Toward a functorial quantum spectrum for noncommutative algebras

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Combining Viewpoints in Quantum Theory
ICMS, Edinburgh — March 20, 2018

1. Do noncommutative rings have a spectrum?
2. From points to contextuality in noncommutative geometry
3. Projection lattices as spectral invariants
4. Toward a quantum spectrum for noncommutative algebras

The spectrum

For this talk, a *spectrum* is an assignment

$$\{\text{commutative algebras}\} \to \{\text{spaces}\}$$

Example: For a commutative ring R, its Zariski spectrum is

$$\mathsf{Spec}(R) = \{\text{prime ideals of } R\},$$

where an ideal $P \subseteq R$ is called *prime* if $1 \notin P$ and

$$ab \in P \implies a \in P \text{ or } b \in P.$$

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It carries a *Zariski topology*, and is nonempty if $R \neq 0$ (by Zorn's lemma).

This is not enough to recover R from $\mathsf{Spec}(R)$, but in algebraic geometry it is equipped with a *structure sheaf* to produce a *scheme*, whose ring of global sections is $\Gamma(\mathsf{Spec}(R), \mathcal{O}_{\mathsf{Spec}(R)}) \cong R$.

Further examples of spectra

Ex. 2: For a (unital) commutative C*-algebra A, its Gelfand spectrum is

$$\mathsf{Spec}(A) = \{\text{maximal ideals of } A\},$$

where an ideal M is *maximal* if it is maximal with respect to $1 \notin M$. This is a *compact Hausdorff space*, and we may recover A from this spectrum as $C(\mathsf{Spec}(A)) = C(\mathsf{Spec}(A), \mathbb{C}) \cong A$.

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Ex. 3: For a Boolean algebra B, its Stone spectrum is

$$\mathsf{Spec}(B) = \{\text{prime ideals of } B\} \cong \{\text{ultrafilters of } B\},$$

where $P \subseteq B$ is an *ideal* if it is non-empty, $\vee$-closed down-set, and it is *prime* if $a \wedge b \in P \implies a \in P$ or $b \in P$. This is a *compact* 0-*dimensional space*, from which we can recover B as the lattice of clopen sets.

Spectrum as a link to noncommutative geometry

Thus we have:

- Commutative rings: $\mathsf{Spec}(R) = \{\text{prime ideals of } R\}$
- Commutative C*-algebras: $\mathsf{Spec}(A) = \{\text{max. ideals of } A\}$
- Boolean algebra: $\mathsf{Spec}(B) = \{\text{prime ideals of } B\}$

Each of these spectra is a (contravariant) functor: each algebra morphism $f \colon A \to B$ yields a geometric map $\mathsf{Spec}(B) \to \mathsf{Spec}(A)$ by $P \mapsto f^{-1}(P)$.

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This partly motivates noncommutative geometry of various flavors:

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In recent decades, NCG proceeds without bothering to construct an actual noncommutative space. . .

A noncommutative spectrum?

Question: What is the actual "noncommutative space" corresponding to a noncommutative algebra?

Why do I care?

- A solution would yield a rich invariant for noncommutative rings.
- Help us "see" which rings are "geometrically nice" (e.g., smooth).
- Quantum modeling: what is the "noncommutative phase space" of a quantum system?

I believe that these and other related questions could benefit if we had an actual "spatial" object to refer to when thinking geometrically about rings.

Modeling all rings via "spaces"?

Question: What is the "noncommutative space" corresponding to a noncommutative algebra?

To make this a rigorous problem, we should first set some ground rules:

(A) Keep the classical construction if the ring is commutative.
(Let's not tell "commutative" geometers how to do their own job!)

(B) Make it a *functorial* construction.
(To ensure it's truly geometric, and to aid computation.)

These rules provide us us with:

- Obstructions proving that certain constructions are impossible;
- Sharpened ideas on how to progress toward useful constructions.

Taking commutative subalgebras seriously

(A) Keep the usual construction if the ring is commutative.

(B) Make it a *functorial* construction.

Applying the criteria: look at *commutative subalgebras* C of any noncommutative algebra A. (Associativity of $A \Rightarrow$ "many" $C \subseteq A$)

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Suppose $F\colon \mathbf{Ring}^{\mathrm{op}} \to \{\text{“spaces”}\}$ is a "spectrum functor."

(A) means that we understand the $F(C)$ very well.
(B) gives us maps $F(A) \to F(C)$, compatible on intersections.

To me, this is reminiscent of the situation in quantum physics:

- $A \leftrightsquigarrow$ algebra of *observables* for quantum system
- $C \leftrightsquigarrow$ "classical viewpoints" of the quantum system

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Naive (and old) idea: Maybe we should assign to each ring a topological space and a sheaf of *noncommutative* rings. To begin this process, we would need a nonempty underlying set.

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There are several candidates for "primes" in noncommutative rings, but:

Problem: Every existing notion of a noncommuative "prime ideal" is either (i) not functorial in any obvious way or (ii) empty for some $R \neq 0$.

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Theorem (R., 2012): Any functor $\textbf{Ring}^{\mathrm{op}} \to \textbf{Set}$ whose restriction to the full subcategory $\textbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec must assign the empty set to $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. (Same holds for Gelfand spectrum of C*-algebras.)

Proving the obstruction

Theorem: Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec has $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

Why? Suppose $F(R) \neq \varnothing$ for some R, so there exists $\mathfrak{q} \in F(R)$.

Commutative subrings $C \subseteq D \subseteq R$ yield "compatible" primes:

$$\begin{aligned} F(R) &\to F(D) \to F(C) \\ \mathfrak{q} &\mapsto \mathfrak{p}_D \mapsto \mathfrak{p}_C = \mathfrak{p}_D \cap C \end{aligned}$$

So $\mathfrak{q}$ yields a subset $\mathfrak{p} = \bigcup \mathfrak{p}_C \subseteq R$ such that, for each commutative subring $C \subseteq R$, we have $\mathfrak{p} \cap C = \mathfrak{p}_C \in \mathrm{Spec}(C)$.

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Def: A subset $\mathfrak{p}$ as above is a prime partial ideal of R, and the set of all prime partial ideals of R is $p\text{-}\mathrm{Spec}(R)$. (Note: $p\text{-}\mathrm{Spec}$ is a functor.)

Colorings from prime partial ideals

Thus: Every F extending Spec maps $F(R) = \varnothing \iff p\text{-}\mathrm{Spec}(R) = \varnothing$

New goal: $p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

What if there were some $\mathfrak{p} \in p\text{-}\mathrm{Spec}(\mathbb{M}_3(\mathbb{C}))$?

Lemma: If $q_1 + q_2 + q_3 = I$ is a sum of orthogonal projections in $\mathbb{M}_3(\mathbb{C})$, then two q_i lie in $\mathfrak{p}$, exactly one lies outside.

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Observation: Any prime partial ideal induces a "010-coloring" on the projections $\mathrm{Proj}(\mathbb{M}_3(\mathbb{C}))$ (those in $\mathfrak{p}$ are "0" and those outside are "1").

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A surprise: This type of coloring has been studied in quantum physics!

The physical motivation was to obstruct hidden-variable theories of quantum mechanics, under the assumption of *non-contextuality*.

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The Kochen-Specker Theorem

Q: (Roughly) Can all observables be simultaneously given definite values, which are independent of the device used to measure them?

- Observables: self-adjoint matrices $\mathbb{M}_n(\mathbb{C})_{sa}$
- Definite values: function $\mathbb{M}_n(\mathbb{C})_{sa} \to \mathbb{R}$
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Def: A function $f\colon \mathrm{Proj}(\mathbb{M}_n(\mathbb{C})) \to \{0, 1\}$ is a Kochen-Specker coloring if, whenever $p_1 + \cdots + p_r = I_n$, we have $f(p_i) = 0$ for all but one i.

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Equivalently: f is "Boolean whenever there is no uncertainty":

1. $f(0) = 0$ and $f(1) = 1$;
2. $f(p \wedge q) = f(p) \wedge f(q)$ and $f(p \vee q) = f(p) \vee f(q)$ if p and q are "commeasurable" (commute).

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Q: (Roughly) "Can all observables be simultaneously given definite values, independent of the device used to measure them?" No!

Kochen-Specker Theorem (1967)

There is no Kochen-Specker coloring of $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$.

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Corollary: For $n \geq 3$, $p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

And as outlined above, this directly proves:

Theorem: Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec has $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

Further observations on spectrum functors

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Question 1: How many rings have this kind of obstruction?

Cor: For F as above and any $\mathbb{C}$-algebra R, $F(M_n(R)) = \varnothing$ for $n \geq 3$.

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Proof: $\mathbb{C} \to R$ yields $\mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(R)$, and thus a function $F(\mathbb{M}_n(R)) \to F(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

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Question 2: What happens for $\mathbb{M}_2(\mathbb{C})$?

Proposition: For the functor $F = p\text{-}\mathrm{Spec}\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ that extends Spec, the set $F(\mathbb{M}_2(\mathbb{C}))$ has cardinality $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$. (It's huge!)

Replacing $\mathbb{C}$ with $\mathbb{Z}$

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What is so special about $\mathbb{C}$? What about other fields? Or universally:

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- As before, reduce to the "universal" functor $F = p\text{-}\mathrm{Spec}$.
- As before, any $\mathfrak{p} \in p\text{-}\mathrm{Spec}(\mathbb{M}_3(\mathbb{Z}))$ yields a Kochen-Specker coloring of the idempotent integer matrices.

Q': Is there an "integer-valued" Kochen-Specker theorem?

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Q': Is there an "integer-valued" Kochen-Specker theorem? Yes!

KS uncolorable vector configurations in the physics literature often use real matrices with irrational entries. So there was real work to be done here.

Functoriality of colorability

Handy obseration: Can think of a KS coloring of idempotents as a morphism $\mathrm{Idpt}(R) \to \{0,1\}$ in a certain category of *partial Boolean algebras* (again, "Boolean when there is no uncertainty).

Lemma: Suppose that there exists a ring homomorphism $R \to S$.

- If $\mathrm{Idpt}(S)$ has a KS coloring, then $\mathrm{Idpt}(R)$ has a KS coloring.
- If $\mathrm{Idpt}(R)$ is KS uncolorable, then $\mathrm{Idpt}(S)$ is KS uncolorable.

Follows by composing partial Boolean algebra morphisms:

$$\mathrm{Idpt}(R) \to \mathrm{Idpt}(S) \to \{0,1\}$$

Similar result holds for $\mathrm{Proj}(\mathbb{M}_n(R)) = \{\text{symmetric idempotents}\}$ and $\mathrm{Proj}(\mathbb{M}_n(S))$ given a **cRing** morphism $R \to S$.

Colorability of projections over various rings

Initial idea: Would suffice for the *partial ring* $\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}}$ of symmetric matrices to have empty partial spectrum, for which it would suffice to show $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/N]))$ uncolorable for two relatively prime values of N.

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Theorem (Ben-Zvi, Ma, R. 2017):

ring R	prime p	$\mathrm{Proj}(\mathbb{M}_3(R))$	$p\text{-}\mathrm{Spec}(\mathbb{M}_3(R)_{\mathrm{sym}})$
$\mathbb{Z}[1/30]$		uncolorable	empty
$\mathbb{F}_p$	$p \geq 5$	uncolorable	empty
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Idea of proof: J. Bub (1996), using observation of Schütte, produced an uncolorable set of integer vectors v such that all $\|v\|^2$ divide 30. Analyze $\mathbb{F}_p$ for $p = 2, 3, 5$ as special cases. Functoriality does the rest.

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- But some orthogonality relations are not preserved, so the proof of uncolorability does not lift.
- Nevertheless, a case-splitting argument shows that $\mathcal{S}$ is uncolorable!

Theorem (Ben-Zvi, Ma, R. 2017) There is no Kochen-Specker coloring of $\mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ for any $n \geq 3$.

Finishing the spectrum obstruction

Theorem (Ben-Zvi, Ma, R.): There is no Kochen-Specker coloring of $\mathsf{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ for any $n \geq 3$.

As mentioned before, this directly implies:

Theorem: Given any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ extending Spec as before, we have $F(\mathbb{M}_n(\mathbb{Z})) = \varnothing$ for any integer $n \geq 3$.

Even better:

Corollary: Let R be any ring, and let $n \geq 3$.

- There is no KS coloring of the idempotents of $\mathbb{M}_n(R)$.
- $p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) = \varnothing$.
- We also get $F(\mathbb{M}_n(R)) = \varnothing$ for any F as above.

Topology without points

We can't find a spectrum built out of points. But there are "point-free" ways to do topology!

$$
\begin{array}{ccc}
\text{topological spaces} & \longrightarrow & \text{sets} \\
\downarrow & & \\
\text{"pointless" spaces} & & \\
\downarrow & & \\
\text{categories of sheaves} & &
\end{array}
$$

Perhaps points are the real problem, so that one of these more exotic approaches could bypass the obstruction?

Avoiding the obstruction with pointless topology?

Pointless topology treats spaces and sheaves purely in terms of their lattices of open subsets, called locales, forming a category **Loc**

The category **Loc** of *locales* has:

- *Objects*: upper-complete lattices satisfying $a \wedge (\bigvee b_i) = \bigvee (a \wedge b_i)$.
- *Morphisms*: $f\colon L_1 \to L_2$ is a function $f^*\colon L_2 \to L_1$ that preserves finite meets and arbitrary joins.

Can we avoid the obstruction by "throwing away points?"

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Can we avoid the obstruction by "throwing away points?" No!

Theorem (van den Berg & Heunen, 2012): Any functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ whose restriction to $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec (considered as a locale) must assign the trivial locale to $\mathbb{M}_n(R)$ for any ring R with $\mathbb{C} \subseteq R$ and any $n \geq 3$. (The same holds for C*-algebras.)

Cor: [Ben-Zvi, Ma, R.] This obstruction still holds with any ring R.

Further "point-free" obstructions

There are several other routes in point-free topology that one might hope to use to escape the obstructions, but which we now know cannot work:

- Viewing $\mathrm{Spec}(A)$ as a quantale (van den Berg & Heunen)
- Replacing $\mathrm{Spec}(A)$ with its topos of sheaves (van den Berg & Heunen)
- Upgrading the structure sheaf to a ring object in a category (R. 2014)
- Extending the "big Zariski topology" on $\mathbf{cRing}^{\mathrm{op}}$ to a compatible Grothendieck topology on $\mathbf{Ring}^{\mathrm{op}}$ (R. 2014)

The first two above still follow from Kochen-Specker.

The last two already fail for $\mathbb{M}_2(k)$ with any ring k. Rather than giving details, I will discuss the inspiration: a simple diagram in **Ring** that's rather strange when interpreted geometrically.

Pullbacks and preimages

Categorical trick in geometry: Suppose that $f\colon X \to Y$ is a function (of sets, spaces,...) and let $y \in Y$ be a point.

We get the following commutative diagram:

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d, hook] & \{y\} \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

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This diagram is a pullback in our favorite category of spaces: the preimage $f^{-}(y)$ is the universal object making the diagram commute.

A strange diagram of "spaces"

Fact: The pushout of $\mathbb{C} \xleftarrow{\pi_i} \mathbb{C} \times \mathbb{C} \xrightarrow{d} \mathbb{M}_2(\mathbb{C})$ in **Ring** is zero, where π_i projects to the ith component and d embeds diagonally.

Let's draw the opposite spectral diagrams, writing $\mathrm{Spec}(\mathbb{M}_2(\mathbb{C}))$ for the imaginary noncommutative space:

$$\begin{tikzcd} \mathrm{Spec}(0) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d] \\ \mathrm{Spec}(\mathbb{M}_2(\mathbb{C})) \arrow[r] & \mathrm{Spec}(\mathbb{C}^2) \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ ?? \arrow[r] & \{\star, \bullet\} \end{tikzcd}$$

Opposite diagrams are pullbacks, so $\varnothing$ is the "preimage" of either point.

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Colorful interpretation: The "quantum space" $\mathrm{Spec}(\mathbb{M}_2(\mathbb{C}))$ maps to the two-point space, without hitting either point!!

This is far from a theorem, but the sheaf obstructions make it precise.

A toy problem

Is there is really no noncommutative spectrum functor after all? I'm not ready to belive so. *We simply need some creativity in how we interpret* Spec*!*

To illustrate, here's an example of a successful noncommutative spectrum.

I'll try to convince you that for a certain class of commutative C*-algebras, the Boolean algebra of projections ($p = p^2 = p^*$) is just as good as its spectrum.

Then we will see how to extend this complete invariant to a kind of "noncommutative Boolean algebra."

C*-algebras with many projections

Recall that $\mathrm{Proj}(A) = \{p \in A \mid p = p^2 = p^*\}$ is partially ordered by $p \leq q \iff p = pq$, has orthocomplement $p^\perp = 1 - p$.

W-algebras* (or *von Neumann algebras*) are famously rich in projections, but there is a larger class, defined algebraically, with similar properties.

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W-algebras* (or *von Neumann algebras*) are famously rich in projections, but there is a larger class, defined algebraically, with similar properties.

Definition: (Kaplansky 1951) An AW*-algebra is a C*-algebra A that satisfies the following equivalent conditions:

- Every maximal commutative $*$-subalgebra is the closure of the linear span of its projections, and $\text{Proj}(A)$ is a complete lattice;
- The left annihilator of any subset of A is of the form Ap for some $p \in \text{Proj}(A)$.

For these algebras, $\text{Proj}(A)$ is complete *orthomodular* lattice, and a complete Boolean algebra when A is commutative.

Category of AW*-algebras

Def: AWstar is the category of AW*-algebras with $*$-homomorphisms that restrict to complete lattice morphisms on projections.

Fact: A commutative C*-algebra A is an AW*-algebra iff $\mathrm{Spec}(A)$ is *Stonean*: closure of each open set is (cl)open.

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Thus Proj(A) is a complete invariant for commutative AW*-algebras!

For noncommutative A, the OML structure of Proj(A) is not a complete invariant: there are anti-isomorphic but not $*$-isomorphic algebras with isomorphic Proj(A).

So we would like a "more noncommutative" invariant than Proj(A)...

In search of "noncommutative Boolean algebras"

Two perspectives on the problem:

OML viewpoint: How can we enrich $\mathrm{Proj}(A)$ to form a complete invariant for AW*-algebras?

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Spectral viewpoint: If we "skip the space," can we find "quantum complete Boolean algebras" to act as a spectrum for noncommutative AW*-algebras?

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Answer (Heunen, R. 2014): Yes!!

How can we “quantize” Boolean algebras?

Say $B = \mathrm{Proj}(A)$ for a commutative AW*-algebra A with $p, q \in B$:

- $p \wedge q = pq$;
- $p \vee q = p + q - pq$;
- “symmetric difference” $p \Delta q = (p \vee q) - (p \wedge q) = p + q - 2pq$ gives an abelian group structure (“addition” operation in the Boolean ring).

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Idea: Think of the noncommutative product $(1 - 2p)(1 - 2q)$ as a "quantum symmetric difference," even though it need not have the form $1 - 2p'$ for any projection $p' \in A$.

Active lattices

Definition (roughly): An active lattice consists of the following data:

- A complete *orthomodular lattice* P
- A *group* G with an injection $P \hookrightarrow G$ onto a generating set of "reflections" (plus an embedding $G \hookrightarrow \mathcal{A}(P)$ in a "partial algebra")
- With an *action* of G on P

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For each AW*-algebra A, we get an active lattice $\mathrm{AProj}(A)$ with:

- Lattice $P = \mathsf{Proj}(A)$
- Group of *symmetries* $G = \mathrm{Sym}(A) = \langle 1 - 2p \mid p \in \mathsf{Proj}(A) \rangle$
- Action of G on P, where $s = 1 - 2p$ acts by conjugation in A: $s(q) = sqs^{-1} = sqs$.

This gives us a functor $\mathrm{AProj} \colon$ **AWstar** $\to$ **Active**

Active lattices determine AW*-algebras

Theorem (Heunen and R., 2014): $\mathrm{AProj}\colon$ **AWstar** $\to$ **Active** is a full and faithful embedding, i.e., there is a bijection between **AWstar** morphisms $A \to B$ and active lattice morphisms $\mathrm{AProj}(A) \to \mathrm{AProj}(B)$.

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Some ideas behind the proof:

- Prove a "$\mathrm{Sym}(A)$-equivariant" version of Dye's theorem to extend $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$ to a Jordan $*$-homomorphism $A \to B$ if there are no type I_2 summands.

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- Use multiplicativity of the map on $\mathrm{Sym}(A)$ to show the Jordan morphism is multiplicative on $\mathsf{Proj}(A)$. Implies multiplicativity for all of A since it is the closed linear span of $\mathsf{Proj}(A)$!
- Treat the type I_2 case $A = \mathbb{M}_2(C)$ with algebraic techniques, inspecting $\mathrm{Sym}(A)$-action arising from a set of matrix units.

Coordinatizing active lattices?

Theorem: $\mathrm{AProj}\colon$ **AWstar** $\to$ **Active** is a full and faithful embedding.

Unfortunately, the result does not show us how to directly recover an algebra from its active lattice.

In fact, it seems likely that there are active lattices that do not arise from an AW*-algebra, but we don't have an explicit example.

This leads to a *coordinatization problem* for active lattices:

Question: Which active lattices L satisfy $L \cong \mathrm{AProj}(A)$ for some AW*-algebra A? (Equivalently, what is the essential image of AProj?)

In search of "noncommutative sets"

The obstructions suggest to me that we don't yet understand discrete noncommutative spaces.

If we strip a "commutative" space of its geometry, we are left with its underlying set. But if we strip a noncommutative space of its geometry, then what *noncommutative* discrete structure remains?

$$\begin{tikzcd}
\{\text{commutative algebras}\} \arrow[r, "\mathsf{Spec}"] \arrow[d, hook] & \{\text{spaces}\} \arrow[r, "U"] & \{\text{sets}\} \arrow[d, hook] \\
\{\text{noncommutative algebras}\} \arrow[rr] & & \{???\}
\end{tikzcd}$$

What category should fill in blank above?

Noncommutative sets via function algebras

Observation 1: The assignment

$$X \mapsto \ell^\infty(X) = \{\text{bounded discrete functions } X \to \mathbb{C}\}$$

is an equivalence between $\mathbf{Set}^{\mathrm{op}}$ and a full subcategory of **cAWstar**.
If we are serious about noncommutative geometry, we might expect:

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Q: Does $C(X) \mapsto \ell^\infty(X)$ extend to a functor $F\colon \mathbf{Cstar} \to \mathbf{Alg}$, with natural embeddings $A \to F(A)$, for some category of $*$-algebras $\mathbf{Alg}$?

Discretization of C*-algebras

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Necessary condition: Applying such a functor F to an arbitrary commutative subalgebra $C(X) \cong C \subseteq A$ induces a commuting square

$$\begin{tikzcd} A \arrow[r, "\phi"] & M = F(A) \\ C(X) \arrow[u, hook] \arrow[r, hook] & \ell^\infty(X) = F(C(X)) \arrow[u, dashed, "\phi_C"'] \end{tikzcd}$$

where ϕ_C is a morphism in $\mathbf{Alg}$.

Def: A morphism $\phi\colon A \to M$ with factorizations ϕ_C as above is called a discretization of A (relative to the category $\mathbf{Alg}$).

Discretization of C*-algebras

$$\begin{tikzcd} A \arrow[r, "\phi"] & M = F(A) \\ C(X) \arrow[u, hook] \arrow[r, hook] & \ell^\infty(X) \arrow[u, dashed, "\phi_C"'] \end{tikzcd}$$

Theorems [Heunen & R., 2017]:

- Every C*-algebra embeds into a non-functorial discretization.
- There is a "profinite completion" functor that discretizes all algebras embedding in $\mathbb{M}_n(C(X))$.
- **Alg** above cannot be the category of AW*-algebras, otherwise every discretization functor gives $F(B(H)) = 0$ for infinite-dimensional H.

But the general question remains open...

What are noncommutative sets "made of"?

If we are replacing sets with algebras of functions, we are still not actually "seeing" our noncommutative sets:

$$\begin{tikzcd}
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What might these objects "look like" in practice? Here is a proposal in the setting of algebras over an arbitrary field k.

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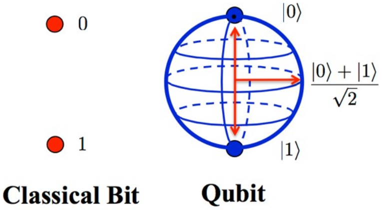
(Apologies in advance for the onslaught of algebraic geometry...)

From sets to "quantum sets"

A cue from the superposition principle: If X is our set of "states," we should also allow linear combinations of states: $X \rightsquigarrow kX = \mathrm{Span}(X)$



http://qoqms.phys.strath.ac.uk/research_qc.html

To stick to our "ground rules," we need to a way to recover X from kX as a kind of distinguished basis. . .

“Quantum sets” for algebras over a field

This vector space $Q = kX$ carries the structure of a coalgebra:

- Comultiplication $\Delta\colon Q \to Q \otimes Q$ given by $x \mapsto x \otimes x$
- Counit $\eta\colon Q \to k$ given by $x \mapsto 1$

Coalgebra maps correspond to set maps: $\mathbf{Set}(X, Y) \cong \mathbf{Coalg}(kX, kY)$.
Gives a full and faithful embedding $\mathbf{Set} \hookrightarrow \mathbf{Coalg}$.

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Therefore: We view a coalgebra (Q, Δ, η) as a “quantum set” (over k). Its *algebra of observables* is the dual algebra $\mathrm{Obs}(Q) = Q^*$.

History: Coalgebras were considered as “discrete objects” by Takeuchi (1974), and in the noncommutative context by Kontsevich-Soibelman (*noncommutative thin schemes*) and Le Bruyn.

Coalgebras in commutative geometry

Every scheme over k has an "underlying coalgebra."

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Observation: A scheme S finite over k is of the form $S \cong \mathrm{Spec}(B)$ for f.d. algebra B. The functor $S \mapsto \Gamma(S, \mathcal{O}_S)^* \cong B^*$ is an equivalence

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Def: For a k-scheme X, the coalgebra of distributions is

$$\mathrm{Dist}(X) = \varinjlim \Gamma(S, \mathcal{O}_S)^*,$$

where S ranges over the closed subschemes of X that are finite over k. This gives a functor $\mathrm{Dist}\colon \mathbf{Sch}_k \to \mathbf{Coalg}$.

Local nature of distributions

It's best to restrict to the case where X is (locally) of finite type over k.

Distributions supported at a closed point x of such X have been defined in the literature on algebraic groups:

$$\mathrm{Dist}(X, x) = \varinjlim(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)^*.$$

This is dual to the completion: $\mathrm{Obs}(\mathrm{Dist}(X, x)) \cong \widehat{\mathcal{O}}_{X,x}$.

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Theorem: Suppose X is of finite type over k, and let X_0 be its set of closed points.

1. There is an isomorphism of coalgebras $\mathrm{Dist}(X) \cong \bigoplus_{x \in X_0} \mathrm{Dist}(X, x)$
2. If $k = \overline{k}$, then $\mathrm{Dist}(X)$ has a subcoalgebra isomorphic to kX_0.

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Moral: $\mathrm{Dist}(X)$ linearizes the set of closed points, and includes the formal neighborhood of each point.

Distributions in the affine case

Suppose $X = \mathrm{Spec}(A)$ with A a finitely generated commutative k-algebra. Distributions given by the Sweedler dual coalgebra

$$\mathrm{Dist}(\mathrm{Spec}(A)) \cong A^\circ := \varinjlim (A/I)^*$$

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Thesis: For "nice" finitely generated algebras over k, then the functor $A \mapsto A^\circ$ is a suitable candidate for a quantized maximal spectrum.

Examples: "Nice" means "many f.d. representations"

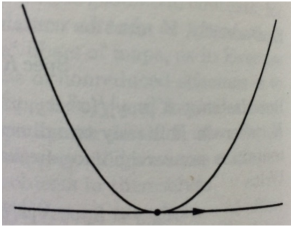
- Finitely generated, noetherian algebras satisfying a polynomial identity (including algebras which have a "large," finitely generated center)
- In particular, lots of "quantum algebras" at roots of unity, such as (algebraic) quantum groups $\mathcal{O}_q(G)$ and quantum planes $k_q[x, y] = k\langle x, y \mid yx = qxy \rangle$ where $q^n = 1$

Glimpses of some quantum spectra

Ex: The ring of dual numbers $A = k[t]/(t^2)$ has $A^\circ = kx \oplus k\varepsilon$ with

- $\Delta(x) = x \otimes x$ and $\Delta(\varepsilon) = x \otimes \varepsilon + \varepsilon \otimes x$
- $\eta(x) = 1$ and $\eta(\varepsilon) = 0$

Here x is like a point and ε is like an "infinitesimal tangent vector."



Eisenbud & Harris, *The Geometry of Schemes*

This is closer to the geometers' picture than $\mathrm{Spec}(A) = \mathrm{Max}(A) = \{pt\}$!

Glimpses of some quantum spectra

Ex: The qubit over k is the matrix coalgebra $\mathbb{M}^2 = (\mathbb{M}_2(k))^\circ$, which has k-basis $\{E^{11}, E^{12}, E^{21}, E^{22}\}$ and structure given by

- $\Delta(E^{ij}) = E^{i1} \otimes E^{1j} + E^{i2} \otimes E^{2j}$
- $\eta(E^{ij}) = \delta_{ij}$

There are no "points," but there is a morphism the classical bit:

$$\mathbb{M}^2 \to \mathrm{Dist}(\mathrm{Spec}(k^2)) = k\{0,1\},$$

given by sending the E^{ii} to the two points and $E^{12}, E^{21} \mapsto 0$.

(But we have *many* morphisms to the bit, one for every basis of k^2!)

This lets us "see" the dual maps from the qubit to its "classical perspectives"

Glimpses of some quantum spectra

Similarly, each matrix algebra has a dual matrix coalgebra $\mathbb{M}^n = (\mathbb{M}_n)^*$.

Ex: Suppose X is a scheme of finite type over k, with coordinate ring $k[X]$. Then $A = \mathbb{M}_n(k[X])$ has dual coalgebra

$$A^\circ \cong \mathbb{M}^n \otimes \mathrm{Dist}(X).$$

Thus we are seeing both quantum and spatial information in the same spectral object!

Note that the Morita equivalent algebras $k[X]$ and $\mathbb{M}_n(k[X])$ seem to have a kind of "Morita equivalence" between their spectral coalgebras.

More questions than answers

Work currently in progress:

- Developing strategies to compute A°
- Describing the underlying coalgebra of a "noncommutative Proj(S)," still assuming that S has "many" f.d. representations.

Several questions that eventually need to be addressed:

- Doing geometry with coalgebras: how to "topologize" them and define sheaves?
- What is a "quantum scheme of finite type over k" in this context?
- Could this approach extend to algebras that are not residually finite?
- Could it even extend to rings that are not algebras over a field?

Thank you! (And some references)

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