Contextuality as a resource

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Combining Viewpoints in Quantum Theory
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Joint work with:

- Samson Abramsky (Oxford)
- Shane Mansfield (Paris VII)

and also:

- Kohei Kishida (Dalhousie)
- Giovanni Carù (Oxford)
- Nadish de Silva (UCL)
- Octavio Zapata (UCL)
Motivation

- **Contextuality and non-locality:** fundamental non-classical phenomena of QM
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- Contextuality as a **resource** for QIP and QC:
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  - **Non-local games**
    - XOR games (CHSH; Cleve–Høyer–Toner–Watrous)
    - quantum graph homomorphisms (Mančinska–Roberson)
    - constraint satisfaction (Cleve–Mittal)
    - etc. (Abramsky–B–de Silva–Zapata)
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    “Contextuality in measurement-based quantum computation”
  - **MSD**
    “Contextuality supplies the ‘magic’ for quantum computation”
Contextuality formulated in a theory-independent fashion
Overview

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- Towards a resource theory of contextuality:
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- Towards a resource theory of contextuality:
  - Combine and transform contextual blackboxes
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- Towards a resource theory of contextuality:
  - Combine and transform contextual blackboxes
  - Measure of contextuality
  - Quantifiable advantages in QC and QIP tasks
Contextuality
Empirical data

\[ o_A \in \{0, 1\} \]

\[ m_A \in \{a_1, a_2\} \]

\[ o_B \in \{0, 1\} \]

\[ m_B \in \{b_1, b_2\} \]
## Empirical data

<table>
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<tr>
<th></th>
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</table>

$\Phi \in \{a_1, a_2\}$

$\Phi_B \in \{b_1, b_2\}$

$\rho \in \{0, 1\}$

$\Phi$ (measurement device) $\rightarrow$ $\Phi_B$ (measurement device) $\rightarrow$ $\rho$ (preparation)
A simple observation
(Abramsky–Hardy)

- Propositional formulae $\phi_1, \ldots, \phi_N$

\[
p_i := \text{Prob}(\phi_i)
\]

Not simultaneously satisfiable, hence
\[
\text{Prob}(\bigwedge \phi_i) = 0
\]

Using elementary logic and probability:
\[
1 = \text{Prob}(\neg \bigwedge \phi_i) = \text{Prob}(\bigvee \neg \phi_i) \leq \sum_{i=1}^{N} \text{Prob}(\neg \phi_i) = \sum_{i=1}^{N} (1 - p_i) = N - \sum_{i=1}^{N} p_i.
\]

Hence,
\[
\sum_{i=1}^{N} p_i \leq N - 1.
\]
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- Hence, $\sum_{i=1}^{N} p_i \leq N - 1$. 
Analysis of the Bell table

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<tr>
<th>A</th>
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These formulae are contradictory. But \( p_1 + p_2 + p_3 + p_4 = 3 \). The inequality is violated by \( 1/4 \).
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<th>A</th>
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$$\phi_1 = a_1 \leftrightarrow b_1$$
$$\phi_2 = a_1 \leftrightarrow b_2$$
$$\phi_3 = a_2 \leftrightarrow b_1$$
$$\phi_4 = a_2 \oplus b_2$$

These formulae are contradictory. But

$$p_1 + p_2 + p_3 + p_4 = 3.35$$

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φ₁ = a₁ ↔ b₁
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What was our unwarranted assumption?
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That all variables could \textit{in principle} be observed simultaneously.
Contextuality

- But the Bell table can be realised in the real world.
- What was our unwarranted assumption?
- That all variables could *in principle* be observed simultaneously.
- **Local consistency vs global inconsistency.**
Abramsky–Brandenburger framework

Measurement scenario \( \langle X, M, O \rangle \):

- \( X \) is a finite set of measurements or variables
- \( O \) is a finite set of outcomes or values
- \( M \) is a cover of \( X \), indicating joint measurability (contexts)

Example: (2,2,2) Bell scenario

- The set of variables is \( X = \{ a_1, a_2, b_1, b_2 \} \).
- The outcomes are \( O = \{ 0, 1 \} \).
- The measurement contexts are: \( \{ \{ a_1, b_1 \}, \{ a_1, b_2 \}, \{ a_2, b_1 \}, \{ a_2, b_2 \} \} \).
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Measurement scenarios

Examples: Bell-type scenarios, KS configurations, and more.
Another example: 18-vector Kochen–Specker

- A set of 18 variables, \( X = \{A, \ldots, O\} \)
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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $M = \{C_1, \ldots, C_9\}$, whose contexts $C_i$ correspond to the columns in the following table:

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```
Empirical Models

Joint outcome or event in a context $C$ is $s \in O^C$, e.g.

$$s = [a_1 \mapsto 0, b_1 \mapsto 1].$$
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\textbf{Empirical model:} family $\{e_C\}_{C \in \mathcal{M}}$ where $e_C \in \text{Prob}(O^C)$ for $C \in \mathcal{M}$.

It specifies a probability distribution over the events in each context. Each distribution is a row of the probability table.
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**Compatibility** condition: the distributions “agree on overlaps”

$$\forall C, C' \in \mathcal{M}. \quad e_C|_{C \cap C'} = e_{C'}|_{C \cap C'} .$$
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In multipartite scenarios, compatibility = the **no-signalling** principle.
Contextuality

A (compatible) empirical model is **non-contextual** if there exists a **global distribution** \( d \in \text{Prob}(O^X) \) on the joint assignments of outcomes to all measurements that marginalises to all the \( e_C \):

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**Contextuality:**
family of data which is **locally consistent** but **globally inconsistent**.
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**Contextuality**: family of data which is **locally consistent** but **globally inconsistent**.

The import of results such as Bell’s and Bell–Kochen–Specker’s theorems is that there are empirical models arising from quantum mechanics that are contextual.
Given an empirical model $e$, define possibilistic model $\text{poss}(e)$ by taking the support of each distributions.

Contains the possibilistic, or logical, information of that model.
Possibilistic collapse

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<table>
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$\rightarrow$

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Logical contextuality: Hardy model

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There are some global sections, but . . .

**Logical contextuality:** Not all sections extend to global ones.
Strong Contextuality:

**no** event can be extended to a global assignment.
Strong contextuality

Strong Contextuality: **no** event can be extended to a global assignment.

E.g. K–S, GHZ, the PR box:

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Cohomological witnesses of contextuality
(Abramsky–B–Mansfield, ABM–Kishida–Lal, Carù, Raussendorf et al.)
Measuring Contextuality
The contextual fraction

Non-contextuality: global distribution $d \in \text{Prob}(O^X)$ such that:

$$\forall C \in \mathcal{M}. \ d|_C = e_C.$$
The contextual fraction

Non-contextuality: global distribution $d \in \text{Prob}(O^X)$ such that:

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Which fraction of a model admits a non-contextual explanation?
The contextual fraction

Non-contextuality: global distribution \( d \in \text{Prob}(O^X) \) such that:

\[
\forall C \in \mathcal{M}. \ d|_C = e_C .
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Which fraction of a model admits a non-contextual explanation?

Consider subdistributions \( c \in \text{SubProb}(O^X) \) such that:

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Non-contextual fraction: maximum weight of such a subdistribution.
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**Non-contextual fraction**: maximum weight of such a subdistribution.

Equivalently, maximum weight $\lambda$ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda)e'$$

where $e^{NC}$ is a non-contextual model.
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Non-contextuality: global distribution \( d \in \text{Prob}(O^X) \) such that:

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\[
e = \lambda e^{NC} + (1 - \lambda) e^{SC}
\]

where \( e^{NC} \) is a non-contextual model. \( e^{SC} \) is strongly contextual!

\[
\text{NCF}(e) = \lambda \quad \text{CF}(e) = 1 - \lambda
\]
Checking contextuality of $e$ corresponds to solving

Find $d \in \mathbb{R}^n$

such that $M d = v^e$

and $d \geq 0$. 

(Non-)contextual fraction via linear programming
Checking contextuality of $e$ corresponds to solving

Find $d \in \mathbb{R}^n$

such that $M d = v^e$

and $d \geq 0$.

Computing the non-contextual fraction corresponds to solving the following linear program:

Find $c \in \mathbb{R}^n$

maximising $1 \cdot c$

subject to $M c \leq v^e$

and $c \geq 0$.
E.g. Equatorial measurements on GHZ\((n)\)

Figure: Contextual fraction of empirical models obtained with equatorial measurements at \(\phi_1\) and \(\phi_2\) on each qubit of \(|\psi_{\text{GHZ}(n)}\rangle\) with: (a) \(n = 3\); (b) \(n = 4\).
Violations of Bell inequalities
Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

- a set of coefficients $\alpha = \{ \alpha(C, s) \}_{C \in \mathcal{M}, s \in O^c}$
- a bound $R$

Wlog we can take $R$ non-negative (in fact, we can take $R = 0$).

It is called a **Bell inequality** if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.

NB: A complete set of inequalities can be derived from the logical approach.
Generalised Bell inequalities

An inequality for a scenario \( \langle X, M, O \rangle \) is given by:

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For a model \( e \), the inequality reads as

\[
B_\alpha(e) \leq R ,
\]

where

\[
B_\alpha(e) := \sum_{C \in M, s \in O} \alpha(C, s)e_C(s) .
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Violation of a Bell inequality

A Bell inequality establishes a bound for the value of $B_{\alpha}(e)$ amongst NC models.
Violation of a Bell inequality

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For a general (no-signalling) model \( e \), the quantity is limited only by

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\| \alpha \| := \sum_{C \in \mathcal{M}} \max \left\{ \alpha(C, s) \mid s \in O^C \right\}
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Violation of a Bell inequality

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$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \left\{ \alpha(C, s) \mid s \in O^C \right\}$$

The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model $e$ is the value

$$\frac{\max\{0, B_{\alpha}(e) - R\}}{\|\alpha\| - R}.$$
Proposition

Let $e$ be an empirical model.
Bell inequality violation and the contextual fraction

**Proposition**
Let $e$ be an empirical model.

- The normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$. 

Moreover, this Bell inequality is tight at “the” non-contextual model $e_{\text{NC}}$ and maximally violated by “the” strongly contextual model $e_{\text{SC}}$ for any decomposition:

$$e = \text{NCF}(e) e_{\text{NC}} + \text{CF}(e) e_{\text{SC}}.$$
Bell inequality violation and the contextual fraction

Proposition

Let $e$ be an empirical model.

- The normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$.

- This bound is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\text{CF}(e)$.
Bell inequality violation and the contextual fraction

Proposition
Let $e$ be an empirical model.

- The normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$.

- This bound is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\text{CF}(e)$.

- Moreover, this Bell inequality is tight at “the” non-contextual model $e^{NC}$ and maximally violated by “the” strongly contextual model $e^{SC}$ for any decomposition:

$$e = \text{NCF}(e)e^{NC} + \text{CF}(e)e^{SC}.$$
Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find \( \mathbf{c} \in \mathbb{R}^n \)
maximising \( 1 \cdot \mathbf{c} \)
subject to \( M\mathbf{c} \leq \mathbf{v}^e \)
and \( \mathbf{c} \geq 0 \).

\[ e = \lambda e^{NC} + (1 - \lambda) e^{SC} \text{ with } \lambda = 1 \cdot x^*. \]
Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find \( c \in \mathbb{R}^n \)

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Dual LP:

Find \( y \in \mathbb{R}^m \)

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\[ a := 1 - |M|y \]

Find \( a \in \mathbb{R}^m \)

maximising \( a \cdot v^e \)

subject to \( M^T a \leq 0 \)

and \( a \leq 1 \).

computes tight Bell inequality (separating hyperplane)
Operations on empirical models
Contextuality as a resource

More than one possible measure of contextuality.

What properties should a good measure satisfy?

Monotonicity wrt operations that do not introduce contextuality.

Towards a resource theory as for entanglement (e.g. LOCC), non-locality, . . .
Contextuality as a resource

- More than one possible measure of contextuality.
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Contextuality as a resource

- More than one possible measure of contextuality.
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- Monotonicity wrt operations that do not introduce contextuality
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Algebra of empirical models

- Think of empirical models as black boxes

\[ e \langle X, M, O \rangle \] to mean that \( e \) is a (compatible) empirical model on \( \langle X, M, O \rangle \).

The operations remind one of process algebras.
Algebra of empirical models

- Think of empirical models as black boxes
- What operations can we perform (non-contextually) on them?
Algebra of empirical models

- Think of empirical models as black boxes

- What operations can we perform (*non-contextually*) on them?

- We write type statements

\[ e : \langle \mathcal{X}, \mathcal{M}, O \rangle \]

...to mean that \(e\) is a (compatible) empirical model on \(\langle \mathcal{X}, \mathcal{M}, O \rangle\).
Algebra of empirical models

- Think of empirical models as black boxes
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- We write type statements
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- The operations remind one of process algebras.
Operations

Relabelling:

\[\langle X, M, O \rangle \xrightarrow{\alpha} (X, M) \sim (X', M') \xrightarrow{e} \langle X', M', O \rangle\]

For \(C \in M\), \(s:\alpha(C) \rightarrow O\), \(e[\alpha](s) := e(C \circ \alpha^{-1})\)

Restriction \(e\):

\[\langle X, M, O \rangle \leq (X, M) \xrightarrow{e\restriction M'}: \langle X', M', O \rangle\]

For \(C' \in M'\), \(s:\ C' \rightarrow O\), \((e\restriction M')(s) := e|_{C'}(s)\)

with any \(C \in M\) s.t. \(C' \subseteq C\)

Coarse-graining \(e\):

\[\langle X, M, O \rangle \xrightarrow{f}: O \rightarrow O' \xrightarrow{e/f}: \langle X, M, O' \rangle\]

For \(C \in M\), \(s:\ C \rightarrow O'\), \((e/f)(s) := \sum t:C \rightarrow O, f \circ t = s e(C)(t)\)

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Contextuality as a resource 25
Operations

Relabelling

\[
ed : \langle X, M, O \rangle
\]
\[
\alpha : (X, M) \cong (X', M') \quad \leadsto \quad e[\alpha] : \langle X', M', O \rangle
\]
Operations

Relabelling

\[ e : \langle X, M, O \rangle \]
\[ \alpha : (X, M) \cong (X', M') \quad \leadsto \quad e[\alpha] : \langle X', M', O \rangle \]

For \( C \in M \), \( s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)
Operations

Relabelling

\[ e : \langle X, M, O \rangle \sim \alpha : (X, M) \cong (X', M') \leadsto e[\alpha] : \langle X', M', O \rangle \]

For \( C \in M \), \( s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

Restriction

\[ e : \langle X, M, O \rangle \sim (X', M') \leq (X, M) \leadsto e \upharpoonright M' : \langle X', M', O \rangle \]
Operations

Relabelling

\[ e : \langle X, M, O \rangle \quad \alpha : (X, M) \cong (X', M') \quad \leadsto \quad e[\alpha] : \langle X', M', O \rangle \]

For \( C \in M \), \( s : \alpha(C) \longrightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

Restriction

\[ e : \langle X, M, O \rangle \quad (X', M') \leq (X, M) \quad \leadsto \quad e \upharpoonright M' : \langle X', M', O \rangle \]

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with any \( C \in M \) s.t. \( C' \subseteq C \)
Operations

Relabelling

\[ e : \langle X, M, O \rangle \]
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Coarse-graining

\[ e : \langle X, M, O \rangle \]
\[ f : O \rightarrow O' \quad \leadsto \quad e/f : \langle X, M, O' \rangle \]
Operations

Relabelling
\[ e : \langle X, M, O \rangle \]
\[ \alpha : ( X, M ) \cong ( X', M' ) \leadsto e[\alpha] : \langle X', M', O \rangle \]
For \( C \in M \), \( s : \alpha(C) \rightarrow O \), \( e[\alpha]_\alpha(C)(s) := e_C(s \circ \alpha^{-1}) \)

Restriction
\[ e : \langle X, M, O \rangle \]
\[ ( X', M' ) \leq ( X, M ) \leadsto e \upharpoonright M' : \langle X', M', O \rangle \]
For \( C' \in M' \), \( s : C' \rightarrow O \), \( (e \upharpoonright M')_{C'}(s) := e_C|_{C'}(s) \)
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Coarse-graining
\[ e : \langle X, M, O \rangle \]
\[ f : O \rightarrow O' \leadsto e/f : \langle X, M, O' \rangle \]
For \( C \in M \), \( s : C \rightarrow O' \), \( (e/f)_C(s) := \sum_{t : C \rightarrow O, f \circ t = s} e_C(t) \)
Operations

Mixing

\[ e, e' : \langle X, M, O \rangle \]
\[ \lambda \in [0, 1] \]
\[ \leadsto e + \lambda e' : \langle X, M, O \rangle \]
Operations

Mixing

\[ e, e' : \langle X, M, O \rangle \]
\[ \lambda \in [0, 1] \]
\[ \leadsto e + \lambda e' : \langle X, M, O \rangle \]

For \( C \in M, s : C \to O' \),
\[ (e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s) \]
Operations

Mixing

\[ e, e' : \langle X, M, O \rangle \quad \sim \quad e + \lambda \ e' : \langle X, M, O \rangle \]

\[ \lambda \in [0, 1] \]

For \( C \in M, s : C \rightarrow O' \),

\[ (e + \lambda \ e')_{C}(s) := \lambda e_{C}(s) + (1 - \lambda) e'_{C}(s) \]

Choice

\[ e : \langle X, M, O \rangle \quad \sim \quad e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

\[ e' : \langle X', M', O \rangle \]
Operations

Mixing

\[ e, e' : \langle X, M, O \rangle \]
\[ \lambda \in [0, 1] \]
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Choice

\[ e : \langle X, M, O \rangle \]
\[ e' : \langle X', M', O \rangle \]
\[ \leadsto e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

For \( C \in M, (e \& e')_C := e_C \)
For \( D \in M', (e \& e')_D := e'_D \)
**Operations**

**Mixing**

\[ e, e' : \langle X, \mathcal{M}, O \rangle \]
\[ \lambda \in [0, 1] \]
\[ \xrightarrow{\lambda} e + \lambda e' : \langle X, \mathcal{M}, O \rangle \]

For \( C \in \mathcal{M}, s : C \longrightarrow O' \),
\[ (e + \lambda e')_c(s) := \lambda e_c(s) + (1 - \lambda)e'_c(s) \]

**Choice**

\[ e : \langle X, \mathcal{M}, O \rangle \]
\[ e' : \langle X', \mathcal{M}', O \rangle \]
\[ \xrightarrow{\text{}} e \& e' : \langle X \sqcup X', \mathcal{M} \sqcup \mathcal{M}', O \rangle \]

For \( C \in \mathcal{M}, (e \& e')_c := e_c \)
For \( D \in \mathcal{M}', (e \& e')_D := e'_D \)

**Tensor**

\[ e : \langle X, \mathcal{M}, O \rangle \]
\[ e' : \langle X', \mathcal{M}', O \rangle \]
\[ \xrightarrow{\text{}} e \otimes e' : \langle X \sqcup X', \mathcal{M} \star \mathcal{M}', O \rangle \]
Operations

Mixing

\[ e, e' : \langle X, \mathcal{M}, O \rangle \]
\[ \lambda \in [0, 1] \]
\[ \rightsquigarrow e + \lambda\ e' : \langle X, \mathcal{M}, O \rangle \]

For \( C \in \mathcal{M}, s : C \rightarrow O' \),
\[ (e + \lambda\ e')_C(s) := \lambda e_C(s) + (1 - \lambda) e'_C(s) \]

Choice

\[ e : \langle X, \mathcal{M}, O \rangle \]
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\[ \rightsquigarrow e \& e' : \langle X \sqcup X', \mathcal{M} \sqcup \mathcal{M}', O \rangle \]

For \( C \in \mathcal{M}, (e \& e')_C := e_C \)
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Tensor

\[ e : \langle X, \mathcal{M}, O \rangle \]
\[ e' : \langle X', \mathcal{M}', O \rangle \]
\[ \rightsquigarrow e \otimes e' : \langle X \sqcup X', \mathcal{M} \star \mathcal{M}', O \rangle \]

\[ \mathcal{M} \star \mathcal{M}' := \{ C \sqcup D \mid C \in \mathcal{M}, D \in \mathcal{M}' \} \]
### Operations

#### Mixing

$$e, e' : \langle X, M, O \rangle \quad \lambda \in [0, 1] \quad \leadsto \quad e + \lambda \ e' : \langle X, M, O \rangle$$

For $C \in M$, $s : C \rightarrow O'$,

$$(e + \lambda \ e')_C(s) := \lambda e_C(s) + (1 - \lambda) e'_C(s)$$

#### Choice

$$e : \langle X, M, O \rangle \quad e' : \langle X', M', O \rangle \quad \leadsto \quad e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle$$

For $C \in M$, $(e \& e')_C := e_C$

For $D \in M'$, $(e \& e')_D := e'_D$

#### Tensor

$$e : \langle X, M, O \rangle \quad e' : \langle X', M', O \rangle \quad \leadsto \quad e \otimes e' : \langle X \sqcup X', M \star M', O \rangle$$

$$M \star M' := \{ C \sqcup D \mid C \in M, D \in M' \}$$

For $C \in M$, $D \in M'$, $s = \langle s_1, s_2 \rangle : C \sqcup D \rightarrow O$,

$$(e \otimes e')_{C \sqcup D} \langle s_1, s_2 \rangle := e_C(s_1) e'_D(s_2)$$
Operations and the contextual fraction

\[ \text{NCF}(e_1 \otimes e_2) = \text{NCF}(e_1) \cdot \text{NCF}(e_2) \]

Sequencing

\[ \text{NCF}(e_1; e_2) \geq \text{NCF}(e_1) \cdot \text{NCF}(e_2) \]
Operations and the contextual fraction

Relabelling $e[\alpha]$
Operations and the contextual fraction

Relabelling \( e[\alpha] \)

Restriction \( e \upharpoonright \mathcal{M}' \)
Operations and the contextual fraction

Relabelling $e[\alpha]$  
Restriction $e \upharpoonright M'$  
Coarse-graining $e/f$
Operations and the contextual fraction

Relabelling \( e[\alpha] \)

Restriction \( e \upharpoonright \mathcal{M}' \)

Coarse-graining \( e/f \)

Mixing \( \lambda e + (1 - \lambda)e' \)
Operations and the contextual fraction

Relabelling  \( e[\alpha] \)

Restriction  \( e \upharpoonright \mathcal{M}' \)

Coarse-graining  \( e/f \)

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Choice  \( e & e' \)
Operations and the contextual fraction

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Operations and the contextual fraction

Relabelling  \( e[\alpha] \)

Restriction  \( e \upharpoonright M' \)

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Choice  \( e \& e' \)

Tensor  \( e_1 \otimes e_2 \)

Sequencing  \( e_1; e_2 \)
Operations and the contextual fraction

Relabelling \( \text{CF}(e[\alpha]) = \text{CF}(e) \)

Restriction \( e \upharpoonright M' \)

Coarse-graining \( e/f \)

Mixing \( \lambda e + (1 - \lambda)e' \)

Choice \( e \& e' \)

Tensor \( e_1 \otimes e_2 \)

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Operations and the contextual fraction

Relabelling \[ \text{CF}(e[\alpha]) = \text{CF}(e) \]

Restriction \[ \text{CF}(e \upharpoonright \mathcal{M}') \leq \text{CF}(e) \]

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Mixing \[ \text{CF}(\lambda e + (1 - \lambda)e') \leq \lambda \text{CF}(e) + (1 - \lambda)\text{CF}(e') \]

Choice \[ e \& e' \]

Tensor \[ e_1 \otimes e_2 \]

Sequencing \[ e_1 ; e_2 \]
Operations and the contextual fraction

**Relabelling** \( \text{CF}(e[\alpha]) = \text{CF}(e) \)

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**Choice** \( \text{CF}(e \& e') = \max\{\text{CF}(e), \text{CF}(e')\} \)

**Tensor** \( e_1 \otimes e_2 \)

\( \text{NCF}(e_1 \otimes e_2) = \text{NCF}(e_1) \text{NCF}(e_2) \)

**Sequencing** \( e_1; e_2 \)
Operations and the contextual fraction

Relabelling $\text{CF}(e[\alpha]) = \text{CF}(e)$

Restriction $\text{CF}(e \upharpoonright M') \leq \text{CF}(e)$

Coarse-graining $\text{CF}(e/f) \leq \text{CF}(e)$

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NCF($e_1 \otimes e_2$) $= \text{NCF}(e_1)\text{NCF}(e_2)$

Sequencing $e_1 ; e_2$
Operations and the contextual fraction

Relabelling \( \text{CF}(e[\alpha]) = \text{CF}(e) \)

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Tensor \( \text{CF}(e_1 \otimes e_2) = \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1)\text{CF}(e_2) \)
\( \text{NCF}(e_1 \otimes e_2) = \text{NCF}(e_1)\text{NCF}(e_2) \)

Sequencing \( \text{CF}(e_1 \otimes e_2) \leq \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1)\text{CF}(e_2) \)
\( \text{NCF}(e_1; e_2) \geq \text{NCF}(e_1)\text{NCF}(e_2) \)
Resource theory of contextuality
(some work in progress)
Resource theory of contextuality
(some work in progress)

- Resource theory *a la* Coecke–Fritz–Spekkens. (resource theory of combinable processes)
Resource theory of contextuality  
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- Resource theory *a la* Coecke–Fritz–Spekkens.  
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- Device-independent processes
Resource theory of contextuality
(some work in progress)

- Resource theory *a la* Coecke–Fritz–Spekkens. (resource theory of combinable processes)

- Device-independent processes
  - Operations remind one of process algebra

Sequencing:
- so far, it hides middle steps
- not doing so leads to notion of causal empirical models.

Allow natural expression of measurement-based computation with feed-forward, in a device-independent form:
- One can measure a non-maximal context (face $\sigma$ of complex)
  - leaving a model on scenario $\sigma$.
Resource theory of contextuality
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  - One can measure a non-maximal context (face $\sigma$ of complex)
Resource theory of contextuality
(some work in progress)

- Resource theory *a la* Coecke–Fritz–Spekkens.
  (resource theory of combinable processes)

- Device-independent processes
  - Operations remind one of process algebra
  - Process calculus:
    operational semantics by (probabilistic) transitions
  - bissimulation, metric / approximation
  - (modal) logic for device-independent processes

- Sequencing:
  - so far, it hides middle steps
  - not doing so leads to notion of causal empirical models.

- Allow natural expression of measurement-based computation with feed-forward, in a device-independent form:
  - One can measure a non-maximal context (face $\sigma$ of complex)
  - leaving a model on scenario $\text{lk}_\sigma \mathcal{M}$
Contextual fraction and quantum advantages
Contextual fraction and advantages

- Contextuality has been associated with quantum advantage in information-processing and computational tasks.
Contextual fraction and advantages

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- Measure of contextuality \( \leadsto \) quantify such advantages.
Contextual fraction and cooperative games

- Game described by $n$ formulae (one for each allowed input).
- These describe the winning condition that the corresponding outputs must satisfy.
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- If the formulae are \( k \)-consistent (at most \( k \) are jointly satisfiable), the **hardness of the task** is \( \frac{n-k}{n} \).

  (cf. Abramsky & Hardy, “Logical Bell inequalities”)
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- If the formulae are $k$-consistent (at most $k$ are jointly satisfiable), the hardness of the task is $\frac{n-k}{n}$.
  (cf. Abramsky & Hardy, “Logical Bell inequalities”)
- We have

\[
1 - \bar{p}_S \geq \text{NCF} \frac{n-k}{n}
\]
Contextuality and MBQC
E.g. Raussendorf (2013) $\ell^2$-MBQC

- Measurement-based quantum computing scheme ($m$ input bits, $l$ output bits, $n$ parties)
  - Classical control:
    - Pre-processes input
    - Determines the flow of measurements
    - Post-processes to produce the output
  - Only $Z_2$-linear computations.
  - Additional power to compute non-linear functions resides in certain resource empirical models.
  - Raussendorf (2013): If an $\ell^2$-MBQC deterministically computes a non-linear Boolean function $f : 2^m \rightarrow 2^l$ then the resource must be strongly contextual.
  - Probabilistic version: non-linear function computed with sufficiently large probability of success implies contextuality.
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Goal: Compute Boolean function $f : 2^m \rightarrow 2^l$ using $\ell^2$-MBQC

Hardness of the problem
$
\nu(f) := \min \{ d(f, g) \mid g \text{ is } \mathbb{Z}_2\text{-linear} \}
$

(average distance between $f$ and closest $\mathbb{Z}_2$-linear function)

Average probability of success computing $f$ (over all $2^m$ possible inputs):
$\bar{p}_S$

Then,
$1 - \bar{p}_S \geq NCF(e) \nu(f)$
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where for Boolean functions $f$ and $g$, $d(f, g) := 2^{-m} | \{ i \in 2^m \mid f(i) \neq g(i) \} |$. 
Contextual fraction and MBQC

- **Goal**: Compute Boolean function \( f : 2^m \rightarrow 2^l \) using \( \ell_2\)-MBQC

- **Hardness of the problem**

  \[ \nu(f) := \min \{ d(f, g) \mid g \text{ is } \mathbb{Z}_2\text{-linear} \} \]

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  where for Boolean functions \( f \) and \( g \),
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- **Average probability of success** computing \( f \) (over all \( 2^m \) possible inputs): \( \bar{p}_S \).
Goal: Compute Boolean function $f : 2^m \rightarrow 2^l$ using $\ell 2$-MBQC

Hardness of the problem

$$\nu(f) := \min \{d(f, g) \mid g \text{ is } \mathbb{Z}_2\text{-linear}\}$$

(average distance between $f$ and closest $\mathbb{Z}_2$-linear function)

where for Boolean functions $f$ and $g$, $d(f, g) := 2^{-m} \mid \{i \in 2^m \mid f(i) \neq g(i)\}$.

Average probability of success computing $f$ (over all $2^m$ possible inputs): $\bar{\rho}_S$.

Then,

$$1 - \bar{\rho}_S \geq \text{NCF}(e) \nu(f)$$
Questions...

“The contextual fraction as a measure of contextuality”
Samson Abramsky, RSB, Shane Mansfield