Choquet order and abelian subalgebras

Jan Hamhalter and Ekaterina Turilova

Czech Technical University in Prague, Czech Republic
Kazan Federal University, Russia

Edinburgh, March 2018

Our aim is to show an interplay between two structures relevant to foundations of quantum theory that have been investigated separately so far. On one side it is the structure of decompositions of states on operator algebras that has an important applications to quantum statistical mechanics, to describing equilibrium of quantum systems and to the convex approach to quantum theory in general.
Let us note that the concept of convex combinations of states is one of the basic ingredients of quantum theory that embodies the principle of superposition.

On the other side, it is the "context structure" $\mathcal{V}(\mathcal{M})$ of abelian subalgebras of a operator algebra $\mathcal{M}$ ordered by set theoretic inclusion. This structure has received a great deal of attention recently. It is vital for topos approach to foundations of quantum theory and plays a central role in "bohrification program" for quantum structures. In this approach abelian subalgebras embody classical local physical subsystems which, according to the Bohr's doctrine, are the only things we can see as macroscopic observers. Therefore, from this standpoint, quantum system given by a von Neumann algebra $\mathcal{M}$ is given in fact by the structure $\mathcal{V}(\mathcal{M})$ of mutually overlapping classical subsystems of $\mathcal{M}$. In other words, $\mathcal{V}(\mathcal{M})$ is the stucture of von Neumann subalgebras that are commutative and so do not admit any kind of uncertainty principle. The relation $X \subseteq Y$ for X and Y in $\mathcal{V}(\mathcal{M})$ can be interpreted as the fact that the subsystem Y contains more information than subsystem X.

Alternatively, abelian subalgebras are related to quantum measurement. Let us recall that von Neumann measurement is modeled by projection valued measure, while more generall approach is given by the positive operator valued measure. But in both cases the range of the measure generates some abelian subalgebras. In this light, the relation $X \subseteq Y$ can be seen as the fact that quantum measurements associated with the subsystem Y are more accurate than quantum measurements given by the subsystem X.

For each $X \in \mathcal{V}(\mathcal{M})$ we associate a certain probability measure μ_X on the state space of $\mathcal{M}$ that has barycenter ω. The relation $X \subseteq Y$ for $X, Y \in \mathcal{V}(\mathcal{M})$ means that the measure μ_Y is closer to a boundary measure than the measure μ_X. More preciasely, μ_Y dominates μ_X in the Choquet order: $\mu_X \prec \mu_Y$.
In statistical terms it can be expressed by saying that for each observable (viewed as an affine function on the state space) the distribution μ_X has smaller dispersion than the distribution μ_Y.

Each measure μ_X is a probability measure on the state space with expectation value ω than governs certain convex decopmosition of the state ω. Moreover, μ_X is an orthogonal measure which means that it gives orthogonal positive functionals by integrating over disjoint regions in the state space. Re-expressing the relation $\mu_X \prec \mu_Y$ one can say that both measures have the common center of mass ω but the support of μ_Y is further removed from its barycenter ω. In an extreme case μ_Y is maximal with respect to the Choquet order. Then μ_Y is supported by the set of pure states. In this situation Y is a maximal abelian subalgebra and μ_Y provides decomposition of ω into pure states.

Orthogonal measures play an important role in decomposition theory of states that aims at expressing complex states as a superposition of simpler components. Besides the decomposition into pure states, the following two decompositions are widely used in quantum statistical mechanics. First one is given by central orthogonal measures, that is by measures corresponding to abelian subalgebras of the center of von Neumann algebra. Since the center corresponding to invariants of the system, the associated decompositions are expressions of a given states in terms of states in which the invariants have special values. Second important class consists of orthogonal measures that are invariant with respect to action of some group G on the algebra. They correspond to abelian subalgebras that are invariant with the respect to automorphism group representing G.

Maximal measures of his kind enable one to express a given state invariant with the respect to G as a superpposition of G-ergodic states. G-ergodig states are extreme points of the set of invariant states. In application to quantum statistical mechanics the group G represents symmetries of the system and invariance of the state ω with recpect to G means that ω reflects well these symmetries. The G-ergodic states correspond to symmetric pure phases of the system. In this manner mathematical concept of orthogonal measures and Choquet ordering we find their applications to studying ground states and equilibrium states of quantum statistical mechanics systems.

Importance of the poset of orthogonal measures equipped with Choquet order for studying the structure of superpositions of states motivates the questions of describing all preservers of this poset. Such preservers leave the structure of decompositions unchanged. Using recent results on the order isomorphisms of the structure of abelian subalgebras $\mathcal{V}(\mathcal{M})$, we establish that all maps between barycentric decompositions of a given state that preserve the Choquet order of corresponding orthogonal measures are induced by Jourdan $*$-isomorphisms. In other words, structure of decompositions of a state ω is a invariant for a quantum system that has both geometric and physical content.

In order to elucidate our results without going into technical details, we exhibit a simple Jordan invariant that follows from our analysis. Let $\mathcal{M}$ be a σ-finite von Neumann algebra with a fixed normal faithful state φ. Let us consider decomposition of φ into convex combinations of another states $\varphi_1 \ldots \varphi_n$ on $\mathcal{M}$:

$$\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i \,, \tag{1}$$

such that the states $\varphi_1 \ldots \varphi_n$ are mutually orthogonal. So (1) is a nice convex mixture of "non overlapping"states.

To reexpress (1) in measure-theoretic language we can use Dirac measures on the state space $\mathcal{S}(\mathcal{M})$ and identify convex combination in (1) with convex combination of Dirac measures on $\mathcal{S}(\mathcal{M})$ of $\mathcal{M}$:

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{\varphi_i} \,.$$

Let us define the barycenter of the measure μ as an element $b(\mu)$ in the state space $\mathcal{S}(\mathcal{M})$ satisfying that

$$b(\mu) = \int_{\mathcal{S}(\mathcal{M})} \omega \, d\mu(\omega)$$

in the weak sense.

It means that for each $a \in \mathcal{M}$ we have

$$b(\mu)(a) = \int_{\mathcal{S}(\mathcal{M})} a(\omega) d\mu(\omega) = \sum_{i=1}^{n} \lambda_i \varphi_i(a) \,.$$

(Consequently, (1) reads as $\varphi = b(\mu)$.)

Denote by $D(\varphi)$ the set of all possible decompositions of φ specified in (1).

We shall introduce an order, $\prec$, on the set $D(\varphi)$ as follows: Let

$$\sum_{i=1}^{n} \lambda_i \varphi_i \text{ and } \sum_{j=1}^{m} \eta_i \psi_i$$

be two convex combinations in $D(\varphi)$. We define

$$\sum_{i=1}^{n} \lambda_i \varphi_i \prec \sum_{j=1}^{m} \eta_j \psi_j$$

if

$$\sum_{i=1}^{n} \lambda_i \varphi_i^2(a) \leq \sum_{j=1}^{m} \eta_j \psi_j^2(a) \qquad (2)$$

for all self-adjoint $a \in \mathcal{M}$.

In order to explain relation (2) let us identify the convex combinations in the question with convex combinations of Dirac measures on the state space $\mathcal{S}(\mathcal{M})$ of $\mathcal{M}$:

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{\varphi_i} \quad \text{and} \quad \nu = \sum_{j=1}^{m} \eta_j \delta_{\psi_j}.$$

In this view, the inequality (2) says that, given any self-adjoint element a on $\mathcal{M}$ (identified with an affine function $\hat{a}$ on $\mathcal{S}(\mathcal{M})$), the dispersion of $\hat{a}$ with respect to μ is smaller than the dispersion of $\hat{a}$ with respect to ν.

Let us denote the dispersion of μ with respect to μ as

$$\Delta_\mu(\hat{a}) = \sqrt{\mu(|\hat{a} - \mu(\hat{a})|^2)}.$$

The set of all such dispersions is a natural measure of the distribution of μ around its mean value φ. General Choquet theory says that the bigger the dispersions are the closer is the measure μ to a boundary measure and so the closer are the states $\varphi_1 \dots \varphi_n$ in (1) to pure states.

It turns out that $(D(\varphi), \prec)$ is a complete Jordan invariant in the following sense:
Let φ and ψ be normal faithful states on von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, respectively. Suppose that $\mathcal{M}$ has no Type I_2 direct summand. Then

$$(D(\varphi), \prec) \text{ is order isomorphic to } (D(\psi), \prec)$$

if and only if

$$\mathcal{M} \text{ is Jordan } *\text{-isomorphic to } \mathcal{N}\,.$$

The main results concern general variant of the above example where "discrete"decompositions are generalized to integral mixtures involving orthogonal Radon measures on the state space with a common barycenter. The order on this structure is determined by values of measures at continuous convex functions. This particular order is widely used in Choquet theory of compact convex sets. We study one-to-one correspondence between ordered sets of orthogonal representing measures on one side and the structures of abelian subalgebras of von Neumann algebras ordered by set theoretic inclusion on the other side. We show that in this correspondence measures with finite support correspond to finite dimensional abelian subalgebras. We prove that order isomorphisms between the sets of orthogonal representing measures are given by Jordan $*$-isomorphisms between corresponding von Neumann algebras.

Basic concepts of ordered sets

Let us recall needed concepts and fix the notation. Let $(X, \leq)$ be a *partially ordered set* (a *poset* for short). Suprema and infima of a subset $S \subseteq X$ (if they exist) will be denoted by $\bigvee S$ or $\bigwedge S$, respectively. In case of $S = \{a, b\}$ we shall use notation $\bigvee S = a \vee b$ or $\bigwedge S = a \wedge b$, respectively. A subset $S \subset X$ is called *upward directed* if for all elements $a, b \in S$ there is an element $c \in S$ such that $c \geq a, b$. *Directed-complete poset* is defined as a poset in which every upward directed subset has a supremum. A *semilattice* is a directed-complete poset in which infimum of any two elements exists. A map $f : X \to Y$ between two posets $(X, \leq)$ and $(Y, \leq)$ is called an *order isomorphism* if it is a bijection preserving order in both directions, that is

$$a \leq b \iff f(a) \leq f(b) \qquad \text{for all } a, b \in X\,.$$

Throughout the talk $\mathcal{A}$ will denote a C^*-algebra. We will always assume that $\mathcal{A}$ is *unital* with the unit $\mathbf{1}$. By $\mathcal{A}_h$ and $\mathcal{A}^+$ we shall mean the self-adjoint and the positive part of $\mathcal{A}$, respectively. The set $\mathcal{A}^+$ is a cone that introduces the vector order, $\leq$, on $\mathcal{A}_h$ as follows:

$$a \leq b \quad \text{if} \quad b - a \in A^+ .$$

Let $\mathcal{A}^*$ stand for the dual of $\mathcal{A}$. The *self-adjoint element* of $\mathcal{A}^*$ is a functional that takes real values on $\mathcal{A}_h$. Let $\mathcal{A}_h^*$ be the set of all self-adjoint elements of $\mathcal{A}^*$. *Positive functional* on $\mathcal{A}$ is a functional φ of $\mathcal{A}$ such that $\varphi(a) \geq 0$, whenever $a \in \mathcal{A}^+$. The set $\mathcal{A}^{*+}$ of all positive functionals on $\mathcal{A}$ is a cone introducing the order, $\leq$, on $\mathcal{A}_h^*$. A positive functional is called a *state* if it has norm one. Further, a positive functional φ on $\mathcal{A}$ is called *faithful* if $\varphi(a^* a) = 0$ implies $a = 0$ for all $a \in \mathcal{A}$. Two positive functionals ψ and φ on $\mathcal{A}$ are called orthogonal if the following holds: If ϱ is a positive functional on $\mathcal{A}$ with $\varrho \leq \varphi$ and $\varrho \leq \psi$, then ϱ is zero.

A *$*$-isomorphism* between C^*-algebras is a linear bijection preserving the star operation and products. A *$*$-anti-isomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is a linear bijection preserving the star operation and reversing the product of elements, that is $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$. Finally, a *Jordan $*$-isomorphism* is a linear bijection $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras preserving the star operation and the Jordan product $(a, b) \to a \circ b = \frac{1}{2}(ab + ba)$:

$$\varphi(a \circ b) = \varphi(a) \circ \varphi(b)\,, \qquad \varphi(a^*) = \varphi(a)^*\,, \qquad a, b \in \mathcal{A}\,.$$

By *projection* of a C^*-algebra we mean the self-adjoint idempotent. The symbol $P(\mathcal{A})$ will denote the set of all projections of $\mathcal{A}$. Two projections p and q are called *orthogonal* if $pq = 0$. If $p \in P(\mathcal{A})$ then $p^\perp = \mathbf{1} - p$ is an *orthocomplement* of p. The structure $P(\mathcal{A})$ endowed with the order, $\leq$, and orthocomplementaion, $\perp$, forms an orthomodular poset. Two elements a, b in such a poset are said to be orthogonal if $a \leq b^\perp$. By an *orthoisomorphism* between orthomodular posets we mean a bijection preserving the orthogonality of elements in both directions.

It can be proved quite easily that any orthoisomorphism preservers automatically the unit, orthocomplementation and order in both directions.

C^*-algebra having a predual is called a *von Neumann algebra*. In the sequel $\mathcal{M}$ we will always denote a von Neumann algebra. A functional ω on $\mathcal{M}$ is called *normal* if it can be identified with an element of the predual of $\mathcal{M}$. Each positive normal functional φ on $\mathcal{M}$ admits the *support* that is a smallest projection, $s(\varphi)$, such that $\varphi(\mathbf{1}-s(\varphi)) = 0$. The double dual $\mathcal{A}^{**}$ of the C^*-algebra $\mathcal{A}$ is a von Neumann algebra and any element of $\mathcal{A}^*$ can be identified with a normal functional on $\mathcal{A}^{**}$. Two positive functionals on $\mathcal{A}$ are orthogonal if and only if their supports in $\mathcal{A}^{**}$ are orthogonal projections. Let us also recall that von Neumann algebra is said to be σ-finite if any system of nonzero mutually orthogonal projections in $\mathcal{M}$ is countable. $\mathcal{M}$ is σ-finite if and only if it admits a faithful normal state.

Basic principles of the theory of compact convex sets

Let K be a compact convex set in a locally convex topological space X

The symbol $C(K)$ will stand for the C^*-algebra of all continuous complex functions on K

Let $A(K)$ represent the set of all continuous affine functions on K

$P(K)$ ... the set of all continuous convex functions on K

Radon measure

By a *Radon measure* μ on K we mean an element of $C(K)^*$ canonically identified with regular Borel measure $d\mu$ on K in the sense of the formula

$$\mu(f) = \int_K f(\omega)\, d\mu(\omega) \qquad f \in C(K)\,.$$

$M^+(K)$ ... set of all positive Radon measures on K

$M_1^+(K)$ ... set of all probability Radon measures on K

The point $b(\mu) \in K$ is called the *barycenter* of $\mu \in M_1^+(K)$, if for each $a \in A(K)$

$$a(b(\mu)) = \int_K a(\omega)\, d\mu(\omega).$$

Measure μ is called *representing* for a given $x \in K$, if x is the barycenter of μ

The set $M_x^+(K)$ is the set of all representing measures of x

Let μ and ν be positive Radon measures. We define the relation $\mu \prec \nu$ as follows:

Choquet order

$$\mu \prec \nu, \quad \text{if} \;\; \mu(f) \leq \nu(f) \text{ for all } f \in P(K)\,.$$

The relation $\prec$ is a partial order on the set of positive Radon measures.

The order $\prec$ is called the Choquet order.

The convex theory to the state spaces of C^*-algebras

$\mathcal{S}(\mathcal{A})$ be the set of all states on C^*-algebra $\mathcal{A}$ endowed with the weak*-topology

φ be a state on $\mathcal{A}$

The triple $(\pi_\varphi, \xi_\varphi, \mathcal{H}_\varphi)$ will represent the GNS data of φ

$\mathcal{M}_\varphi = \pi_\varphi(\mathcal{A})''$ ... von Neumann algebra generated by $\pi_\varphi(\mathcal{A})$

Then $\mathcal{M}'_\varphi = \pi_\varphi(\mathcal{A})'$

Let C_φ be the space of all functionals in $\mathcal{A}^*$ spanned by positive functionals dominated by φ. In other words,

$$C_\varphi = lin\{\psi \in \mathcal{A}^{*+} \,|\, 0 \leq \psi \leq \varphi\}$$

There is a bijective positive map between C_φ and $\pi_\varphi(\mathcal{A})'$, sending each element $\psi \in C_\varphi$ to an operator $a'_\psi \in \mathcal{M}'_\varphi$ such that, for each $a \in \mathcal{A}$

$$\psi(a) = \langle a'_\psi \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$$

Let $\mu \in M^+_\varphi(\mathcal{S}(\mathcal{A}))$. For $f \in L^\infty(\mathcal{S}(\mathcal{A}), \mu)$ there is unique element $\theta_\mu(f) \in \mathcal{M}'_\varphi$, such that for each $a \in \mathcal{A}$

$$\langle \theta_\mu(f)\pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle = \int_{\mathcal{S}(A)} f(\omega)a(\omega)\, d\mu(\omega)$$

The measure $\mu \in M^+_\varphi(\mathcal{S}(\mathcal{A}))$ is called *orthogonal*, if for each Borel set $E \subseteq \mathcal{S}(\mathcal{A})$ the positive functionals φ_E and φ_{E^c} on $\mathcal{A}$ given by

$$\varphi_E(a) = \int_E a(\omega)\, d\mu(\omega),\ \varphi_{E^c}(a) = \int_{E^c} a(\omega)\, d\mu(\omega)$$

are orthogonal.
μ is an orthogonal measure if and only if θ_μ is a $*$-isomorphism that maps $L^\infty(\mathcal{S}(\mathcal{A}), \mu)$ onto von Neumann abelian subalgebra

$$C_\mu = \theta_\mu(L^\infty(\mathcal{S}(\mathcal{A}), \mu))$$

of $\mathcal{M}'_\varphi$.

Let we denote $O_\varphi(\mathcal{A})$ the set of all orthogonal measures in $M_\varphi^+(\mathcal{S}(\mathcal{A}))$

$O(\mathcal{A})$ the set of all orthogonal probability Radon meusures on $\mathcal{S}(\mathcal{A})$

By Θ_φ we will denote the map

$$\Theta_\varphi : O_\varphi(\mathcal{A}) \to \mathcal{V}(\mathcal{M}'_\varphi) : \mu \to C_\mu$$

Given a von Neumann algebra $\mathcal{M}$ we shall denote by $\mathcal{V}(\mathcal{M})$ the poset of all abelian von Neumann subalgebras containing the unit of $\mathcal{M}$. The order is given by the set theoretic inclusion.

The basic theorem for us is the following

Tomita Theorem

The map $\Theta_\varphi : \mu \to C_\mu$ is a bijection of $O_\varphi(\mathcal{A})$ onto $\mathcal{V}(\mathcal{M}'_\varphi)$. Moreover, the following conditions are equivalent for each $\mu, \nu \in O_\varphi(\mathcal{A})$.

1. $\mu \prec \nu$
2. $C_\mu \subseteq C_\nu$
3. $\int_{\mathcal{S}(\mathcal{A})} a(\omega)^2 \, d\mu(\omega) \leq \int_{\mathcal{S}(\mathcal{A})} a(\omega)^2 \, d\nu(\omega)$ for every $a \in \mathcal{A}$.

In particular, the posets $(O_\varphi(\mathcal{A}), \prec)$ and $(\mathcal{V}(\mathcal{M}'_\varphi), \subseteq)$ are order isomorphic.

According to the Tomita-Takesaki Theorem we have following

Theorem

Let φ be a faithful state on $\mathcal{A}$. Then the poset $(O_\varphi(\mathcal{A}), \prec)$ is isomorphic to the poset $(\mathcal{V}(\mathcal{M}'_\varphi), \subseteq)$.
Especially, if φ is a normal faithful state on a von Neumann algebra $\mathcal{M}$, then $(O_\varphi(\mathcal{M}), \prec)$ is isomorphic to $(\mathcal{V}(\mathcal{M}), \subseteq)$.

An important role in the poset $\mathcal{V}(\mathcal{M})$ is played by finite dimensional subalgebras. They are suprema of finitely many atoms in $\mathcal{V}(\mathcal{M})$. Atom in $\mathcal{V}(\mathcal{M})$ is the set $\{\mathbf{1}, \mathbf{0}, p, \mathbf{1} - p\}$, if $p \in \mathcal{M}^{pr}$ nontrivial

Besides, finite dimensional abelian subalgebras are related to so-called compact elements of the directed-complete posets of abelian subalgebras. Compact elements are important in the structure theory of these posets and related questions of information content.
C^*-algebra: finite dimensional abelian subalgebras are precisely compact elements in the directed-complete poset of abelian C^*-subalgebras (Heunen, Lindenhovius).

However, the structure of von Neumann abelian subalgebras behave differently in this respect.
von Neumann algebra: if $\mathcal{M}$ is an infinite dimensional von Neumann algebra, then there are finite dimensional elements of $\mathcal{V}(\mathcal{M})$ that are not compact (*Döring*, *Barbosa*).

Therefore, the following natural question arises: What kind of measures corresponds to finite dimensional abelian subalgebras?

Let φ be a state on $\mathcal{A}$.

By $O_\varphi^{fin}(\mathcal{A})$ we denote the set of all elements of $O_\varphi(\mathcal{A})$ that have finite support

Further, given a von Neumann algebra $\mathcal{M}$ we shall denote by $\mathcal{V}^{fin}(\mathcal{M})$ the set of all finite dimensional abelian subalgebras of $\mathcal{M}$ containing $\mathbf{1}$.

We denote by Θ_φ^{fin} the restriction of the map Θ_φ to $O_\varphi^{fin}(\mathcal{A})$

Theorem

Let φ be a faithful state on $\mathcal{A}$. Then the map

$$\Theta_\varphi^{fin} : \mu \to \mathcal{C}_\mu$$

is an order isomorphism between $(O_\varphi^{fin}(\mathcal{A}), \prec)$ and $(\mathcal{V}^{fin}(\mathcal{M}'_\varphi), \subseteq)$.

Especially, if φ is a faithful normal state on von Neumann algebra $\mathcal{M}$, then $(O_\varphi^{fin}(\mathcal{M}), \prec)$ is isomorphic to $(\mathcal{V}^{fin}(\mathcal{M}), \subseteq)$.

Since order isomorphisms of posets of abelian algebras have been studied intensely we can benefit from this analysis and Tomita type theorems to describe order isomorphisms for Choquet order.

Theorem

Let φ and ψ be states on C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let one of the following statements is true:
(i) $\mathcal{M}'_\varphi$ is a von Neumann algebra without type I_2 direct summand;
(ii) $\mathcal{M}_\varphi$ has no nonzero Type I direct summand.

Then for each order isomorphism $F : O_\varphi(\mathcal{A}) \to O_\psi(\mathcal{B})$ there is a unique Jordan isomorphism $J : \mathcal{M}'_\varphi \to \mathcal{M}'_\psi$ such that

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\varphi(\mu)]$$

for each $\mu \in O_\varphi(\mathcal{A})$.

A discrete variant of this result is also true.

Theorem

Let φ and ψ be states on C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let one of the following statements is true:

(i) $\mathcal{M}'_\varphi$ is a von Neumann algebra without type I_2 direct summand;

(ii) $\mathcal{M}_\varphi$ has no nonzero Type I direct summand.

Then for each order isomorphism $F : O_\varphi^{fin}(\mathcal{A}) \to O_\psi^{fin}(\mathcal{B})$ there is a unique Jordan $*$-isomorphism $J : \mathcal{M}'_\varphi \to \mathcal{M}'_\psi$ such that

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\varphi(\mu)]$$

for each $\mu \in O_\varphi^{fin}(\mathcal{A})$.

Let us recall that a state φ on a C^*-algebra $\mathcal{A}$ is Type I, II, or III if the von Neumann algebra $\mathcal{M}_\varphi = \pi_\varphi(\mathcal{A})''$ is of Type I, II, or III. As a corollary of the foregoing theorem we can see that whenever the state φ above is of Type II or III, then theorem holds. Theorem implies that equivalence of Choquet posets implies isomorphisms of commutant algebras as Jordan algebras. We show that in a relatively typical situation this can be said in terms of original algebras as well.

Let φ is a faithful normal state on a von Neumann algebra $\mathcal{M}$. Then π_φ is an isomorphism mapping $\mathcal{M}$ onto $\mathcal{M}_\varphi = \pi_\varphi(\mathcal{M})$. The vector ξ_φ is now cyclic both for $\mathcal{M}_\varphi$ and $\mathcal{M}'_\varphi$. By deep modular theory of von Neumann algebras there is a $*$-antiisomorphisms between $\mathcal{M}_\varphi$ and $\mathcal{M}'_\varphi$. It is clear that thanks to this map the posets $\mathcal{V}(\mathcal{M}_\varphi)$ and $\mathcal{V}(\mathcal{M}'_\varphi)$ are isomorphic. The same holds for $\mathcal{V}^{fin}(\mathcal{M}_\varphi)$ and $\mathcal{V}^{fin}(\mathcal{M}'_\varphi)$. Therefore by applying the previous results we obtain that $O_\varphi(\mathcal{M})$, and $O_\varphi^{fin}(\mathcal{M})$, where φ is normal and faithful, are Jordan invariant in many cases:

In case of σ-finite von Neumann algebras it turns out that Choquet poset is even a complete Jordan invariant.

Corollary

Let $\mathcal{M}$ and $\mathcal{N}$ be σ-finite von Neumann algebras with $\mathcal{M}$ not having Type I_2 direct summand. The following statements are equivalent:

1. There is a Jordan $*$-isomorphism between $\mathcal{M}$ and $\mathcal{N}$.
2. There are faithful normal states φ and ψ on $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $(O_\varphi(\mathcal{M}), \prec)$ and $(O_\psi(\mathcal{N}), \prec)$ are isomorphic as posets.
3. For all faithful normal states φ and ψ on $\mathcal{M}$ and $\mathcal{N}$, respectively, $(O_\varphi(\mathcal{M}), \prec)$ and $(O_\psi(\mathcal{N}), \prec)$ are isomorphic as posets.

Corollary

Let $\mathcal{A}$ and $\mathcal{B}$ be separable C^*-algebras. Let φ and ψ be states on a C^*-algebra $\mathcal{A}$ and $\mathcal{B}$, respetively, that are of Type III. The following statements are equivalent:

1. $O_\varphi(\mathcal{A})$ is isomorphic to $O_\psi(\mathcal{B})$.
2. $O_\varphi^{fin}(\mathcal{A})$ is isomorphic to $O_\psi^{fin}(\mathcal{B})$.
3. $\pi_\varphi(\mathcal{A})''$ and $\pi_\psi(\mathcal{B})''$ are isomorphic as Jordan algebras.

We can describe oder isomorphisms of the set of all orthogonal measures.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Suppose that for every state φ of $\mathcal{A}$ the algebra $\mathcal{M}'_\varphi$ has no Type I_2 direct summand. Let $F : O(\mathcal{A}) \to O(\mathcal{B})$ be an order isomorphism. Then there are bijection $\gamma : S(\mathcal{A}) \to S(\mathcal{B})$ and collection $(J_\varphi)_{\varphi \in S(\mathcal{A})}$ of Jordan $*$-isomorphisms between $\mathcal{M}'_\varphi$ and $\mathcal{M}'_{\gamma(\varphi)}$ such that for each $\mu \in O(\mathcal{A})$, we have

1. $b(\mu) = \varphi$ if and only if $b(F(\mu)) = \gamma(\varphi)$.
2. $F(\mu) = \Theta^{-1}_{\gamma(b(\mu))} J_{b(\mu)}[\Theta_{b(\mu)}(\mu)]$.

Finally, let me present now applications of the previous ideas to describing preservers of Choquet order of orthogonal measures that are invariant with respect to action of some group. Let G be a group. By action of G we mean a representation of G by $*$-automorphisms of the C^*-algebra $\mathcal{A}$. We denote the action of G by $g \to \tau_g$. A state ω on $\mathcal{A}$ is said to be *G-invariant* if

$$\omega(a) = \omega(\tau_g(a)) \qquad a \in \mathcal{A}, g \in G\,.$$

Let $\mathcal{S}^G(\mathcal{A})$ be the set of all G-invariant states. It is a compact convex subset of the state space. We shall call a measure $\mu \in M^+(\mathcal{S}(\mathcal{A}))$ G-invariant if it is supported by $\mathcal{S}^G(\mathcal{A})$.

Bratteli: μ is G-invariant if and only if it enjoys the following algebraic property:

$$\mu(\tau_g(f_1)\, f_2) = \mu(f_1\, f_2)\,,$$

for all $f_1, f_2 \in L^\infty(\mu)$ and $g \in G$.

Here $\tau_g(f)$ denotes the function $s \to f(\tau_{g^{-1}}(s))$.

For example, finitely supported measure is G-invariant if and only if it is a convex combination of Dirac measures concentrated at G-invariant states. In particular, their barycenters are G-invariant states as well.

Given a G-invariant state ω we shall denote by $O_\omega^G(\mathcal{A})$ (resp. $O_\omega^{G,fin}(\mathcal{A})$) the set of all orthogonal G-invariant probability measures having barycenter ω (resp. the set of all finitely supported orthogonal G-invarinat probability measures having barycenter ω). Let us fix a G-invariant state ω. It is well known that there is system of unitary operators $U(g)$, $g \in G$, acting on H_ω such that

$$U(g)\pi_\omega U(g)^{-1} = \pi_\omega(\tau_g(a))$$

$$U(g)\xi_\omega = \xi_\omega$$

for all $a \in \mathcal{A}$, $g \in G$. We shall denote by $\mathcal{M}_\omega^G$ von Neumann algebra acting on H_ω generated by $\mathcal{M}_\omega$ and the set $\{U(g) \mid g \in G\}$. That is, $\mathcal{M}_\omega^G = (\mathcal{M}_\omega \cup \{U(g) \mid g \in G\})''$. We shall now identify Choquet order of the sets of G-invariant measures with structure of abelian subalgebras of the commutant of $\mathcal{M}_\omega^G$ and use it for describing preservers of G-invariant measures.

Theorem

Let ω and ψ be a G-invariant states of $\mathcal{A}$ and $\mathcal{B}$, respectively. The following statements hold:

1. Suppose that $F : O_\omega^G(\mathcal{A}) \to O_\psi^G(\mathcal{B})$ is an isomorphisms with respect to Choquet order. Then there is a piecewise Jordan $*$-isomorphism $J : \mathcal{M}_\omega^{G'} \to \mathcal{M}_\psi^{G'}$ such that for all $\mu \in O_\omega^G(\mathcal{A})$

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\omega(\mu)].$$

2. Suppose that $\varphi : O_\omega^{G,fin}(\mathcal{A}) \to O_\psi^{G,fin}(\mathcal{B})$ is an isomorphisms with respect to the Choquet order. Then there is a piecewise Jordan $*$-isomorphism $J : \mathcal{M}_\omega^{G'} \to \mathcal{M}_\psi^{G'}$ such that for all $\mu \in O_\omega^{G,fin}(\mathcal{A})$

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\omega(\mu)].$$

The piecewise Jordan isomorphisms in the above statements are linear if the algebra $\mathcal{M}_\omega^{G'}$ has no Type I_2 direct summand.

Let us remark that whenever the commutant of covariance algebra $\mathcal{M}_\omega^{G\prime}$ is without Type I_2 direct summand then J in the previous theorem is a Jordan $*$-isomorphisms. As we showed above this is guaranteed when the covariance algebra is Type III and acting on separable Hilbert space. This is often satisfied for dynamical systems occuring in quantum field theory. Let us remark in conclusion that if the group G is trivial then $\mathcal{M}_\omega^G = \mathcal{M}_\omega$. Therefore previous investigation concerns special example of the discussed structure.

- *Bratteli*, *O.*, *Robinson*, *D.W.*: Operator Algebras and Quantum Statistical Mechanics, vol 1, 2. Springer, Berlin (1997)
- Davidson K., Kennedy M. Choquet order and hyperrigidity for function systems.– arXiv:1608.02334v1, 8 August 2016.
- Hamhalter J., Turilova E. Orthogonal measures on state spaces and context structures of quantum theory. – International Journal of Theoretical Physics. – 2016. – V. 55. – N 7.– P. 3353–3365.
- *Takesaki*, *M.*: Theory of Operator Algebras I, II, III. Springer, Berlin (2001)