Quantum functions and the Morita theory of quantum graph isomorphisms

David Reutter

University of Oxford

Combining Viewpoints in Quantum Theory
ICMS, Heriot-Watt University

March 20, 2018

Overview

This talk is based on joint work with Ben Musto and Dominic Verdon:

A compositional approach to quantum functions

The Morita theory of quantum graph isomorphisms

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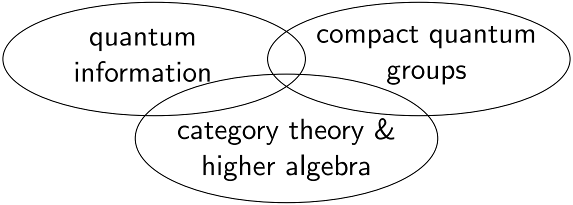

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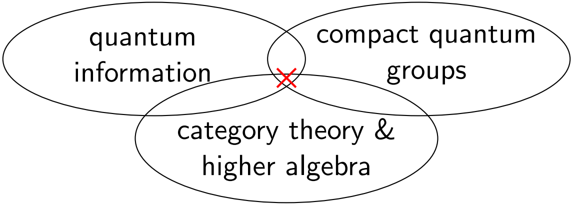

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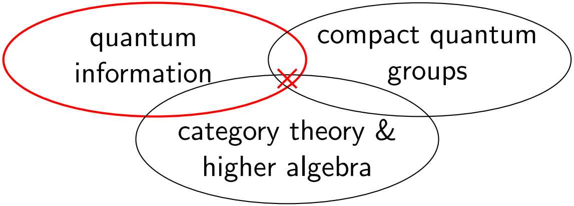

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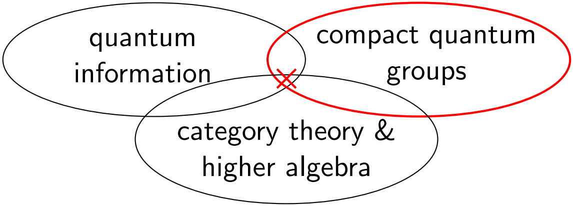

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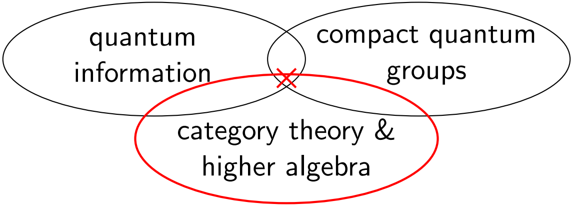

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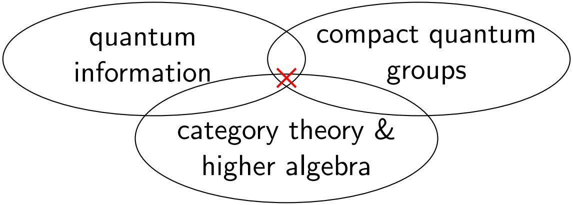

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Part 1: Getting started

Part 2: Quantum functions

Part 3: The Morita theory of quantum graph isomorphisms

Part 1
Getting started

Quantum Functions

$$\left\{ \begin{array}{c} \text{finite sets} \\ \text{functions} \end{array} \right\}$$

Quantum Functions

fSet

Quantum Functions

$$\mathbf{fSet} \qquad \simeq \left\{\begin{matrix}\text{commutative} \\ \text{f.d. } C^*\text{-algebras} \\ *\text{-homomorphisms}\end{matrix}\right\}^{op}$$

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Quantum set of quantum functions $\leftrightsquigarrow$ internal hom $[A, B]$ in $\mathbf{C^*Alg}^{op}$

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fS | fqSet

set (| $B)$

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X, Y finite sets. $[X, Y] =$ universal C^*-algebra with generators $\{p_{xy}\}_{x \in X, y \in Y}$ and relations:

$$p_{xy}^* = p_{xy} = p_{xy}^2 \quad p_{xy} p_{xy'} = \delta_{y,y'} p_{xy} \quad \sum_{y \in Y} p_{xy} = 1$$

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- physical meaning and applications?

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A perfect winning strategy is a graph isomorphisms.

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Definition (Quantum graph isomorphism game [1])

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Alice and Bob play against a verifier and share an entangled state.
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There are graphs that are quantum but not classically isomorphic!

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Pseudo-telepathy and quantum graph isomorphisms

Suppose Alice and Bob share a maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$. Quantum graph isomorphism = projectors $\{P_{xy}\}_{x \in V(G), y \in V(H)}$ on $\mathcal{H}$ such that:

$$P_{xy} P_{xy'} = \delta_{y,y'} P_{xy} \qquad \sum_{y \in V(H)} P_{xy} = \mathrm{id}_{\mathcal{H}}$$

$$P_{xy} P_{x'y} = \delta_{x,x'} P_{xy} \qquad \sum_{x \in V(G)} P_{xy} = \mathrm{id}_{\mathcal{H}}$$

If $x, x' \in V(G)$, $y, y' \in V(H)$ with $x \sim_G x'$ XOR $y \sim_H y'$, then

$$P_{xy} P_{x'y'} = 0$$

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Hilb — the category of finite-dimensional Hilbert spaces and linear maps.

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String diagrams: read from bottom to top.

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Finite Gelfand duality: [1]

$$\text{finite set } X \quad \leftrightsquigarrow \quad \text{commutative finite-dimensional } C^*\text{-algebra } \mathbb{C}X$$

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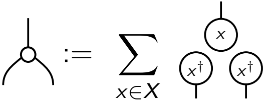

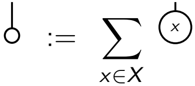

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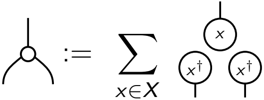

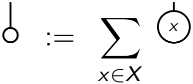

This makes $\mathbb{C}X$ into a commutative special †-Frobenius algebra in Hilb.

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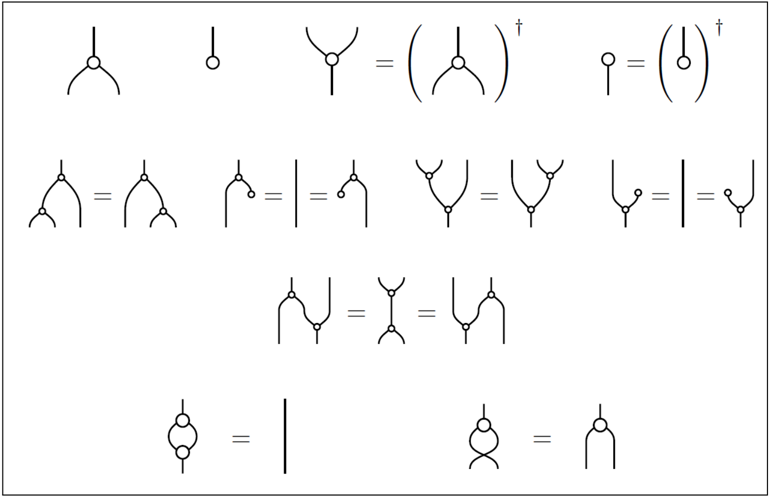

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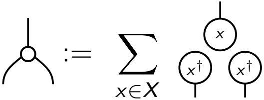

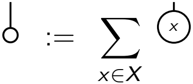

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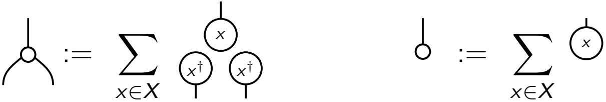

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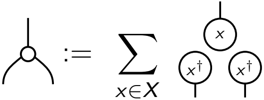
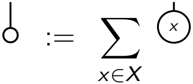




This makes $\mathbb{C}X$ into a commutative special †-Frobenius algebra in Hilb.

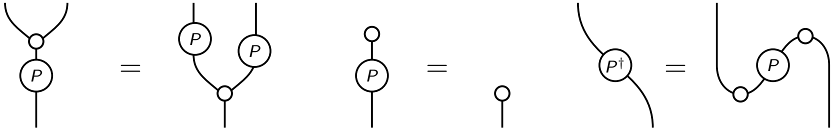
Philosophy: Do finite set theory with string diagrams in Hilb.

[1] Coecke, Pavlovic, Vicary — A new description of orthogonal bases. 2008

Part 2
Quantum functions

Quantizing functions

Function P between finite sets:



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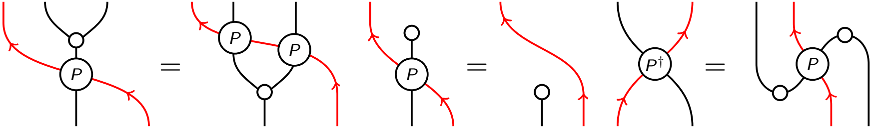

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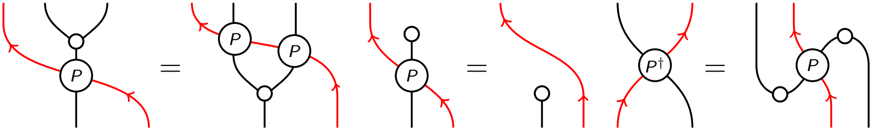

$$\delta_{y,y'} P_{xy} = P_{xy} P_{xy'} \qquad \sum_y P_{xy} = \mathrm{id}_{\mathcal{H}} \qquad P_{xy}^\dagger = P_{xy}$$

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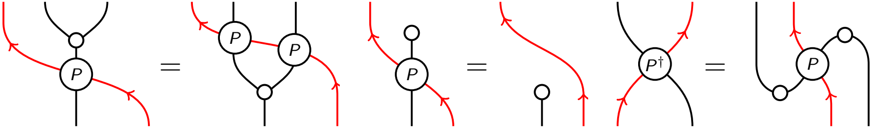

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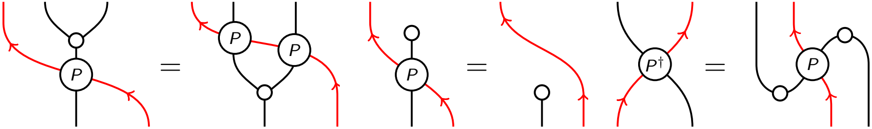

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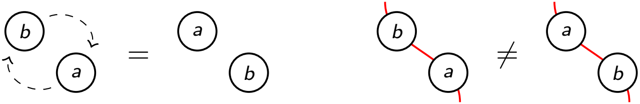

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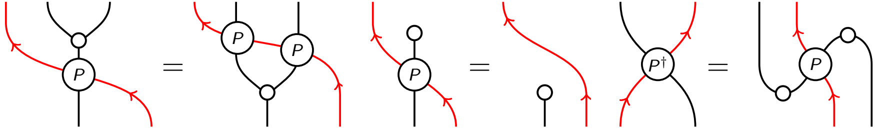

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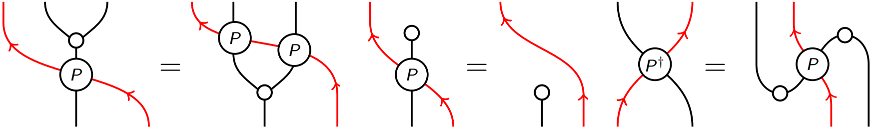

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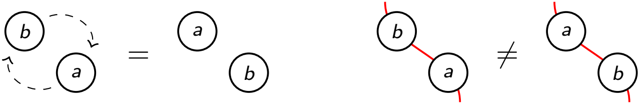

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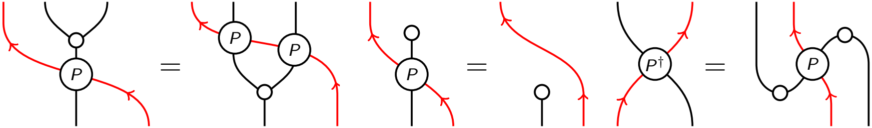

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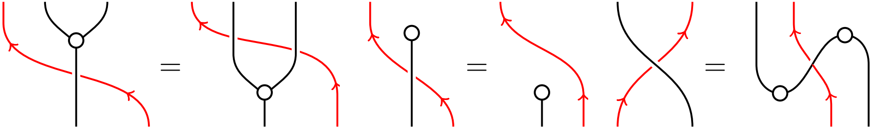

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These look like the equations satisfied by a braiding.

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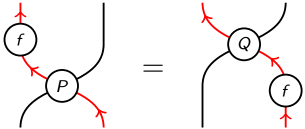

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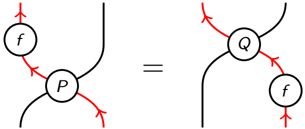

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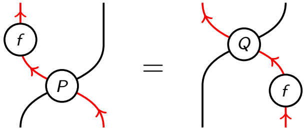
a linear map $f : \mathcal{H} \to \mathcal{H}'$ such that 

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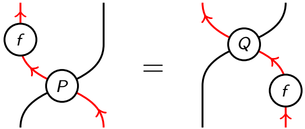
a linear map $f : \mathcal{H} \to \mathcal{H}'$ such that 

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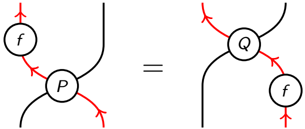

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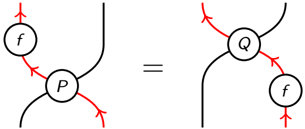

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$\mathrm{QSet}(A, B)$: Category of quantum functions between finite sets A and B

Connection to previous work in noncommutative topology:
$\mathrm{QSet}(A, B)$ is the category of f.d. representations of the internal hom $[A, B]$

The 2-category QSet

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Definition

The 2-category QSet is built from the following structures:

- **objects** are finite sets $A, B, \ldots$;
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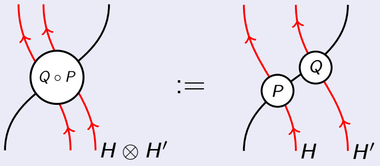
The composition of two quantum functions $(H, P) : A \to B$ and $(H', Q) : B \to C$ is a quantum function $(H \otimes H', Q \circ P)$ defined as follows:



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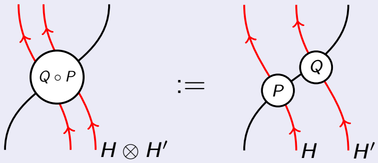
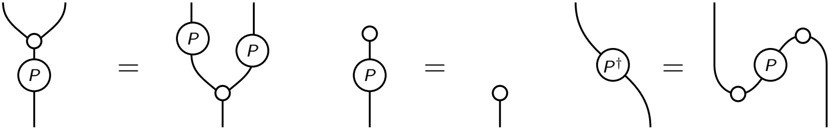


2-morphisms compose by tensor product and composition of linear maps.

Can be extended to also include quantum sets as objects.

Quantum bijections

Function P between finite sets:



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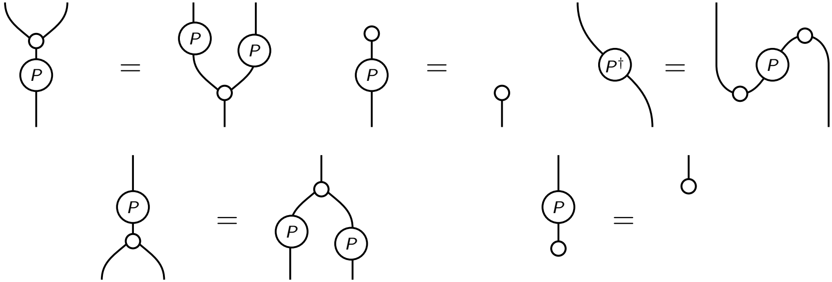

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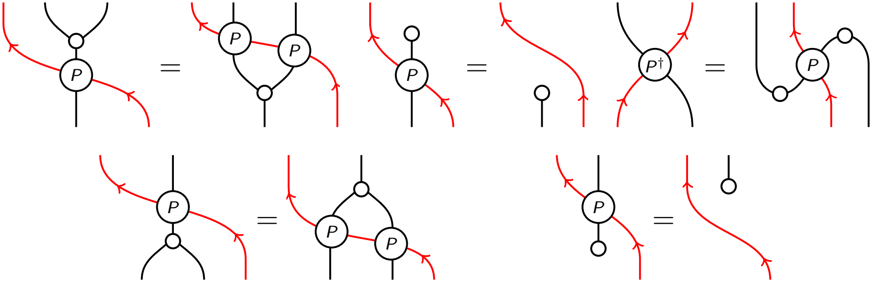

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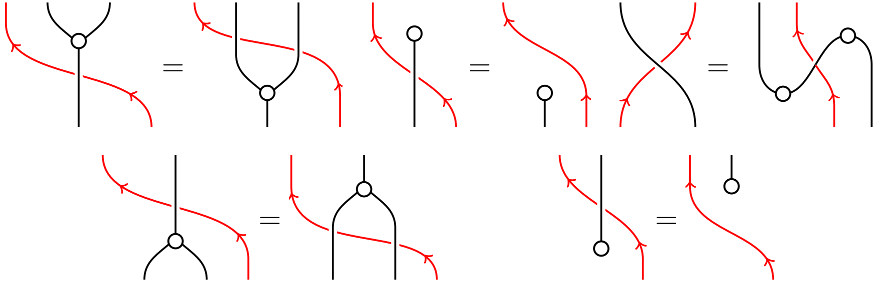

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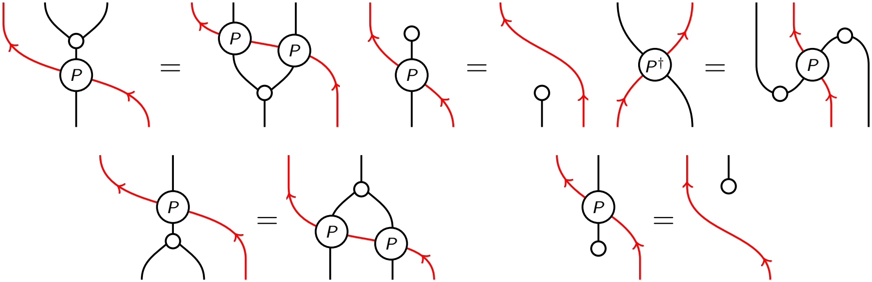

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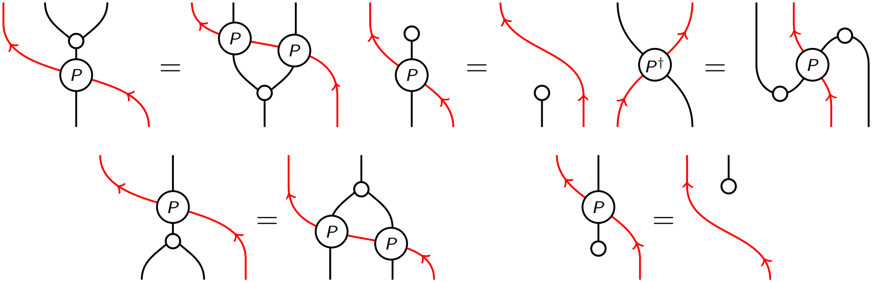

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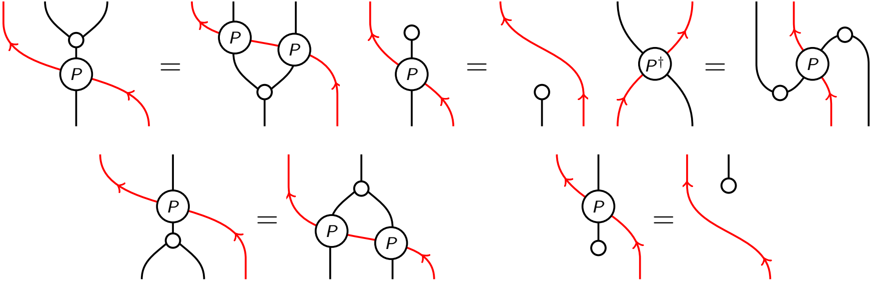

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Theorem. Quantum bijections are *dagger dualizable* quantum functions.

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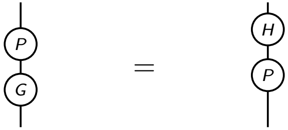

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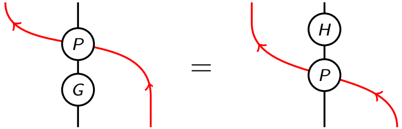

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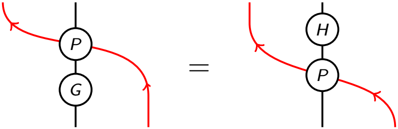

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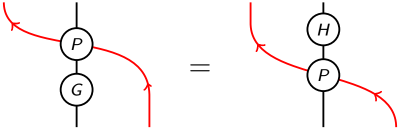

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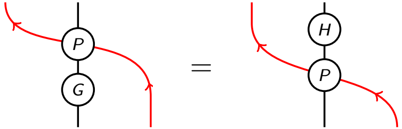

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Can we understand quantum isomorphisms in terms of the quantum automorphism categories $\mathrm{QAut}(G)$?

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Part 3
The Morita theory of quantum graph isomorphisms

Morita theory on a slide

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Morita classes of Frobenius algebras in fusion categories are well studied.

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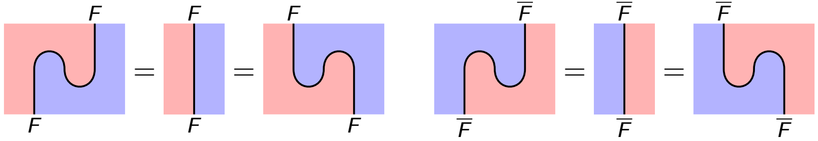

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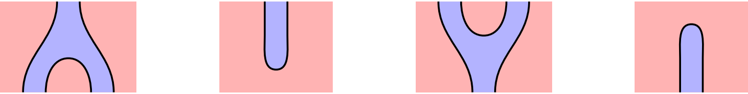

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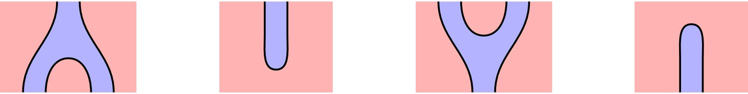

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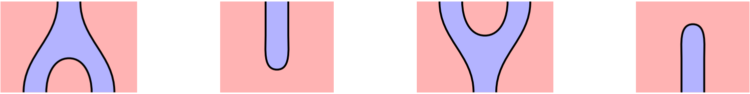

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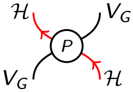
Use this to classify graphs quantum isomorphic to a given graph.

Classifying quantum isomorphic graphs

There is a monoidal forgetful functor $F : \mathrm{QAut}(G) \to \mathrm{Hilb}$:



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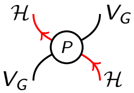

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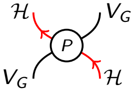

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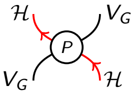

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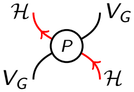

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drop commutativity condition $\rightsquigarrow$ classify quantum graphs [1,2]

[1] Weaver — Quantum graphs as quantum relations. 2015

[2] Duan, Severini, Winter — Zero error communication [...] theta functions. 2010

Frobenius algebras in classical subcategories

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What about the commutativity condition?

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- Reconnect the disjoint components of $\sqcup_O G_O$ as follows:

$$v \in O \sim_{G'} w \in O' \quad \Leftrightarrow \quad h_{O,O} v \sim_G w$$

An example: binary magic squares (BMS)

BMS: a 3×3 square with $\begin{cases} \text{entries in } \{0, 1\} \\ \text{rows and columns add up to } 0 \bmod 2 \end{cases}$

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- vertices: *partial BMS* — only one row or column filled — 24 vertices

$$\begin{pmatrix} \cancel{0} & \cancel{0} & 0 \\ \cancel{0} & \cancel{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \cancel{0} & \cancel{0} \\ \cancel{1} & 1 & \cancel{0} \\ \cancel{1} & \cancel{1} & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & \cancel{1} & \cancel{0} \\ 0 & 1 & 1 \\ 1 & \cancel{0} & \cancel{1} \end{pmatrix}$$

- edge between two vertices if the partial BMS *contradict* each other

Bit-flip symmetries

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Thanks for listening!