Categories of Physical Processes

Stanisław Szawiel

University of Warsaw

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Part I
A non-topological TQFT

Idea

The category of physical processes, $\mathbf{Phys}$ is

- All states of all physical systems (objects)
- All physical processes between them (arrows)
  (time evolution, asymptotic scattering, etc.)

Axioms for $\mathbf{Phys}$

1. $\mathbf{Phys}$ has noninteracting composites ($\otimes$-structure)
2. Physical processes act on observables, preserve composites:

$$\mathcal{O} : \mathbf{Phys} \longrightarrow C^*\mathbf{Alg}^{op}$$

3. States $\varphi \in \mathbf{Phys}$ determine expectation values

$$\langle - \rangle_\varphi : \mathcal{O}(\varphi) \longrightarrow \mathbb{C}$$

4. Processes $f : \varphi \longrightarrow \psi$ preserve expectation values:

$$\begin{tikzcd} \mathcal{O}(\psi) \arrow[rr, "\mathcal{O}(f)"] \arrow[dr, "\psi"'] & & \mathcal{O}(\varphi) \arrow[dl, "\varphi"] \\ & \mathbb{C} & \end{tikzcd}$$

5. Weak independence: $\langle - \rangle_{\varphi \otimes \psi} = \langle - \rangle_\varphi \otimes \langle - \rangle_\psi$

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1. $\mathbf{Phys}$ has noninteracting composites ($\otimes$-structure)
2. Physical processes act on observables, preserve composites:

$$\mathcal{O} : \mathbf{Phys} \longrightarrow C^*\mathbf{Alg}^{op} \text{ (but gauge theory!)}$$

3. States $\varphi \in \mathbf{Phys}$ determine expectation values

$$\langle - \rangle_\varphi : \mathcal{O}(\varphi) \longrightarrow \mathbb{C}$$

Observables without expectation values!!

4. Processes $f : \varphi \longrightarrow \psi$ preserve expectation values:

$$\begin{tikzcd} \mathcal{O}(\psi) \arrow[rr, "\mathcal{O}(f)"] \arrow[dr, "\psi"'] & & \mathcal{O}(\varphi) \arrow[dl, "\varphi"] \\ & \mathbb{C} & \end{tikzcd}$$

(this is not unitarity!)

5. Weak independence: $\langle - \rangle_{\varphi\otimes\psi} = \langle - \rangle_\varphi \otimes \langle - \rangle_\psi$

Theorem

There is a terminal category satisfying these axioms.

- *Call it* $\mathbf{Phys}$*.*

Proof.

It's the category of pairs $(A, \varphi), \varphi : A \longrightarrow \mathbb{C}$. $\square$

The GNS Construction

Definition

A pointed A-module (H, v) represents $\varphi : A \longrightarrow \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$

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The Gelfand-Naimark-Segal Theorem

- Positive φ have an initial representation
- A representation is initial iff it is cyclic
  (cyclic = generated by the chosen vector)

Notation

- Initial representation of $\varphi = GNS(\varphi)$
- Representing vector $= \Omega$
- Write H for (H, v)

The GNS Functor

H represents $\varphi \Longrightarrow f^*H$ represents $f^*\varphi$

$$
\begin{array}{ccccc}
f^*H & \longrightarrow & H & & \\
B & \xrightarrow{\quad f \quad} & A & \xrightarrow{\ \varphi\ } & \mathbb{C}
\end{array}
$$

The GNS Functor

$$\begin{tikzcd}
GNS(\psi) & & \\
& & GNS(\varphi) \\
\mathcal{O}(\psi) \arrow[rr, "\mathcal{O}(f)"] & & \mathcal{O}(\varphi)
\end{tikzcd}$$

The GNS Functor

$$
\begin{tikzcd}
GNS(\psi) & \\
\mathcal{O}(f)^*GNS(\varphi) \arrow[r] & GNS(\varphi) \\
\mathcal{O}(\psi) \arrow[r, "\mathcal{O}(f)"] & \mathcal{O}(\varphi)
\end{tikzcd}
$$

The GNS Functor

$$
\begin{tikzcd}
GNS(\psi) \arrow[d, "\exists!"] & \\
\mathcal{O}(f)^* GNS(\varphi) \arrow[r] & GNS(\varphi)
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$$
\mathcal{O}(\psi) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(\varphi)
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Theorem

This gives a symmetric monoidal functor

$$GNS : \mathbf{Phys}^{op} \longrightarrow *\mathbf{Mod}$$

Proof.

Things exist by initiality. Diagrams commute by cyclicity. $\square$

The GNS Functor

$$\begin{tikzcd}
GNS(\psi) \arrow[d, "\exists!"] \arrow[drr, "GNS(f)"] & & \\
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$$\psi \xleftarrow{\quad f \quad} \varphi$$

Theorem

This gives a symmetric monoidal functor

$$GNS : \mathbf{Phys}^{op} \longrightarrow *\mathbf{Mod}$$

It's going the wrong way!

The Covariant GNS Functor

Physically Correct Direction

$$
\begin{tikzcd}
\mathbf{Phys} \arrow[r, "GNS^{op}"] \arrow[rr, bend right, "GNS_c"'] & {*\mathbf{Mod}^{op}} \arrow[r, "\text{adjoint}"] & {*\mathbf{Mod}_{adj}}
\end{tikzcd}
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Definition

- $*\mathbf{Mod}_{adj}$ is $*$-modules with adjoint homomorphisms
- Adjoint homomorphisms: coisometries h such that

$$ah(v) = h(f(a)v)$$

Part II
Physics From a Functor

The Schrödinger Picture

Example Factory

- $U : H \longrightarrow H'$ unitary
- $A \subseteq End(H)$ chosen observables
- $\varphi \in H$ determines state $\langle (-)\varphi, \varphi \rangle : A \longrightarrow \mathbb{C}$

Lifting Schrödinger

For any choice of A and $\varphi \in H$ there exists a unique lift $f : \varphi \longrightarrow \psi$ to $\mathbf{Phys}$, such that $\mathcal{O}(\varphi) = A$ and:

$$\begin{tikzcd} GNS(\varphi) \arrow[r, "GNS_c(f)"] \arrow[d, hook] & GNS(\psi) \arrow[d, hook] \\ H \arrow[r, "U"] & H' \end{tikzcd}$$

Symmetries and Unitary Representations

Why does a G-equivariant state give a unitary representation of G?

$$G \longrightarrow \mathbf{Phys} \xrightarrow{GNS_c} *\mathbf{Mod}_{adj}$$

Symmetries and Unitary Representations

Why does a G-equivariant state give a unitary representation of G?

Because of composition!

$$\begin{tikzcd}[column sep=huge] G \arrow[r] \arrow[rr, bend right, "\text{Unitary representation of } G!"'] & \mathbf{Phys} \arrow[r, "GNS_c"] & *\mathbf{Mod}_{adj} \end{tikzcd}$$

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Bonus items:

- Groupoids of symmetries
- Equivariant GNS:

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$$\mathbf{Phys}^{G} \xrightarrow{GNS_c^{G}} *\mathbf{Mod}_{adj}^{G}$$

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- Compatibility with composite systems:

$$\varphi \otimes \psi \text{ has symmetry } G \times G'$$

Relation to Probability Theory

(X, μ) – compact probability space.

- $\mathbb{E}_\mu(a) = \int_X a\, d\mu$ – a state on $C(X)$
- $L^2(\mu)$, a $C(X)$-module

Theorem

The following diagram of symmetric monoidal functors commutes

$$\begin{tikzcd} & \mathbf{Prob}^{op} \arrow[dl, "C^{op}"'] \arrow[dr, "L^2"] & \\ \mathbf{Phys}^{op} \arrow[rr, "GNS"] & & *\mathbf{Mod} \end{tikzcd}$$

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Proof.

1. $L^2(\mu)$ is cyclic
2. $1 \in L^2(\mu)$ represents the expectation value $\mathbb{E}_\mu$ $\square$

Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_\varphi(a) = (Spec(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Eigenvalue-Eigenvector Link

The following are equivalent:

1. $a\Omega = \lambda\Omega$
2. $a = \lambda$ a.e. in $P_\varphi(a)$

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Proof.

The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R : \varphi \longrightarrow P_\varphi(a) \in \mathbf{Phys}$
Previous theorem computes $GNS(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \longrightarrow GNS(\varphi)$$

Thus: $a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$ in $L^2 \iff a = \lambda$ a.e. $\square$

Classical Markov Processes

Markov Processes

- $M(X) =$ probability measures on X
- Markov process $X \longrightarrow Y =$ map $X \longrightarrow M(Y)$
- Category of Markov processes $= Kleisli(M)$

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Generalized Gelfand Duality (Furber & Jacobs 2015)

Compact spaces + Markov processes

$=$

C^*-algebras + completely positive unital maps

Quantum Markov Processes

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Quantum Markov Processes

Axioms for $\mathbf{Phys}_M$

1. $\mathbf{Phys}_M$ has noninteracting composites ($\otimes$-structure)
2. Physical processes act on observables, preserve composites:

$$\mathcal{O} : \mathbf{Phys}_M \longrightarrow \mathbf{CompPos}$$

3. States $\varphi \in \mathbf{Phys}_M$ determine expectation values

$$\langle - \rangle_\varphi : \mathcal{O}(\varphi) \longrightarrow \mathbb{C}$$

4. Processes $f : \varphi \longrightarrow \psi$ preserve expectation values:

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Quantum Markov Processes

$$
\begin{tikzcd}
 & \mathbf{Prob}^{op} \arrow[dl, "C^{op}"'] \arrow[dr, "L^2"] \arrow[dd] & \\
\mathbf{Phys}^{op} \arrow[rr, crossing over, "GNS"] \arrow[dd] & & *\mathbf{Mod} \arrow[dd, "U"] \\
 & \mathbf{Prob}_M^{op} \arrow[dl, "C^{op}"'] \arrow[dr, "L^2"] & \\
\mathbf{Phys}_M^{op} \arrow[rr, "GNS_M"] & & \mathbf{Hilb}
\end{tikzcd}
$$

Example: State Vector Collapse

- $P \in A$ – self-adjoint projection (idempotent)
- $\Phi : A \longrightarrow A$ given by $a \longmapsto PaP$

Theorem

- φ *represented by* $\Omega \Longrightarrow \Phi^*\varphi$ *represented by* $P\Omega$
- $GNS_{M,c}(\Phi)$ *acts as*

$$GNS(\varphi) \xrightarrow{\;P\;} GNS(\varphi) \xrightarrow{\text{orth. proj.}} GNS(\Phi^*\varphi)$$

Example: Particle Scattering

$$\begin{tikzcd}
H \arrow[r, "S"] & H \\
H_\alpha \arrow[u, hook] & H_\beta \arrow[u, hook]
\end{tikzcd}$$

Proposition

There is a process $S_{\alpha\beta} : \alpha \longrightarrow \beta \in \mathbf{Phys}_M$ such that

$$\begin{tikzcd}
H_\alpha \arrow[r, hook, "\text{inclusion}"] \arrow[rrr, bend right, "GNS_{M,c}(S_{\alpha\beta})"'] & H \arrow[r, "S"] & H \arrow[r, "\text{projection}"] & H_\beta
\end{tikzcd}$$

If you believe in QED: $\gamma + \gamma \longrightarrow e^- + e^+$

Part III
Work in Progress

Differential Geometry of the GNS functor

Hocus pocus work in a topos

Inside a model of SDG:

- Differentiate symmetric state $\varphi : G \longrightarrow \mathbf{Phys}$ and get

$$Lie(G) \longrightarrow Der(\mathcal{O}(\varphi))$$

Differential Geometry of the GNS functor

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GNS on infinitesimal symmetries

$$GNS(X) = Q \Longleftrightarrow Q\Omega = 0$$

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GNS on infinitesimal symmetries

$$GNS(X) = Q \Longleftrightarrow Q\Omega = 0$$

- Differentiate family of algebras $A_\hbar : \mathbb{R} \longrightarrow *\mathbf{Alg}$
  Result: a class in $HH^2(A_0)$
  Classical limit of observables = Poisson structure!

Why?

Idea

“Path integral argument”

$=$

isomorphism of vacua in $\mathbf{Phys}$

Families of Vacua

Needed to use S-duality:

“The $\hbar$-family of spaces vacua of super Yang-Mills theory is trivial”

Conclusion

Need smooth subcategory $\mathbf{Vac} \subset \mathbf{Phys}$ of vacua

Thank You!