QUANTA AND QUALIA

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  - Here a physical interpretation in terms of the philosophical notion of *quale/qualia* — resembles finite observations on computational systems as in [Abramsky–Vickers 1993, R 2001, R–Vickers 2003]

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  The partial bijections on X form the *symmetric pseudogroup* $\mathcal{I}(X)$.

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- **Example:** The topology $\Omega(X)$ of a topological space X (*spatial locale*).

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- Let $V, W \in \mathrm{Max}\, A$ and (V_i) a family in $\mathrm{Max}\, A$:

$$V \,\&\, W = \overline{\mathrm{span}\{ab \mid a \in V,\ b \in W\}} \qquad V^* = \{a^* \mid a \in V\}$$

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- These operations make $\mathrm{Max}\, A$ an *involutive quantale* (= involutive semigroup in the monoidal category of sup-lattices):

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U \,\&\, (V \,\&\, W) & = & (U \,\&\, V) \,\&\, W & \text{(Associativity)} \\
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- $\mathrm{Max}\, A$ is *stably Gelfand*: $V \,\&\, V^* \,\&\, V \subset V \iff V \,\&\, V^* \,\&\, V = V$ [R 2018b]

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- Let A be a C*-algebra and $B \subset A$ an abelian sub-C*-algebra.

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- **Theorem:** [R 2018a] *$S(B)$ is a spatial pseudogroup and $E(S(B)) = I(B)$.*
- **Back to the qubit example:** If $A = M_2(\mathbb{C})$ and $B = D_2(\mathbb{C})$ then

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- $W(B)$ is a quotient of $S(B)$, so in general contains less information.

2. Groupoids

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- From here on G is always a topological étale groupoid.

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- The topology $\Omega(G)$ is a *unital* stably Gelfand quantale with unit G_0 (and also a locale):

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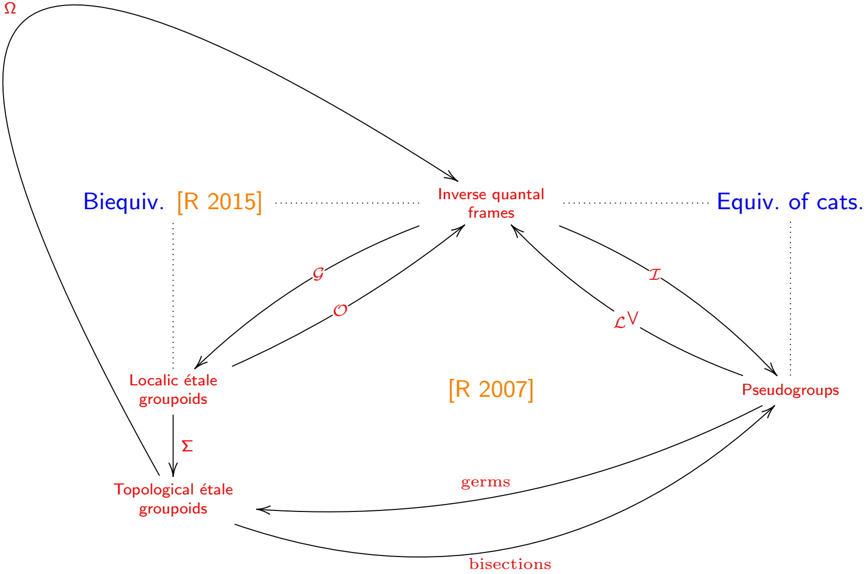
Ω
Biequiv. [R 2015]
Inverse quantal frames
Equiv. of cats.
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3. Groupoid C*-algebras

From here on G is second countable, locally compact and Hausdorff.

- *Convolution algebra* $C_c(G) = \{\phi : G \to \mathbb{C} \mid \operatorname{supp} f \text{ is compact}\}$

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$$\begin{aligned} \|\phi\| &\geq \|\phi\|_\infty \\ \|\phi\| &= \|\phi\|_\infty \qquad \text{if } \phi \in C_c(U) \text{ for some } U \in \mathcal{I}(\Omega(G)) \end{aligned}$$

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- $B := \overline{C_c(G_0)}$ is an abelian sub-C*-algebra. $[B \cong C_0(G_0)]$
- Using $W(B)$ instead of $S(B)$:

  Theorem: [Renault 2008] *There is a bijective correspondence (up to isomorphisms) between twisted topologically principal groupoids and C*-algebras A equipped with Cartan subalgebras B.*

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  Proof sketch: $\overline{C_c(-)} : \Omega(G) \to \mathrm{Max}\, A$ is an *injective homomorphism* of involutive quantales [R 2018b], and it has a left adjoint $\mathrm{supp}^\circ : \mathrm{Max}\, A \to \Omega(G)$ (*open support*) defined by $\mathrm{supp}^\circ(V) = \bigcup_{a \in V} \mathrm{int}\,\mathrm{supp}\, a$ which is *symmetry preserving*: $\mathrm{supp}^\circ(S(B)) = \mathcal{I}(\Omega(G))$. ∎

  Compactness is only a sufficient condition and probably can be replaced by something weaker.

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These conditions imply that $\triangle$ is isomorphic to $\Omega(G)$ for a locally compact Hausdorff étale groupoid G, and that $\mathcal{I}(\triangle) = S(B)$.

Question: Comparison to Cartan subalgebras:

- Is maximality of B dropped?
- Can G be more general than topologically principal?
- On the other hand, does the existence of σ place restrictions that do not exist in Renault's theorem? (cf. aforementioned compactness of G)

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- $\mathrm{Max}\, A$ is a *stably Gelfand quantale*.
- Diagonals $\triangle \subset \mathrm{Max}\, A$ correspond to *classical observers* represented by locally compact Hausdorff étale groupoids G.
- Each diagonal $\triangle$ is an actual *classical (intuitionistic) logic* attached to $\mathrm{Max}\, A$, which itself is non-classical because it lacks distributivity.

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  Examples:
  - Measure the spin of a electron along the z axis using a Stern–Gerlach apparatus;
  - Press a keyboard and watch something appear on the computer screen, etc.
- $a \& b$ means *a and then b* (noncommutative and nonidempotent *conjunction*).
- $a \vee b$ means *a or b* (*disjunction*).
- More generally, $\bigvee_i a_i$ means *a_i for some i* (*infinitary disjunction*).
- Will address involution later.
- In a diagonal $\triangle$ we have $a \& a = a$ and $a \& b = b \& a = a \wedge b$: *intuitionistic logic* of classical observers.

6. A logical interpretation

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- **Problem:** this interpretation relies on *undefined* (or hard to define) *concepts*: *system* and *observer*.
- This brings about derived problems, such as, if by observer we mean *human observer*, that of making the interpretation *anthropocentric*.

7. Qualia

Wikipedia reads:

In philosophy and certain models of psychology, *qualia* (singular form: *quale*) are claimed to be individual instances of subjective, conscious experience.

Clarence Irving Lewis, in his book Mind and the World Order (1929), was the first to use the term "qualia" in its generally agreed upon modern sense:

> *There are recognizable qualitative characters of the given, which may be repeated in different experiences, and are thus a sort of universals; I call these "qualia". But although such qualia are universals, in the sense of being recognized from one to another experience, they must be distinguished from the properties of objects.*

Such a concept seems to correspond to a fundamental observable phenomenon, yet it does not appear in any theories of physics!

7. Qualia

An attempt at a definition of qualia (again from Wikipedia):

Daniel Dennett identifies four properties that are commonly ascribed to qualia. According to these, qualia are:

- ineffable; that is, they cannot be communicated, or apprehended by any other means than direct experience.
- intrinsic; that is, they are non-relational properties, which do not change depending on the experience's relation to other things.
- private; that is, all interpersonal comparisons of qualia are systematically impossible.
- directly or immediately apprehensible in consciousness; that is, to experience a quale is to know one experiences a quale, and to know all there is to know about that quale.

Regardless of whether or not one enjoys such a definition on intuitive grounds, a mathematical description is hard to find!

7. Qualia

Alternatively, do NOT attempt to explain what a single quale is, but rather ask:

WHAT IS A "SPACE OF QUALIA"?

Analogy: a *vector* is defined to be an element of a *vector space*.

8. The space of qualia

- Let us denote the "space" of all qualia by $\mathcal{Q}$, and let us assume that this is a *set*.

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- Let us also assume $\&$ is associative, so that *$\mathcal{Q}$ is a semigroup*.

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- The existence of such time reversed qualia is less easy to justify (how do we interpret, say, $a \,\&\, a^*$?) but we shall keep it for mathematical convenience.

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- $\emptyset$ is the *impossible concept*, and $\mathcal{Q}$ is the *trivial concept*.

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- $\mathcal{Q}$ is called *stably Gelfand* if such untraceable irreversibility is the only possible kind of irreversibility:

$$a \,\&\, a^* \,\&\, a \leq a \iff a \,\&\, a^* \,\&\, a = a$$

  As with some of our previous choices, the validity of this law can be debated, but, at least for now, we shall assume it due to mathematical convenience and because it holds in our examples.

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- Let a and b be two qualia, and suppose the supremum $a \vee b$ exists in the specialization order: $a \vee b$ is the most specific quale which is less specific than both a and b. Think of it as a *disjunction*, to be read *a or b*.

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- **Example:** $\mathrm{Max}\, A$ with the *lower Vietoris topology*.

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- For instance take $\triangle_x$ and $\triangle_z$ to correspond to spin measurements of a spin-1/2 particle along the x and the z axis.
- These diagonals are obtained from the abelian subalgebras B_x and B_z generated by the observables $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_z$, respectively:

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

9. Qualia and diagonals

$$
\begin{tikzcd}
B_x = X^{\downarrow} \vee X^{\uparrow} \arrow[d, dash] \arrow[dr, dash] & & & & B_z = Z^{\downarrow} \vee Z^{\uparrow} \arrow[dl, dash] \arrow[d, dash] \\
X^{\downarrow} \arrow[drr, dash] & X^{\uparrow} \arrow[dr, dash] & & Z^{\downarrow} \arrow[dl, dash] & Z^{\uparrow} \arrow[dll, dash] \\
& & \mathbf{0} & &
\end{tikzcd}
$$

$$B_x = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$X^{\uparrow} = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad X^{\downarrow} = \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle$$

$$B_z = D_2(\mathbb{C}) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

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\end{tikzcd}$$

$X^{\uparrow} \,\&\, X^{\downarrow} = \mathbf{0}$	IMPOSSIBLE
$X^{\uparrow} \,\&\, X^{\uparrow} = X^{\uparrow}$	REDUNDANT
$X^{\uparrow} \,\&\, B_x = X^{\uparrow}$	GENERIC x-SPIN QUESTION
$X^{\uparrow} \,\&\, Z^{\uparrow} \neq \mathbf{0}$	POSSIBLE
$X^{\uparrow} \,\&\, Z^{\downarrow} \neq \mathbf{0}$	POSSIBLE
$X^{\uparrow} \,\&\, B_z \neq \mathbf{0}$	GENERIC z-SPIN QUESTION...
$X^{\uparrow} \,\&\, B_z = X^{\uparrow} \,\&\, Z^{\uparrow} \vee X^{\uparrow} \,\&\, Z^{\downarrow}$	... HAS TWO POSSIBLE ANSWERS

9. Qualia and diagonals

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- This is also the spectrum of the locale

$$\begin{tikzcd}
 & B_x \arrow[dl, dash] \arrow[dr, dash] & \\
x^{\downarrow} \arrow[dr, dash] & & x^{\uparrow} \arrow[dl, dash] \\
 & \mathbf{0} &
\end{tikzcd}$$

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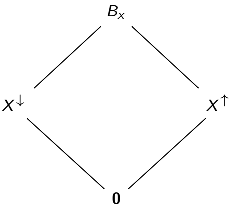

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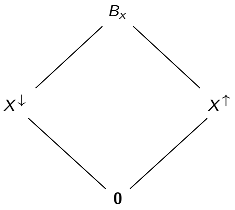

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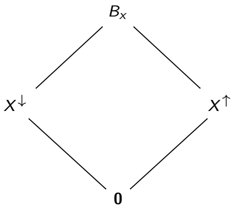

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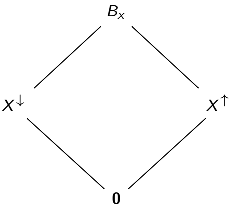

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- No collapse...
- No many worlds...

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- If qualia are physical phenomena surely they cannot be redundant.

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