

Categories of Physical Processes

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Part I
A non-topological TQFT

Idea

The category of physical processes, **Phys** is

- ▶ All states of all physical systems (objects)
- ▶ All physical processes between them (arrows)
(time evolution, asymptotic scattering, etc.)

Axioms for **Phys**

1. **Phys** has noninteracting composites (\otimes -structure)
2. Physical processes act on observables, preserve composites:

$$\mathcal{O} : \mathbf{Phys} \longrightarrow C^* \mathbf{Alg}^{op}$$

3. States $\varphi \in \mathbf{Phys}$ determine expectation values

$$\langle - \rangle_{\varphi} : \mathcal{O}(\varphi) \longrightarrow \mathbb{C}$$

4. Processes $f : \varphi \longrightarrow \psi$ preserve expectation values:

$$\begin{array}{ccc} \mathcal{O}(\psi) & \xrightarrow{\mathcal{O}(f)} & \mathcal{O}(\varphi) \\ & \searrow \psi & \swarrow \varphi \\ & \mathbb{C} & \end{array}$$

5. Weak independence: $\langle - \rangle_{\varphi \otimes \psi} = \langle - \rangle_{\varphi} \otimes \langle - \rangle_{\psi}$

Axioms for Phys

1. **Phys** has noninteracting composites (\otimes -structure)
2. Physical processes act on observables, preserve composites:

$$\mathcal{O} : \mathbf{Phys} \longrightarrow C^* \mathbf{Alg}^{op} \text{ (but gauge theory!)}$$

3. States $\varphi \in \mathbf{Phys}$ determine expectation values

$$\langle - \rangle_{\varphi} : \mathcal{O}(\varphi) \longrightarrow \mathbb{C}$$

Observables without expectation values!!

4. Processes $f : \varphi \longrightarrow \psi$ preserve expectation values:

$$\begin{array}{ccc} \mathcal{O}(\psi) & \xrightarrow{\mathcal{O}(f)} & \mathcal{O}(\varphi) \\ & \searrow \psi & \swarrow \varphi \\ & \mathbb{C} & \end{array} \quad \text{(this is not unitarity!)}$$

5. Weak independence: $\langle - \rangle_{\varphi \otimes \psi} = \langle - \rangle_{\varphi} \otimes \langle - \rangle_{\psi}$

Theorem

There is a terminal category satisfying these axioms.

▶ *Call it **Phys**.*

Proof.

It's the category of pairs $(A, \varphi), \varphi : A \rightarrow \mathbb{C}$. □

The GNS Construction

Definition

A pointed A -module (H, v) **represents** $\varphi : A \rightarrow \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$

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The Gelfand-Naimark-Segal Theorem

- ▶ Positive φ have an **initial representation**
- ▶ A representation is initial iff it is cyclic (cyclic = generated by the chosen vector)

Notation

- ▶ Initial representation of $\varphi = GNS(\varphi)$
- ▶ Representing vector = Ω
- ▶ Write H for (H, v)

The GNS Functor

H represents $\varphi \implies f^* H$ represents $f^* \varphi$

$$f^* H \longrightarrow H$$

$$B \xrightarrow{f} A \xrightarrow{\varphi} \mathbb{C}$$

The GNS Functor

$GNS(\psi)$

$GNS(\varphi)$

$$\mathcal{O}(\psi) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(\varphi)$$

The GNS Functor

$$GNS(\psi)$$

$$\mathcal{O}(f)^*GNS(\varphi) \longrightarrow GNS(\varphi)$$

$$\mathcal{O}(\psi) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(\varphi)$$

The GNS Functor

$$\begin{array}{ccc} GNS(\psi) & & \\ \downarrow \exists! & & \\ \mathcal{O}(f)^*GNS(\varphi) & \longrightarrow & GNS(\varphi) \\ & & \\ \mathcal{O}(\psi) & \xrightarrow{\mathcal{O}(f)} & \mathcal{O}(\varphi) \end{array}$$

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Theorem

This gives a symmetric monoidal functor

$$GNS : \mathbf{Phys}^{op} \longrightarrow *Mod$$

Proof.

Things exist by initiality. Diagrams commute by cyclicity. \square

The GNS Functor

$$\begin{array}{ccc} GNS(\psi) & & \\ \downarrow \exists! & \searrow GNS(f) & \\ \mathcal{O}(f)^*GNS(\varphi) & \longrightarrow & GNS(\varphi) \end{array}$$

$$\psi \longleftarrow \xrightarrow{f} \varphi$$

Theorem

This gives a symmetric monoidal functor

$$GNS : \mathbf{Phys}^{op} \longrightarrow *Mod$$

It's going the wrong way!

The Covariant GNS Functor

Physically Correct Direction

$$\mathbf{Phys} \xrightarrow{GNS^{op}} *Mod^{op} \xrightarrow{\text{adjoint}} *Mod_{adj}$$

GNS_c

The diagram illustrates the relationship between three categories: \mathbf{Phys} , $*Mod^{op}$, and $*Mod_{adj}$. A horizontal arrow labeled GNS^{op} points from \mathbf{Phys} to $*Mod^{op}$. A second horizontal arrow labeled adjoint points from $*Mod^{op}$ to $*Mod_{adj}$. A curved arrow labeled GNS_c points directly from \mathbf{Phys} to $*Mod_{adj}$, representing the composition of the two horizontal arrows.

The Covariant GNS Functor

Physically Correct Direction

$$\begin{array}{ccc} \mathbf{Phys} & \xrightarrow{GNS^{op}} & *Mod^{op} \xrightarrow{\text{adjoint}} *Mod_{adj} \\ & \searrow & \nearrow \\ & & GNS_c \end{array}$$

Definition

- ▶ $*Mod_{adj}$ is $*$ -modules with adjoint homomorphisms
- ▶ Adjoint homomorphisms: coisometries h such that

$$ah(v) = h(f(a)v)$$

Part II
Physics From a Functor

The Schrödinger Picture

Example Factory

- ▶ $U : H \rightarrow H'$ unitary
- ▶ $A \subseteq \text{End}(H)$ chosen observables
- ▶ $\varphi \in H$ determines state $\langle (-)\varphi, \varphi \rangle : A \rightarrow \mathbb{C}$

Lifting Schrödinger

For any choice of A and $\varphi \in H$ there exists a unique lift $f : \varphi \rightarrow \psi$ to **Phys**, such that $\mathcal{O}(\varphi) = A$ and:

$$\begin{array}{ccc} GNS(\varphi) & \xrightarrow{GNS_c(f)} & GNS(\psi) \\ \downarrow & & \downarrow \\ H & \xrightarrow{U} & H' \end{array}$$

Symmetries and Unitary Representations

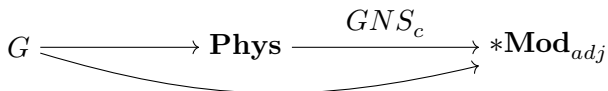
Why does a G -equivariant state give a unitary representation of G ?

$$G \longrightarrow \mathbf{Phys} \xrightarrow{GNS_c} * \mathbf{Mod}_{adj}$$

Symmetries and Unitary Representations

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Because of composition!



Unitary representation of G !

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$$\begin{array}{ccc} G & \xrightarrow{\quad} & \mathbf{Phys} \xrightarrow{GNS_c} \mathbf{*Mod}_{adj} \\ & \searrow & \nearrow \\ & & \end{array}$$

Unitary representation of G !

Bonus items:

- ▶ Groupoids of symmetries
- ▶ Equivariant GNS:

$$\mathbf{Phys} \xrightarrow{GNS_c} \mathbf{*Mod}_{adj}$$

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$$G \begin{array}{c} \xrightarrow{\quad} \mathbf{Phys} \xrightarrow{GNS_c} * \mathbf{Mod}_{adj} \\ \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad \text{Unitary representation of } G! \end{array}$$

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$$\mathbf{Phys}^G \xrightarrow{GNS_c^G} * \mathbf{Mod}_{adj}^G \xrightarrow{U} \mathbf{Rep}(G)$$

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- ▶ Equivariant GNS:

$$\mathbf{Phys}^G \xrightarrow{GNS_c^G} * \mathbf{Mod}_{adj}^G \xrightarrow{U} \mathbf{Rep}(G)$$

- ▶ Compatibility with composite systems:

$$\varphi \otimes \psi \text{ has symmetry } G \times G'$$

Relation to Probability Theory

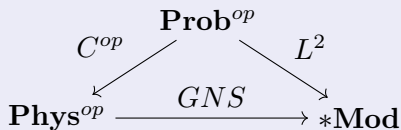
(X, μ) – compact probability space.

▶ $\mathbb{E}_\mu(a) = \int_X a d\mu$ – a state on $C(X)$

▶ $L^2(\mu)$, a $C(X)$ -module

Theorem

The following diagram of symmetric monoidal functors commutes



Relation to Probability Theory

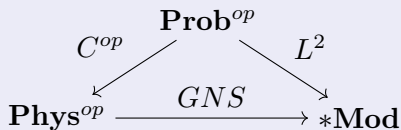
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Proof.

1. $L^2(\mu)$ is cyclic

2. $1 \in L^2(\mu)$ represents the expectation value \mathbb{E}_μ



Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_\varphi(a) = (\text{Spec}(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Eigenvalue-Eigenvector Link

The following are equivalent:

1. $a\Omega = \lambda\Omega$
2. $a = \lambda$ a.e. in $P_\varphi(a)$

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Proof.

The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R : \varphi \rightarrow P_\varphi(a) \in \mathbf{Phys}$
Previous theorem computes $GNS(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \rightarrow GNS(\varphi)$$

Thus: $a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$ in $L^2 \iff a = \lambda$ a.e. □

Classical Markov Processes

Markov Processes

- ▶ $M(X)$ = probability measures on X
- ▶ Markov process $X \rightarrow Y = \text{map } X \rightarrow M(Y)$
- ▶ Category of Markov processes = $Kleisli(M)$

Classical Markov Processes

Markov Processes

- ▶ $M(X)$ = probability measures on X
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- ▶ Category of Markov processes = $Kleisli(M)$

Generalized Gelfand Duality (Furber & Jacobs 2015)

Compact spaces + Markov processes

=

C^* -algebras + **completely positive unital maps**

Quantum Markov Processes

Axioms for \mathbf{Phys}

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Quantum Markov Processes

Axioms for \mathbf{Phys}_M

1. \mathbf{Phys}_M has noninteracting composites (\otimes -structure)
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$$\mathcal{O} : \mathbf{Phys}_M \longrightarrow \mathbf{CompPos}$$

3. States $\varphi \in \mathbf{Phys}_M$ determine expectation values

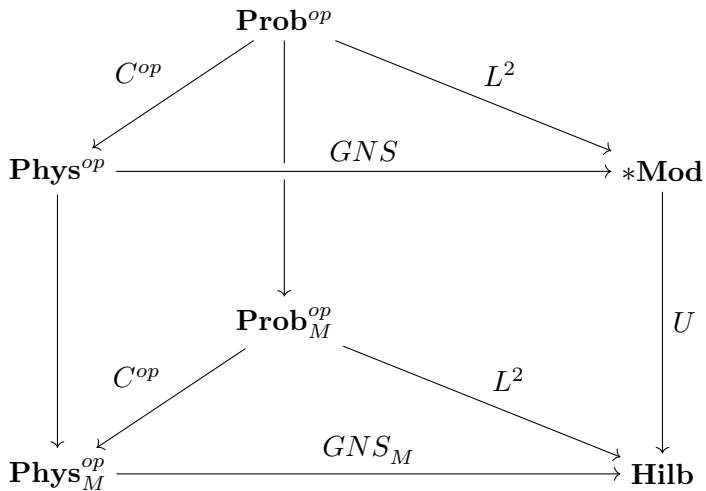
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Quantum Markov Processes



Example: State Vector Collapse

- ▶ $P \in A$ – self-adjoint projection (idempotent)
- ▶ $\Phi : A \rightarrow A$ given by $a \mapsto PaP$

Theorem

- ▶ φ represented by $\Omega \implies \Phi^*\varphi$ represented by $P\Omega$
- ▶ $GNS_{M,c}(\Phi)$ acts as

$$GNS(\varphi) \xrightarrow{P} GNS(\varphi) \xrightarrow{\text{orth. proj.}} GNS(\Phi^*\varphi)$$

Example: Particle Scattering

$$\begin{array}{ccc} H & \xrightarrow{S} & H \\ \uparrow & & \uparrow \\ H_\alpha & & H_\beta \end{array}$$

Proposition

There is a process $S_{\alpha\beta} : \alpha \rightarrow \beta \in \mathbf{Phys}_M$ such that

$$\begin{array}{ccccccc} H_\alpha & \xleftarrow{\text{inclusion}} & H & \xrightarrow{S} & H & \xrightarrow{\text{projection}} & H_\beta \\ & & & & & \searrow & \\ & & & & & & GNS_{M,c}(S_{\alpha\beta}) \end{array}$$

If you believe in QED: $\gamma + \gamma \rightarrow e^- + e^+$

Part III
Work in Progress

Differential Geometry of the GNS functor

Hocus pocus work in a topos

Inside a model of SDG:

- ▶ Differentiate symmetric state $\varphi : G \rightarrow \mathbf{Phys}$ and get

$$Lie(G) \rightarrow Der(\mathcal{O}(\varphi))$$

Differential Geometry of the GNS functor

Hocus pocus work in a topos

Inside a model of SDG:

► Differentiate symmetric state $\varphi : G \rightarrow \mathbf{Phys}$ and get

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GNS on infinitesimal symmetries

$$GNS(X) = Q \iff Q\Omega = 0$$

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GNS on infinitesimal symmetries

$$GNS(X) = Q \iff Q\Omega = 0$$

- ▶ Differentiate family of algebras $A_\hbar : \mathbb{R} \rightarrow *Alg$
Result: a class in $HH^2(A_0)$
Classical limit of observables = Poisson structure!

Why?

Idea

“Path integral argument”
=
isomorphism of vacua in **Phys**

Families of Vacua

Needed to use S-duality:

“The \hbar -family of spaces vacua of super Yang-Mills theory is trivial”

Conclusion

Need smooth subcategory **Vac** \subset **Phys** of vacua

Thank You!