

$$\Gamma_H(u, v) \in K_G^\mathbb{R} cx(\{pq \top : p \in \{-1, 1\}^m, q \in \{-1, 1\}^n\})$$

$$= K_G^\mathbb{R} C_{loc}(m \times n; \mathbb{R}) .$$
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Theorem (Grothendieck (1953), Tsirelson (1987), Pisier (2001))

Let $H$ be a separable Hilbert space and $m, n \in \mathbb{N}$. Let $u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$. Then

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Corollary (Tsirelson (1987, 1993))

Let $m, n \in \mathbb{N}$. Then

$$QC(m \times n; \mathbb{R}) \subseteq K_G^\mathbb{R} C_{loc}(m \times n; \mathbb{R}).$$

Moreover, $C_{loc}(m \times n; \mathbb{R}) \subseteq QC(m \times n; \mathbb{R})$. The latter set inclusion is strict.
It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as Bell’s inequalities.
Bell’s inequalities and GT I

It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as *Bell’s inequalities*. Purely in terms of a very elementary application of classical Kolmogorovian probability theory and a bit of elementary algebra - and completely independent of any modelling assumptions in physics - Bell’s inequalities can be represented in form of an inequality originating from *J. F. Clauser, M. A. Horne, A. Shimony* and *R. A. Holt* in 1969.
It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as *Bell’s inequalities*. Purely in terms of a very elementary application of classical Kolmogorovian probability theory and a bit of elementary algebra - and “without the annoying adherence to physics” (as we have learnt from Niel 😊) - Bell’s inequalities can be represented in form of an inequality originating from *J. F. Clauser, M. A. Horne, A. Shimony* and *R. A. Holt* in 1969.
Lemma (BCHSH Inequality)

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an arbitrary probability space. Let \(X_1, X_2, X_3\) and \(X_4\) be arbitrary random variables with values in \([-1, 1]\) \(\mathbb{P}\)-a.s., all defined on \(\Omega\). Then

\[
|\mathbb{E}[X_iX_2] - \mathbb{E}[X_iX_3]| \leq 1 - \mathbb{E}[X_2X_3] \quad \text{for all } i \in \{1, 4\}
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Lemma (BCHSH Inequality)

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$$|\mathbb{E}_\mathbb{P}[X_iX_2] - \mathbb{E}_\mathbb{P}[X_iX_3]| \leq 1 - \mathbb{E}_\mathbb{P}[X_2X_3] \text{ for all } i \in \{1, 4\}$$

and

$$|\mathbb{E}_\mathbb{P}[X_iX_2] + \mathbb{E}_\mathbb{P}[X_iX_3]| \leq 1 + \mathbb{E}_\mathbb{P}[X_2X_3] \text{ for all } i \in \{1, 4\}.$$
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In particular,

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Lemma (BCHSH Inequality)

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In particular,

$$|\mathbb{E}_\mathbb{P}[X_1X_2] + \mathbb{E}_\mathbb{P}[X_1X_3] + \mathbb{E}_\mathbb{P}[X_4X_2] - \mathbb{E}_\mathbb{P}[X_4X_3]| \leq 2.$$ 

In other words:
Bell’s inequalities and GT III

Observation (BCHSH Inequality in matrix form)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space (in the sense of Kolmogorov).
Bell’s inequalities and GT III

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Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an arbitrary probability space (in the sense of Kolmogorov).

Put

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A^{\text{Had}} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (= \sqrt{2} \cdot \text{Hadamard matrix} \sim \text{“quantum gate”})
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Then

\[ |\langle A^{\text{Had}}, \Gamma \rangle| = |tr(A^{\text{Had}} \Gamma)| \leq 2 \text{ for all } \Gamma \in C_{\text{loc}}(2 \times 2; \mathbb{R}). \]
Bell’s inequalities and GT IV

Let us turn to the left “quantum correlation side” of GT!
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**Theorem (Tsirelson (1980))**

Let $H$ be an arbitrary Hilbert space, $u \in S^2_H$ and $v \in S^2_H$. Then

$$|\langle A^{Had}, \Gamma_H(u, v) \rangle| = |tr(A^{Had} \Gamma_H(u, v))| \leq 2\sqrt{2}$$
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Even more holds!
To this end, we recall the main ideas underlying the EPR/Bell-CHSH experiment.
Bell’s inequalities and GT IV

Let us turn to the left “quantum correlation side” of GT!

Theorem (Tsirelson (1980))

Let $H$ be an arbitrary Hilbert space, $u \in S_H^2$ and $v \in S_H^2$. Then

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Even more holds!
To this end, we recall the main ideas underlying the EPR/Bell-CHSH experiment.
Bear also Rui’s talk in mind!
A source emits in opposite directions two spin $\frac{1}{2}$ particles created from one particle of spin 0. By rotating magnets perpendicular to the directions of the two spin $\frac{1}{2}$ particles, both, Alice and Bob measure the spin in 2 different directions, leading to angles $-\frac{\pi}{2} \leq \alpha_1, \alpha_2 < \frac{\pi}{2}$ for Alice and $-\frac{\pi}{2} \leq \beta_1, \beta_2 < \frac{\pi}{2}$ for Bob. Only one angle per measurement can be chosen on both sides. The outcome of this experiment is a “random” pair of observables belonging to the set

$$\{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2)\}.$$ 

Any of these observables takes its values in $\{-1, +1\}$. 

Describing this experiment purely in terms of mathematics we immediately recognise that the Bell-Tsirelson constant $2\sqrt{2}$ is attained by the Hadamard matrix, since:

$$\frac{43}{54}$$
Bell’s inequalities and GT V

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Bell’s inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$. Let $H \ni x := \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2)$ (“entangled Bell state”).
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$$\alpha_1 := \frac{\pi}{2}, \quad \alpha_2 := 0, \quad \beta_1 := \frac{\pi}{4} \quad \text{and} \quad \beta_2 := -\frac{\pi}{4}. $$
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$$\alpha_1 := \frac{\pi}{2}, \quad \alpha_2 := 0, \quad \beta_1 := \frac{\pi}{4} \quad \text{and} \quad \beta_2 := -\frac{\pi}{4}.$$ 

Put

$$\Gamma^{EPR} := \begin{pmatrix} \langle x, (A_1 \otimes B_1)x \rangle_H & \langle x, (A_1 \otimes B_2)x \rangle_H \\ \langle x, (A_2 \otimes B_1)x \rangle_H & \langle x, (A_2 \otimes B_2)x \rangle_H \end{pmatrix},$$

where $A_i := R(\alpha_i), \; B_j := R(\beta_j)$ and

$$O(2; \mathbb{R}) \ni R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix} \quad ("\text{rotary reflections}").$$

Bell’s inequalities and GT VI

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Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$. Let

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$$O(2; \mathbb{R}) \ni R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix}$$ ("rotary reflections").

Then $\Gamma^{EPR} \in QC(2 \times 2; \mathbb{R})$ and

$$|\langle A^{Had}, \Gamma^{EPR} \rangle| = |tr(A^{Had} \Gamma^{EPR})| = 2\sqrt{2} > 2.$$
1. A very short glimpse at A. Grothendieck’s work in functional analysis

2. Grothendieck’s inequality in matrix formulation

3. Grothendieck’s inequality rewritten

4. Grothendieck’s inequality and its relation to non-locality in quantum mechanics

5. Towards a determination of Grothendieck’s constant $K^G_R$
Schur product and the matrix $f[A]$

Definition
Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i, j) \in [m] \times [n]$. 

Guiding Example
The Schur product (or Hadamard product) $(a_{ij}) \ast (b_{ij}) := (a_{ij}b_{ij})$ of matrices $(a_{ij})$ and $(b_{ij})$ leads to $f[A]$, where $f(x) := x^2$. 

Remark
The notation "$f[A]\)$ is used to highlight the difference between the matrix $f(A)$ originating from the spectral representation of $A$ (for normal matrices $A$) and the matrix $f[A]$, defined as above!
Schur product and the matrix $f[A]$

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Grothendieck’s identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?
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Theorem (Grothendieck’s identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ and $(\xi, \eta)^\top \sim N_2(0, \Sigma_\rho)$, where

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E[\text{sign}(\xi) \text{sign}(\eta)] = 4P(\xi \geq 0, \eta \geq 0) - 1
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Corollary

Let $2 \leq k \in \mathbb{N}$. Let $\Sigma \in C(k; \mathbb{R})$ an arbitrarily given correlation matrix. Then there exists a Gaussian random vector $\xi \sim N_k(0, \Sigma)$ such that

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$$C(k; \mathbb{R}) \ni \frac{2}{\pi} \arcsin[\Sigma] = \mathbb{E}[\Theta(\xi)] ,$$

where

$$\Theta(\xi(\omega))_{ij} := \text{sign}(\xi_i(\omega))\text{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$, and for all $i, j \in [k]$. 

Grothendieck’s identity II
Corollary

Let \( 2 \leq k \in \mathbb{N} \). Let \( \Sigma \in C(k; \mathbb{R}) \) an arbitrarily given correlation matrix. Then there exists a Gaussian random vector \( \xi \sim N_k(0, \Sigma) \) such that

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for all \( \omega \in \Omega \), and for all \( i, j \in [k] \). \( \Theta(\xi(\omega)) \) is a correlation matrix of rank 1 for all \( \omega \in \Omega \), and we have
Corollary

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\[
\max_{\Theta \in C(k; \mathbb{R}) \text{ rank}(\Theta) = 1} |\langle \hat{A}, \Theta \rangle| \geq \mathbb{E}[|\langle \hat{A}, \Theta(\xi) \rangle|] \geq |\langle \hat{A}, \mathbb{E}[\Theta(\xi)] \rangle| = \frac{2}{\pi} |\langle \hat{A}, \arcsin[\Sigma] \rangle| .
\]
More generally, we may list the following two “Schoenberg-type” results (applied to non-linear correlation matrix transforms) which are implied by the Schur product theorem:
Grothendieck’s identity III

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**Theorem (Schoenberg (1942), Rudin (1959))**

Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be a function that admits a power series representation \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) for some sequence \( (a_n) \) of non-negative numbers on \([-1, 1]\).
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Let \( k \in \mathbb{N} \) and \( \Sigma \) be an arbitrary \((k \times k)\)-correlation matrix. Then also \( \frac{1}{f(1)} f [\Sigma] \) is a \((k \times k)\)-correlation matrix.
Grothendieck’s identity III

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Let $k \in \mathbb{N}$ and $\Sigma$ be an arbitrary $(k \times k)$-correlation matrix. Then also $\frac{1}{f(1)} f[\Sigma]$ is a $(k \times k)$-correlation matrix.

Conversely, we have:
Grothendieck’s identity IV

Theorem (Schoenberg (1942), Rudin (1959), Christensen/Ressel (1978))

Let $0 \neq g : [-1, 1] \rightarrow \mathbb{R}$ be a function such that $g[\Sigma]$ is a $(k \times k)$-correlation matrix for all $(k \times k)$-correlation matrices $\Sigma$ and all $k \in \mathbb{N}$.
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Let \( 0 \neq g : [-1, 1] \rightarrow \mathbb{R} \) be a function such that \( g[\Sigma] \) is a \((k \times k)\)-correlation matrix for all \((k \times k)\)-correlation matrices \( \Sigma \) and all \( k \in \mathbb{N} \). Then \( g(1) = 1 \) and \( |g(\rho)| \leq 1 \) for all \( \rho \in [-1, 1] \), and
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Let $0 \neq g : [-1, 1] \rightarrow \mathbb{R}$ be a function such that $g[\Sigma]$ is a $(k \times k)$-correlation matrix for all $(k \times k)$-correlation matrices $\Sigma$ and all $k \in \mathbb{N}$. Then $g(1) = 1$ and $|g(\rho)| \leq 1$ for all $\rho \in [-1, 1]$, and $g[A] \in PSD(m; \mathbb{R})$ for all $A \in PSD(m; [-1, 1])$ and all $m \in \mathbb{N}$. 
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Let $0 \neq g : [-1, 1] \rightarrow \mathbb{R}$ be a function such that $g[\Sigma]$ is a $(k \times k)$-correlation matrix for all $(k \times k)$-correlation matrices $\Sigma$ and all $k \in \mathbb{N}$. Then $g(1) = 1$ and $|g(\rho)| \leq 1$ for all $\rho \in [-1, 1]$, and $g[A] \in PSD(m; \mathbb{R})$ for all $A \in PSD(m; [-1, 1])$ and all $m \in \mathbb{N}$. Moreover, $g : [-1, 1] \rightarrow [-1, 1]$ has to be a function that admits a power series representation $g(x) = \sum_{n=0}^{\infty} b_n x^n$ for some sequence $(b_n)$ of non-negative numbers on $[-1, 1]$. 
A seemingly fruitful approach is the following one:

(i) Transform an arbitrarily given correlation matrix \( \Sigma_0 \) non-linearly - and entrywise - to another correlation matrix \( \Sigma_1 = \Phi[\Sigma_0] \) for some \( \Phi : \mathbb{C}(k; \mathbb{R}) \rightarrow \mathbb{C}(k; \mathbb{R}) \) such that this non-linear transformation \( \Phi \) strongly reduces the impact of the \text{arcsin} function (up to a given small error).

(ii) Apply Grothendieck's identity to the so obtained correlation matrix \( \Sigma_1 \) and apply the estimation above - to \( \text{arcsin}[\Sigma_1] \).

(iii) A reiteration of the steps (i) and (ii) could lead to an iterative algorithm which might converge to a "suitable" - upper - bound of \( K_{RG} \).
Grothendieck’s identity V

A seemingly fruitful approach is the following one:

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(ii) Apply Grothendieck’s identity to the so obtained correlation matrix $\Sigma_1$ and apply the estimation above - to $\text{arcsin}[\Sigma_1]$.

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(iii) A reiteration of the steps (i) and (ii) could lead to an iterative algorithm which might converge to a “suitable” - upper - bound of $K^\mathbb{R}_G$. 
“... at present I will say only that if a chess problem is, in the crude sense, ’useless’, then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The ’seriousness’ of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects...”

– A Mathematician’s Apology (1940)
Only a - very - few references

Approximating the cut-norm via Grothendieck’s inequality.  

The Grothendieck problem with rank constraint.  

Extrait d’une lettre adressé à M. Hermite.  

Some results and problems on quantum Bell-type inequalities.  
Thank you for your attention!
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Are there any questions, comments or remarks?