# AF C*-algebras, Many-valued Logics, and Effect Algebras

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# Many-Valued Logics and their algebras: every 30 years

- **1920's**: Polish school: **Łukasiewicz** , Tarski , Post.
- **1950's**: R. McNaughton, **C.C. Chang (MV Algebras)**
- **1980's**: D. Mundici, et.al.
  - MV-Algebras: rich algebraic, topological, & geometric theory.
  - Closely related to (AF) C*-algebras (Bratteli, Elliott).
  - Deep connections with analysis, alg. geometry & topology.
- **2010–:**
  - Sheaf Representation: Dubuc/Poveda (2010), Gehrke (2014).
  - Toposes, Morita Equiv. & MV-algebras (Caramello: 2014–),
  - Łukasiewicz $\mu$-calculus, M. Mio & A. Simpson (2013)
  - Coordinatization (Lawson-Scott, Wehrung, Mundici ) (2015-) (via Boolean Inverse Monoids)

# What are MV Algebras? (C.C. Chang, 1950's)

MV algebras are structures $\mathcal{M} = \langle M, \oplus, \neg, 0\rangle$ satisfying:

- $\langle M, \oplus, 0\rangle$ is a commutative monoid.
- $\neg$ is an involution: $\neg\neg x = x$, for all $x \in M$.
- $1 := \neg 0$ is absorbing: $x \oplus 1 = 1$ , for all $x \in M$.
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

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Example: a Boolean algebra $\mathcal{B} = (B, \vee, \overline{(\ )}, 0)$, where we define $x \oplus y := x \vee y$ and $\neg x = \overline{x}$. The last equation says: $x \vee y = y \vee x$

# Fundamental Example of an MV Algebra: $[0,1]$

For $x, y \in [0,1]$, define:

1. $\neg x = 1 - x$
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Similarly consider the same operations on:

- $\mathbb{Q} \cap [0, 1]$ and $\mathbb{Q}_{\text{dyad}} \cap [0, 1]$.
- Finite MV algebras $\mathcal{M}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ (subalgebras of [0,1]). Note: $\mathcal{M}_2 = \{0, 1\}$.

# Example 2: Lattice-Ordered Abelian Groups

- Let $\langle G, +, -, 0, \leqslant \rangle$ be a partially ordered abelian group, i.e. an abelian group with translation invariant partial order.
- If $G$ is lattice-ordered, call $G$ an *$\ell$-group*, $G^+$ its positive cone.

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- If $G$ is an $\ell$-group with order unit $u$, define the $G$-interval

$$[0, u]_G = \{g \in G \mid 0 \leqslant g \leqslant u\} \quad \text{(just a poset)}$$

$G$-**Chain**: totally ordered $G$-interval $[0, u]$.

# $G$-interval MV algebras

$G$ an $\ell$-group. $\Gamma(G,u) = ([0,u]_G, \oplus, \otimes, {}^*, 0, 1)$ is an MV algebra:

$$
\begin{aligned}
x \oplus y \quad &:= \quad u \wedge (x+y) \\
x^* \quad &:= \quad u - x \\
x \otimes y \quad &:= \quad (x^* \oplus y^*)^* \\
0 := 0_G \quad &\text{and} \quad 1 := u
\end{aligned}
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Let $\mathcal{MV}$ = the category of MV-algebras and MV-morphisms.
$\ell\mathcal{G}_u$ = the category of $\ell$-groups and structure preserving homs.

### Theorem (Mundici I, 1986)

$\Gamma$ *induces an equivalence of categories* $\ell\mathcal{G}_u \cong \mathcal{MV}$ : $\quad G \mapsto [0,u]_G$

$\therefore$ For each MV algebra $A$, there exists $\ell$-group G with order unit $u$, unique up to iso, s.t. $A \cong [0,u]_G$ and $|G| \leqslant max(\aleph_0, |A|)$.

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**Corollary (Existence of Free MV-Algebras)**

*The free MV algebra* $\mathcal{F}_\kappa$ *on* $\kappa$ *free generators is the smallest MV-algebra of functions* $[0, 1]^\kappa \to [0, 1]$ *containing all projections (as generators) and closed under the pointwise operations.*

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Theorem (McNaughton, 1950: earlier than Chang!)

*The free MV algebra* $\mathcal{F}_n$ *is exactly the algebra of* McNaughton Functions*: continuous, piecewise (affine-)linear polynomial functions (in n vbls, with integer coefficients):* $[0,1]^n \to [0,1]$.

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*Corollary: an MV equation holds in* $[0,1]$ *iff it holds in* $[0,1] \cap \mathbb{Q}$

# Some Algebra of MV algebras

Analogs of all standard algebra in MV form:

1. Usual theory of ideals/kernels/congruence/HSP theorems, etc.
2. Direct & sub-direct products, tensor products, ultra products, limits, colimits.
3. Radical Ideals, spectral spaces, etc.

# Some Geometry of MV-Algebras

Mundici & colleagues (Marra, Cabrer, Spada, et.al.) have shown deep connections to algebraic geometry and topology.

1. If $P \subseteq \mathbb{R}^n$, the convex hull $conv(P) = \{\sum_i r_i v_i \mid v_i \in P, r_i \in \mathbb{R}^+, \sum_i r_i = 1\}$.
2. $P$ is called:
   - 2.1 *convex* iff $P = conv(P)$.
   - 2.2 *a polytope* iff $P = conv(F)$, $F \subseteq \mathbb{R}^n$ finite.
   - 2.3 *a rational polytope* iff it's a polytope and $F \subseteq \mathbb{Q}^n$.
   - 2.4 *a (compact) polyhedron* iff it's a union of finitely many polytopes in $\mathbb{R}^n$.
   - 2.5 *a rational polyhedron* iff it's a union of finitely many rational polytopes. (These are subsets of $\mathbb{R}^n$ definable by MV-terms.)

What about maps between rational polyhedra?

## Some Geometry of MV-Algebras

- For $P \subseteq \mathbb{R}^n$, $f : P \to \mathbb{R}$ is a $\mathbb{Z}$-*map* if it's a McNaughton Function into $\mathbb{R}$ (instead of $[0,1]$)). Ditto, if $P, Q \subseteq \mathbb{R}^n$, $P \xrightarrow{f} Q$ is a $\mathbb{Z}$-map if its components are. (These are the continuous transformations of polyhedra definable by tuples of MV terms!)

**Theorem (Marra& Spada, APAL, 2012)**

*The category of finitely presented MV-algebras and homs is equivalent to the opposite of the category of rational polyhedra and* $\mathbb{Z}$*-maps:* $MV_{fp} \cong Poly_{\mathbb{Q}}^{op}$

There is a strong analogy with a remarkable independent series of papers by the algebraic topologist W. M. Beynon (1974-77) on related topological dualities for $\ell$-groups.

# Typical Beynon Theorem

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*The full subcategory of the category of finitely generated lattice-ordered Abelian groups consisting of projective lattice-ordered Abelian groups is equivalent to the dual of the category whose objects are rational Euclidean closed polyhedral cones, and whose morphisms are piecewise homogeneous linear maps with integer coefficients.*

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1. W. M. Beynon, Combinatorial aspects of piecewise linear maps, J. London Math. Soc. (2) (1974), 719-727.
2. W. M. Beynon: Duality theorems for finitely generated vector lattices, Proc. London Math. Soc. (3) 31 (1975), 114-128.
3. W. M. Beynon, Applications of Duality in the theory of finitely generated lattice-ordered abelian groups, Can.J. Math, 1977

# From Marra & Mundici, 2003: MV- vs $\ell-$

| MV | $\ell$ |
| --- | --- |
| **Chang's Theorem** (1959) [22] | **Weinberg's Theorem** (1963) [102] |
| The variety of MV algebras is generated by $[0,1] \cap \mathbb{Q}$. (Corollary 3.3.) | The variety of $\ell$-groups is generated by $\mathbb{Z}$. (Corollary 5.5.) |
| **McNaughton's Theorem** (1951) [67] | **Beynon's Theorem, I** (1974) [13] |
| Every McNaughton function of $n$ variables belongs to $\mathcal{M}_n$. (Theorem 8.1.) | Every $\ell$-function of $n$ variables belongs to $\mathcal{A}_n$. (Subsection 4.4, *passim.*) |
| **Free representation** (1951-59) [22, 67] | **Free representation** (1963-74) [102, 13] |
| $\mathcal{M}_n$ is the free MV algebra over $n$ free generators, i.e. projection functions. (Subsection 3.1, *passim.*) | $\mathcal{A}_n$ is the free $\ell$-group over $n$ free generators, i.e. projection functions. (Subsection 4.4, *passim.*) |
| **MV Nullstellensatz** (1959) [104, 22] | **$\ell$-Nullstellensatz** (1975) [14] |
| TFAE: 1. $A$ is fin. gen. semisimple. 2. $\mathbb{I}(\mathbb{V}(J)) = J$ if $A \cong \mathcal{M}_n/J$. (Theorem 3.2.) | TFAE: 1. $G$ is fin. gen. Archimedean. 2. $\mathbb{I}(\mathbb{V}(\mathfrak{o})) = \mathfrak{o}$ if $G \cong \mathcal{A}_n/\mathfrak{o}$. (Subsection 4.4, *passim.*) |
| **Wójcicki's Theorem** (1973) [103] | **Baker's Theorem** (1968) [9] |
| Every finitely presented MV algebra is semisimple. (Theorem 3.4.) | Every finitely presented $\ell$-group is Archimedean. (Subsection 4.4, *passim.*) |

# Effect Algebras: quantum effects

Let $H$ be a complex Hilbert space of a quantum system $\mathcal{S}$. In the theory of quantum measurement, *effects* represent certain kinds of measurements.

# Effect Algebras (of Quantum Effects)

Foulis & Bennet (1994): an abstraction of algebraic structure of (*quantum effects*).

An Effect Algebra is a *partial* algebra $\langle E; 0, 1, \widetilde{\oplus} \rangle$ satisfying: $\forall a, b, c \in E$ (Using Kleene directed equality $\succeq$ )

$$\left.\begin{array}{l} 1.\ a \mathbin{\widetilde{\oplus}} b \succeq b \mathbin{\widetilde{\oplus}} a. \\ 2.\ \text{If } a \mathbin{\widetilde{\oplus}} b \downarrow \text{ then } (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \succeq a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \\ 3.\ 0 \mathbin{\widetilde{\oplus}} a \downarrow \text{ and } 0 \mathbin{\widetilde{\oplus}} a = a \end{array}\right\} \text{PCM}$$

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3. $0 \,\widetilde{\oplus}\, a \downarrow$ and $0 \,\widetilde{\oplus}\, a = a$

$$\left.\begin{array}{l} \text{4. } \forall_{a \in E}\ \exists!_{a' \in E} \text{ such that } a \,\widetilde{\oplus}\, a' = 1. \\ \text{5. } a \,\widetilde{\oplus}\, 1 \downarrow \text{ implies } a = 0. \end{array}\right\} \text{Orthocomp. \& 0-1 Law}$$

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Eastern European School: Dvurecenskij, Jenca, Pulmannova, ...

Nijmegen: Bart Jacobs and his school (Effectus Theory)

# Posetal Examples of Effect Algebras

- **Boolean Algebras**: Let $\mathcal{B} = (B, \wedge, \vee, \overline{(\ )}, 0, 1)$ be a Boolean algebra. For $x, y \in B$, define $x' = \overline{x}$ and

$$x \mathbin{\widetilde{\oplus}} y = \begin{cases} x \vee y & \text{if } x \wedge y = 0 \\ \uparrow & \text{else} \end{cases}$$

- **Orthomodular Lattices**:

  Bounded lattices $\mathcal{L}$ with an operation $(\ )^{\perp} : \mathcal{L} \to \mathcal{L}$ satisfying:

  1. $x \leqslant y$ implies $y^{\perp} \leqslant x^{\perp}$.
  2. $x^{\perp\perp} = x$
  3. $x \vee x^{\perp} = 1$
  4. $x \leqslant y$ implies $x \vee (x^{\perp} \wedge y) = y$.

  For $x, y \in \mathcal{L}$, define $x \mathbin{\widetilde{\oplus}} y = x \vee y$ , if $x \leqslant y^{\perp}$; undefined else.

# More Examples of Effect Algebras

- **Interval Effect Algebras:** Let $(G, G^+, u)$ be an ordered abelian group with order unit $u$. Consider

$$G^+[0,u] = \{a \in G \mid 0 \leqslant a \leqslant u\}.$$

For $a, b \in G^+[0,u]$, set $a \mathbin{\widetilde{\oplus}} b := a + b$ if $a + b \leqslant u$; otherwise undefined. Also set $a' := u - a$. e.g. [0,1] as a partial algebra.

- E.g.: **Standard Effect Algebra** $\mathcal{E}(H)$ of a quantum system.

$G := \mathcal{B}_{sa}(H)$, (self-adj) bnded linear operators on $H$,
$G^+ :=$ the positive operators. Let $\mathbb{O} =$ constant zero ,
$\mathbb{I} =$ identity. $\mathcal{E}(H) := G^+[\mathbb{O}, \mathbb{I}]$.

  - $A \in \mathcal{E}(H)$ represent unsharp (fuzzy) measurements
  - Projections $\mathcal{P}(H) \subset \mathcal{E}(H)$ represent sharp measurements

# Effect Algebras of Predicates (B. Jacobs, 2012-2015)

**Predicates in $\mathcal{C}$**: let $\mathcal{C}$ be a category with "good" finite coprods and terminal object 1. Define $\mathit{Pred}_{\mathcal{C}}(X) := \mathcal{C}(X, 1+1)$ .

## Proposition (Jacobs)

*If $\mathcal{C}$ satisfies reasonable p.b. conditions on $+$, $\mathit{Pred}_{\mathcal{C}}(X)$, $X \in \mathcal{C}$, forms an effect algebra. (Such a $\mathcal{C}$ is called an "effectus"). Get an indexed category $\mathit{Pred} : \mathcal{C}^{op} \to \mathit{Eff}$ .*

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**Examples:**

- Predicates on Kleisli categories of various distribution monads (e.g. Discrete, Continuous, etc.)
- Predicates on various concrete categories: **Set**, $\mathbf{SemiRing}^{op}$, $\mathbf{Ring}^{op}$, $\mathbf{DL}^{op}$, $(C^*_{PU})^{op}$, ....

# Effect Algebras: Additional Properties

Let $E$ be an effect algebra. Let $a, b, c \in E$. Denote $a'$ by $a^\perp$ or $a^*$.

1. Partial Order: $a \leqslant b$ iff for some $c$, $a \widetilde{\oplus} c = b$.
2. $0 \leqslant a \leqslant 1$, $\forall a \in E$.
3. $a^{\perp\perp} = a$.
4. $0^\perp = 1$ and $1^\perp = 0$.
5. $a \leqslant b$ implies $b^\perp \leqslant a^\perp$
6. (Cancellation) $a \widetilde{\oplus} c_1 = a \widetilde{\oplus} c_2$ implies $c_1 = c_2$.
7. (Positivity / conical) $a \widetilde{\oplus} b = 0$ implies $a = b = 0$

# Effect Algebras: Morphisms

Effect Algebras form a category **Eff**.

A function $f : \mathcal{A} \to \mathcal{B}$ is a *morphism* if:

1. $f$ preserves 1.
2. If $a \mathbin{\widetilde{\oplus}} b$ is defined, then also $f(a) \mathbin{\widetilde{\oplus}} f(b)$ is defined, and $f(a \mathbin{\widetilde{\oplus}} b) = f(a) \mathbin{\widetilde{\oplus}} f(b)$.

- Such maps automatically preserve 0 and $(\ )^{\perp}$.

## MV versus Effect Algebras I

- An effect algebra satisfies RDP (*Riesz Decomposition Property*) iff

$$\begin{array}{ccc} a \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_n & \Rightarrow & \exists a_1, \ldots, a_n \ s.t. \\ a = a_1 \oplus a_2 \oplus \cdots \oplus a_n & \text{with} & a_i \leqslant b_i, i \leqslant n \end{array}$$

**Proposition (Bennett & Foulis, 1985)**

*An effect algebra is an MV-effect algebra iff it is lattice ordered and has RDP.*

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But morphisms are different!

$|\, Hom_{\mathbf{MV}}([0,1],[0,1])\,| = 1$, $|\, Hom_{\mathbf{MV}}([0,1]^2,[0,1])\,| = 2$

$|\, Hom_{\mathbf{EA}}([0,1],[0,1])\,| = 1$, $|\, Hom_{\mathbf{EA}}([0,1]^2,[0,1])\,| = 2^{\aleph_0}$

# Universal Groups of Effect Algebras: Mundici Anew

- If $(E, +, 0, 1)$ is an effect algebra with RDP, there is a universal monoid $E \hookrightarrow M_E$. This (total) monoid $M_E$ is abelian, cancellative, satisfies a universal property.
- Every cancellative abelian monoid $\mathcal{M}$ has a Grothendieck group $\mathcal{M} \hookrightarrow G_{\mathcal{M}}$ satisfying a universal property (essentially the INT construction yielding $\mathbb{Z}$ from $\mathbb{N}$).

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**Theorem (Ravindran,1996)**

*Let $E$ be an effect algebra with RDP and $E \xrightarrow{\gamma} G_E$ its universal (Groth.) group. Then $G_E$ satisfies:*

1. *(i) $G_E$ is partially ordered,*
2. *(ii) $u = \gamma(1)$ is an order unit and (iii) $\gamma : E \cong [0, u]_{G_E}$.*
3. *If $E$ is an MV-algebra, then $G_E$ is an $\ell$-group (cf. Mundici).*

# Ravindran's Theorem–some details

Essentially an independent approach to Mundici's theorem, via effect algebras. Technique goes back to R. Baer (1949).

**Theorem**

*Let $E$ be an effect algebra satisfying RDP. Then it is an interval effect algebra, with universal group an interpolation group.*

Let $E^+$ be the free (word) semigroup on $E$. Take the smallest congruence $\sim$ such that the word $(a, b) \sim (a \oplus b)$, whenever $(a \oplus b) \downarrow$. i.e. Take the congruence relation on words generated as: $(a_1, a_2, \cdots a_n) \sim (a_1, a_2, \cdots, a_{k-1}, a_k \oplus a_{k+1}, a_{k+2}, \cdots, a_n)$, whenever $a_k \oplus a_{k+1} \downarrow$. Then $E^+/\sim$ is a positive abelian monoid (get commutativity for free!) with RDP. Its Grothendieck Group is its universal group. If $E$ satisfies RDP, this is the universal group $\gamma : E \to G_E$ of the effect algebra, which is a po-group with $u = \gamma(1)$ an order unit. If $E$ is MV, then $[0, u]$ is lattice and $G_E$ is an $\ell-$group.

# Matrix algebras and AF C*-algebras: Mundici II

(*Notes on Real and Complex C*-algebras* by K. R. Goodearl.)

- A finite dimensional C*-algebra is one isomorphic (as a *-algebra) to a direct sum of matrix algebras over $\mathbb{C}$: $\cong M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C})$.
- The ordered list $(m(1), \cdots, m(k))$ is an invariant.
- (Bratteli, 1972) An *AF C*-algebra* (approximately finite C*-algebra) is a countable colimit

$$\varinjlim \left( \mathcal{A}_1 \xrightarrow{\alpha_1} \mathcal{A}_2 \xrightarrow{\alpha_2} \mathcal{A}_3 \xrightarrow{\alpha_3} \cdots \right)$$

  of finite-dimensional C*-algebras and *-algebra maps.

  Bratteli showed AF C*-algebras have a *standard form*:

# Matricial C*-algebras: standard maps

$$\mathcal{A} := M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C}) \text{ and}$$
$$\mathcal{B} := M_{n(1)}(\mathbb{C}) \oplus \cdots \oplus M_{n(l)}(\mathbb{C}) .$$

- Define *-algebra maps $\mathcal{A} \to M_{n(i)}(\mathbb{C})$

$$(A_1, \cdots, A_k) \mapsto DIAG_{n(i)}(\overbrace{A_1, \cdots, A_1}^{s_{i1}}, \overbrace{A_2, \cdots, A_2}^{s_{i2}}, \cdots, \overbrace{A_k, \cdots, A_k}^{s_{ik}})$$

determined by $s_{ik} \in \mathbb{N}$ where $\boxed{s_{i1}m(1) + \cdots + s_{ik}m(k) = n(i)}$.

- A *standard ∗-map* $\mathcal{A} \to \mathcal{B}$ is an $l$-tuple of such DIAGs:

$$(A_1, \cdots, A_k) \mapsto (DIAG_{n(1)}(\cdots), \ldots, DIAG_{n(l)}(\cdots))$$

determined by $l \times k$ matrix $(s_{ij})$ s.t. $\sum_{j=1}^{k}(s_{ij}m(j)) = n(i)$,

# Bratteli's Theorem

### Theorem (Bratteli)

*Any AF C*-algebra is isomorphic (as a C*-algebra) to a colimit of a system of matricial C*-algebras and standard maps.*

Bratteli introduced an important graphical language to handle the difficult combinatorics: Bratteli Diagrams.

# Bratteli's Diagrams: a combinatorial structure

A Bratteli diagram as an infinite directed multigraph $B = (V, E)$, where $V = \cup_{i=0}^{\infty} V(i)$ and $E = \cup_{i=0}^{\infty} E(i)$.

- Assume $V(0)$ has one vertex, the *root*.
- Edges are only defined from $V(i)$ to $V(i+1)$.

$$\begin{array}{ll} V(i) & \overbrace{m(1) \quad m(2) \quad \cdots \quad m(k)}^{(\mathbb{Z}^k,(m(1),\cdots,m(k)))} \\ & \\ V(i+1) & \quad n(1) \quad n(2) \quad \cdots \quad n(l) \end{array}$$

  Draw $s_{ij}$-many edges between $m(j)$ to $n(i)$ . (Of course, for adjacent levels, the $s_{ij}$ must satisfy the combinatorial conditions.)

- Vertices now assigned $\ell\mathbf{AB}_u$ groups $(\mathbb{Z}^k, u)$.

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Colimits along standard maps induces colimits of associated $\mathbb{Z}^k$, called dimension groups.

# $K_0$: Grothendieck group functors

A very general construction:

- $K_0$: **Ring**$\to$ **Ab** and $K_0$ : **AF** $\to$ **Po-Ab**$_u$
- Roughly: turn the isomorphism classes (of idempotents) in the Karoubi Envelope into an abelian cancellative monoid and then by INT into an abelian group.
- Tricky for AF C*-algebras: technicalities of self-adjoint idempotents (= projections)

# AF C*-algebras & Mundici's Theorem II

Approx. finite (AF) C*-algebras classified in deep work of G. Elliott (studied further by Effros, Handelman, Goodearl, et. al).

## Theorem (Mundici)

*Let* $\ell\mathbf{AF}_u$ = *category of AF-algebras, st* $K_0(\mathcal{A})$ *is lattice-ordered with order unit. Let* $\mathcal{MV}_\omega$ = countable *MV-algebras.*

*We can extend* $\Gamma : \ell\mathcal{G}_u \cong \mathcal{MV}$ *to a functor* $\hat{\Gamma} : \ell\mathbf{AF}_u \to \mathcal{MV}_\omega$,

$$\hat{\Gamma}(\mathcal{A}) := \Gamma(K_0(\mathcal{A}), [1_\mathcal{A}]) = [\,0, [1_\mathcal{A}]\,]_{K_0(\mathcal{A})}$$

(i) $\mathcal{A} \cong \mathcal{B}$ *iff* $\hat{\Gamma}(\mathcal{A}) \cong \hat{\Gamma}(\mathcal{B})$

(ii) $\hat{\Gamma}$ *is full.*

# Some Mundici Examples (1991):

| MV Algebra | AF C*-correspondent |
|---|---|
| $\{0, 1\}$ | $\mathbb{C}$ |
| Chain $\mathcal{M}_n$ | $Mat_n(\mathbb{C})$ |
| Finite | Finite Dimensional |
| Dyadic Rationals | CAR algebra of a Fermi gas |
| $\mathbb{Q} \cap [0, 1]$ | Glimm's universal UHF algebra |
| Chang Algebra | Behncke-Leptin algebra |
| Real algebraic numbers in [0,1] | Blackadar algebra $B$. |
| Generated by an irrational $\rho \in [0, 1]$ | Effros-Shen Algebra $\mathfrak{F}_p$ |
| Finite Product of Post MV-algebras | Continuous Trace |
| Free on $\aleph_0$ generators | Universal AF C*-algebra $\mathfrak{M}$ |
| Free on one generator | Farey AF C*-algebra $\mathfrak{M}_1$. Mundici (1988), Boca (2008) |

# Coordinatization: von Neumann's Continuous Geometry

- In an article in PNAS (US) (1936) "Continuous Geometry" von Neumann says "The purpose of the investigations, ... reported briefly in this note, was to complete the elimination of the notion of point (and line and plane) from geometry."

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- What's left?

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- In an article in PNAS (US) (1936) "Continuous Geometry" von Neumann says "The purpose of the investigations, ... reported briefly in this note, was to complete the elimination of the notion of point (and line and plane) from geometry."
- What's left? A (complemented, modular) lattice of subspaces of a space and a dimension function (into [0,1] or $\mathbb{R}$). The subspaces correspond to the principal right ideals of a von-Neumann regular ring.
  **Ref.**
  https://en.wikipedia.org/wiki/Continuous_geometry

# Some Mundici Examples (1991): Coordinatizations (LS)

+ MSc. Thesis of Wei Lu

+ Mundici

| Denumerable MV Algebra | AF C*-correspondent |
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# Inverse Semigroups and Monoids

## Definition (Inverse Semigroups)

*Semigroups (resp. monoids) satisfying: "Every element $x$ has a unique pseudo-inverse $x^{-1}$."*

- $\forall_x \exists!_{x^{-1}} (xx^{-1}x = x \;\;\&\;\; x^{-1}xx^{-1} = x^{-1})$

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**Fundamental Examples**

- $\mathcal{I}_X = \mathbf{PBij}(X)$, Symmetric Inverse Monoid. These are partial bijections on the set $X$, i.e. partial functions $f : X \rightharpoonup X$ which are bijections $dom(f) \to ran(f)$.
  - For each subset $A \subseteq X$, there are partial identity functions $1_A \in \mathcal{I}_X$. These are the idempotents.
  - $f^{-1} \circ f = 1_{dom(f)}$ and $f \circ f^{-1} = 1_{ran(f)}$ , partial identities on $X$ .
- **Semisimple:** $=$ Finite Cartesian Products of *finite* symmetric inverse monoids $\mathcal{I}_{X_1} \times \cdots \times \mathcal{I}_{X_n}$

# Inverse Monoids: Basic Definitions

Let $S$ be an inverse monoid with zero element 0.
Let $E(S)$ be the set of **idempotents** of $S$.

- In analogy with $S = \mathcal{I}_X$, if $a \in S$, define $dom(a) = a^{-1}a$, $ran(a) = aa^{-1}$.

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- $S$ is *boolean* if:
  (i) $E(S)$ is a boolean algebra,
  (ii) "compatible" elements have joins,
  (iii) multiplication distributes over (finite) $\vee$'s.

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**Example:** In $\mathcal{I}_X$, $\leqslant$ is inclusion, and two partial bijections will be "compatible" iff their union is a partial bijection. $\mathcal{I}_X$ is a Boolean $\wedge$-monoid, since partial bijections have meets.

# Inverse Monoids: More Basic Definitions and Facts

Let $S$ be an inverse monoid.

- $U(S)$ is the Group of Units (i.e. invertible elements) of $S$. For example, $U(\mathcal{I}_X)$ is the symmetric group $Sym(X)$.
- $S$ is *factorizable* if every element is $\leqslant u$, for some $u \in U(S)$. ($\mathcal{I}_X$ is factorizable iff $X$ is finite).
- $S$ is *fundamental* if the centralizer$(E(S)) = E(S)$. ($\mathcal{I}_X$ is always fundamental).

# Non-Commutative Stone Duality

Boolean Inverse monoids arise in various recent areas of noncommutative Stone Duality.

### Theorem (Lawson, 2009,2011)

*The category of Boolean inverse $\wedge$-semigroups is dual to the category of Hausdorff Boolean groupoids.*

### Theorem (Kudryavtseva,Lawson 2012)

*The category of Boolean inverse semigroups is dual to the category of Boolean groupoids.*

# Green's Relations and Type Monoid

Let $S$ be an inverse monoid. Define:

1. $\mathcal{J}$ on $S$: $a\mathcal{J}b$ iff $SaS = SbS$ (i.e. equality of principal ideals).
2. $\mathcal{D}$ on $E(S)$: $e\mathcal{D}f$ iff $\exists_{a\in S}(e = dom(a), f = ran(a),\ e \xrightarrow{a} f)$

**The Type Monoid of** $S$. Consider $E(S)/\mathcal{D}$, $S$ boolean. For idempotents $e, f \in E(S)$, define $[e] \widetilde{\oplus} [f]$ as follows: *if* we can find orthogonal idempotents $e' \in [e], f' \in [f]$, let $[e] \widetilde{\oplus} [f] := [e' \vee f']$. Otherwise, undefined.

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### Proposition

*Let $S$ be a factorizable Boolean inverse monoid. Then:*

- $\mathcal{D}$ *preserves complementation and* $(E(S)/\mathcal{D}, \widetilde{\oplus}, [0], [1])$ *is an effect algebra w/ RDP.*

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*Let $S$ be a factorizable Boolean inverse monoid. Then:*

- *$\mathcal{D}$ preserves complementation and $(E(S)/\mathcal{D}, \widetilde{\oplus}, [0], [1])$ is an effect algebra w/ RDP.*

Call these Foulis Monoids.

# Coordinatizing MV Algebras: Main Theorem

- For Foulis monoids $S$ as in the Proposition, $\mathcal{D} = \mathcal{J}$.
- Can identify $E(S)/\mathcal{D}$ with the poset of principal ideals $S/\mathcal{J}$.
- We say $S$ *satisfies the lattice condition* if $S/\mathcal{J}$ is a lattice. It is then in fact an MV-algebra (by Bennet & Foulis).

Theorem (Coordinatization Theorem for MV Algebras: L& S)

*For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid $S$ satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$, as MV algebras.*

# Towards AF inverse monoids

Methodology: redo Bratteli theory, using rook (or boolean) matrices

- A *rook matrix* in $Mat_n(\{0,1\})$ is one where every row and column have at most one 1. Let $R_n :=$ rook matrices.
- There's bijection $\mathcal{I}_n \xrightarrow{\cong} R_n$: $f \mapsto M(f)$, where $M(f)_{ij} = 1$ iff $i = f(j)$.

Up to isomorphism, it's possible to redo the entire theory of Bratteli diagrams using rook matrices and $\mathcal{I}_n$'s instead of $\mathbb{Z}$'s.

# Bratteli Diagrams, AF Inverse Monoids and colimits of $\mathcal{I}_n$s

Recall $B = (V, E)$ a Bratteli diagram.

$$\begin{array}{llllll} V(i) & & m(1) & m(2) & \cdots & m(k) \\ & & & & & \\ V(i+1) & & n(1) & n(2) & \cdots & n(l) \end{array}$$

Draw $s_{ij}$-many edges between $m(j)$ to $n(i)$ .

$$\text{Now associate} \quad \begin{array}{ccc} V(0) & \leftrightarrow & S_0 = \mathcal{I}_1 \cong \{0,1\} \\ \vdots & \vdots & \vdots \\ V(i) & \leftrightarrow & S_i = \mathcal{I}_{m(1)} \times \cdots \times \mathcal{I}_{m(k)} \end{array}$$

Monomorphisms $\sigma_i : S_i \to S_{i+1}$ are induced by standard maps.
$\boxed{\text{Combinatorial Conditions are true}}$

An AF Inverse Monoid $I(B) := colim(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$, for Bratteli diagram $B$.

# AF Inverse Monoids and colimits of $\mathcal{I}_n$s

## Lemma

*(1) Colimits of $\omega$-chains $(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$ of boolean inverse $\wedge$-monoids with monos inherit all the nice features of the factors. In particular, the groups of units are direct limits of groups of units of the $S_i$.*

*(2) Given any $\omega$-sequence of semisimple inverse monoids and injective morphisms, the colim$(S_i)$ is isomorphic to $I(B)$, for some Bratteli diagram $B$.*

## Theorem

*AF inverse monoids are Dedekind finite Boolean inverse monoids in which $\mathcal{D}$ preserves complementation. Their groups of units are direct limits of finite direct products of finite symmetric groups.*

# The General Coordinatization Theorem

### Theorem (Coordinatization Theorem for MV Algebras: L& S)

*For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid $S$ satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$.*

Proof sketch: We know from Mundici every MV algebra $\mathcal{A}$ is isomorphic to an MV-algebra $[0, u]_G$, an interval effect algebra for some order unit $u$ in a countable $\ell$-group $G$. It turns out that $G$ is a countable dimension group. Thus there is a Bratteli diagram $B$ yielding $G$. Take then $I(B)$, the AF inverse monoid of $B$. It turns out that $I(B)/\mathcal{J}$ is isomorphic to $[0, u]$ as an MV effect-algebra, and the latter will be a lattice, thus a Foulis monoid. So, we have coordinatized $\mathcal{A}$.

# New Results: Characterizing AF Inverse monoids

**Goal:** characterize AF inverse monoids abstractly and connect with Krieger & Wehrung's work.

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Consider $\alpha \in \mathcal{I}_{\{1,2,3,4,5,6,7\}}$, where $\alpha = id_{\{2,3\}} \cup \{(4,5,6)\}$.

$$\therefore \quad \alpha = id_{\{2,3\}} \vee \left(\begin{smallmatrix}4\\5\end{smallmatrix}\right) \vee \left(\begin{smallmatrix}5\\6\end{smallmatrix}\right) \vee \left(\begin{smallmatrix}6\\4\end{smallmatrix}\right) \qquad \text{(orthogonal join)}$$

$\alpha =$ idempotent $\vee$ infinitesimals (i.e. $s^2 = 0$): **Basic**.

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**Krieger monoid** $=$ locally finite, basic Boolean inverse monoid

**Theorem** (i) Countable Krieger Monoids $=$ AF Inverse Monoids
(ii) Groups of units of Krieger monoids $=$ Krieger's ample groups.
(iii) Kreiger Monoids $=$ Wehrung's locally matricial B.Inv.Monoids

# Corollary: AF Monoids vs Boolean algebras

We can now answer Mundici's challenge:

**Theorem:** *Countable Krieger Monoids = AF Inverse Monoids*

**Corollary**: Commutative AF inverse monoids = countable Boolean algebras

**Proof:** Suppose $S$ is a commutative AF inverse monoid and $s^2 = 0$. Then $s^{-1}sss^{-1} = 0$. By commutativity, $s^{-1}s = ss^{-1}$. Then $s^{-1}s = 0$, so $s = 0$. So there are no nonzero infinitesimals. But the monoid is basic, so all elements are idempotents. But the idempotents $E(S)$ form a Boolean algebra!

# Example 1: Coordinatizing Finite MV-Algebras

Let $\mathcal{I}_n = \mathcal{I}_X$ be the inverse monoid of partial bijections on $n$ letters, $|X| = n$. One can show that all the $\mathcal{I}_n$'s are Foulis monoids. The idempotents in this monoid are partial identities $1_A$, where $A \subseteq X$. Two idempotents $1_A \mathcal{D} 1_B$ iff $|A| = |B|$. Indeed we get a bijection $\mathcal{I}_n/\mathcal{J} \xrightarrow{\cong} \mathbf{n+1}$, where $\mathbf{n+1} = \{0, 1, \cdots, n\}$. This induces an order isomorphism, where $\mathbf{n+1}$ is given its usual order, and lattice structure via *max, min*.

The effect algebra structure of $\mathcal{I}_n/\mathcal{J}$ becomes: let $r, s \in \mathbf{n+1}$. $r \mathbin{\widetilde{\oplus}} s$ is defined and equals $r + s$ iff $r + s \leqslant n$. The orthocomplement $r' = n - r$. The associated MV algebra: $r \oplus s = r + min(r', s)$, which equals $r + s$ if $r + s \leqslant n$ and $r \oplus s$ equals $n$ if $r + s > n$.

We get an iso $\mathcal{I}_n/\mathcal{J} \cong \mathcal{M}_n$, the Łukasiewicz chain. But every *finite* MV algebra is a product of such chains, which are then coordinatized by a product of $\mathcal{I}_n$'s.

# Example 2: Coordinatizing Dyadic Rationals–Cantor Space

Cuntz (1977) studied C*-algebras of isometries (of a sep. Hilbert space). Also arose in wavelet theory & formal language theory (Nivat, Perrot). We'll describe $C_n$ the $n$th Cuntz inverse monoid.

*Cantor Space* $A^\omega$, $A$ finite. For $C_n$, pick $|A| = n$. For $C_2$, pick $A = \{a, b\}$. Given the usual topology, we have:

1. Clopen subsets have the form $XA^\omega$, where $X \subseteq A^*$ are *Prefix codes* : finite subsets s.t. $x \precsim y$ ($y$ prefix of $x$) $\Rightarrow x = y$.
2. Representation of clopen subsets by prefix codes is not unique. E.g. $aA^\omega = (aa + ab)A^\omega$.
3. We can make prefixes $X$ in clopens uniquely representable: define *weight* by $w(X) = \sum_{x \in X} |x|$. **Theorem:** *Every* clopen is generated by a unique prefix code $X$ of minimal weight.

# Cuntz and $n$-adic AF-Inverse Monoids

### Definition (The Cuntz inverse monoid, Lawson (2007))

*$C_n \subseteq \mathcal{I}_{A^\omega}$ consists of those partial bijections on prefix sets of same cardinality: $(x_1 + \cdots x_r)A^\omega \longrightarrow (y_1 + \cdots y_r)A^\omega$ such that $x_i u \mapsto y_i u$, for any right infinite string $u$.*

### Proposition (Lawson (2007))

*$C_n$ is a Boolean inverse $\wedge$-monoid, whose set of idempotents $E(C_n)$ is the unique countable atomless B.A. Its group of units is the Thompson group $V_n$.*

### Definition ( $n$-adic inverse monoid $Ad_n \subseteq C_n$)

*$Ad_n$ = those partial bijections in $C_n$ of the form $x_i \mapsto y_i$, where $|x_i| = |y_i|$, $i \leqslant r$. $Ad_2$ = the dyadic inverse monoid.*

# Cuntz and Dyadic AF-Inverse Monoids

**Theorem**

*The MV-algebra of dyadic rationals is co-ordinatized by $Ad_2$.*

The proof uses Bernoulli measures on Cantor spaces.

**Proposition (Characterizing $Ad_2$ as an AF monoid)**

*The dyadic inverse monoid is isomorphic to the direct limit of the sequence of symmetric inverse monoids (partial bijections)*

$$\mathcal{I}_2 \to \mathcal{I}_4 \to \mathcal{I}_8 \to \cdots$$

*called the CAR inverse monoid. The group of units is a colimit of symmetric groups: $Sym(1) \to Sym(2) \to \cdots Sym(2^r) \to \cdots$.*

# Cuntz and Dyadic AF-Inverse Monoids: Invariant Measures

General theory of measures on Cantor Space is recent research (Akin, Handelman, ...). Look at simple *Bernoulli Measures*.

## Definition

*Let $S$ be a Boolean inverse monoid. An* invariant measure *is a function $\mu : E(S) \to [0,1]$ satisfying: (i) $\mu(1) = 1$,*
*(ii) $\forall s \in S(\mu(s^{-1}s) = \mu(ss^{-1}))$,*
*(iii) If $e, f \in E(S), e \perp f$ then $\mu(e \vee f) = \mu(e) + \mu(f)$.*

A good invariant measure $\mu$ is an invariant measure such that:
$\mu(e) \leqslant \mu(f) \ \Rightarrow \exists e'[e' \leqslant f \wedge \mu(e) = \mu(e')]$

**Example** If $|A| = n$ and $a \in A$, let $\mu(a) = \frac{1}{n}$. If $x \in A^*$, let $\mu(x) = \frac{1}{n^{|x|}}$. For prefix set $X$, let $\mu(X) = \sum_{x \in X} \mu(x)$.
(If $n = 2$, $\mu$ is called *Bernoulli measure*.)

# Bernoulli Measures

A general property:

**Lemma**

*If $S$ is a boolean inverse monoid with a good invariant measure $\mu$ that reflects the $\mathcal{D}$ relation (i.e. $\mu(e) = \mu(f) \Rightarrow e\mathcal{D}f$) then $S$ is (i) Dedekind finite, (ii) $\mathcal{D}$ preserves complementation, and (iii) $S/\mathcal{J}$ is linearly ordered.*

**Lemma**

*$Ad_2$ has a good invariant measure that reflects the $\mathcal{D}$ relation. Hence $Ad_2/\mathcal{J}$ is linearly ordered.*

The main coordinatization theorem in this example then follows:

M. Lawson , P. Scott, AF Inverse Monoids and the structure of Countable MV Algebras, *J. Pure and Applied Algebra* 221 (2017), pp. 45–74. (also extended arXiv version).

# Coordinatizing $\mathbb{Q} \cap [0,1]$ : thesis of Wei Lu

## Definition (Omnidivisional sequence)

*A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is omnidivisional if it satisfies the following properties.*

- *For all $i$, $n_i \mid n_{i+1}$.*
- *For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.*

## Example

*The sequence $\{n!\}_{n=1}^{\infty}$.*

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## Example

*The sequence $\{n!\}_{n=1}^{\infty}$.*

## Theorem (Coordinatization of the rationals)

*Let $D = \{n_i\}_{n=1}^{\infty}$ be an omnidivisional sequence. Then, (for certain "standard embeddings" $\tau_i$) the directed colimit of the sequence*

$$Q\colon \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \dots,$$

*coordinatizes $\mathbb{Q} \cap [0,1]$.*

# Coordinatizing the Chang Algebra

**Theorem**[Decomposition Theorem I] Let $A$ be an MV algebra. Suppose that $A$ has subalgebras forming a chain of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$ and each $A_i$ is coordinatized by an inverse semigroup $S_i$. Suppose there are injective maps $\tau_i \colon S_i \longrightarrow S_{i+1}$ well-defined on $\mathcal{D}$-classes. Then, $A$ is coordinatized by the directed colimit of $S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots$.

**Theorem**[Decomposition Theorem II] : "Converse" of Theorem I.

For the Chang Algebra, the Foulis monoid is interesting: $\mathcal{I}(\mathbb{N})_{fc} =$ the subinverse monoid of $\mathcal{I}(\mathbb{N})$ of those partial bijections on $\mathbb{N}$ whose domain are either finite or balanced cofinite.