Ortholattices and their automorphisms: towards a simple description of the Hilbert space

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Once upon a time ...

The Birkhoff - von Neumann legacy

Can we derive the basic model of quantum theory

the complex Hilbert space,

from some underlying logico-algebraic structure?

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- Accordingly, the closed subspaces of a Hilbert space form an orthoposet – in fact an orthomodular lattice.

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An outdated, irrelevant approach, or still a useful viewpoint on quantum theory??

In the meantime ...

... numerous further (more sophisticated?) approaches were proposed.

- G. Niestegg, Composite systems and the role of complex numbers in quantum mechanics.
- F. M. Lev, Why is quantum physics based on complex numbers?
- S. Davis, Quantum theory and the category of complex numbers.
- A. Ivanov, D. Caragheorgheopol, Spectral automorphisms in quantum logics.
- J. Vicary, Completeness of †-categories and the complex numbers.
- V. Moretti, M. Oppio, Quantum theory in real Hilbert space: how the complex Hilbert space structure emerges from Poincaré symmetry.

An old approach with a still unexhausted potential

Our relaxation of the issue

Does the complex Hilbert space arise from any simpler, easier comprehensible structure?

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Peculiarities of the logico-algebraic approach:

- It deals with "propositions" but not with actual physical contents.
- It deals with the model of a single system, not caring about interrelations.

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Peculiarities of the logico-algebraic approach:

- It deals with “propositions” but not with actual physical contents.
- It deals with the model of a single system, not caring about interrelations.

Charmingness of the approach:

- With some effort, we may reconstruct precisely the model in question.

Definition

Let E be an abelian group, let K be a division ring, and assume that an action of K on E makes E into a left K-module.

Linear spaces

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Related issue

Why is this notion so fundamental?
Can we reduce this structure to something more tangible?

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Can we reduce this structure to something more tangible?

We can identify linear spaces with certain lattices.

Lattices

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A lattice is a partially ordered set such that the greatest lower bound and least upper bound of any pair of elements exist.

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Example

The set $\Pi(X)$ of equivalence relations on a set X, partially ordered by inclusion, is a lattice.

For $R, S \in \Pi(X)$, we have

$$\begin{aligned} R \wedge S &= R \cap S, \\ R \vee S &= R;S \,\cup\, R;S;R \,\cup\, R;S;R;S \,\cup\, \ldots \qquad (\star). \end{aligned}$$

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Theorem

Every lattice L is a lattice of equivalence relations.
L is linear iff $(\star)$ is "$R \vee S = R;S$".
L is modular iff $(\star)$ is "$R \vee S = R;S;R$".

Theorem

Let E be a linear space over a division ring K.
Then, under inclusion, the set $\mathcal{L}(E)$ of subspaces of E is a geomodular lattice:
a complemented modular lattice which is moreover compactly atomistic and irreducible.

Linear spaces and lattices

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Then, under inclusion, the set $\mathcal{L}(E)$ of subspaces of E is a geomodular lattice:
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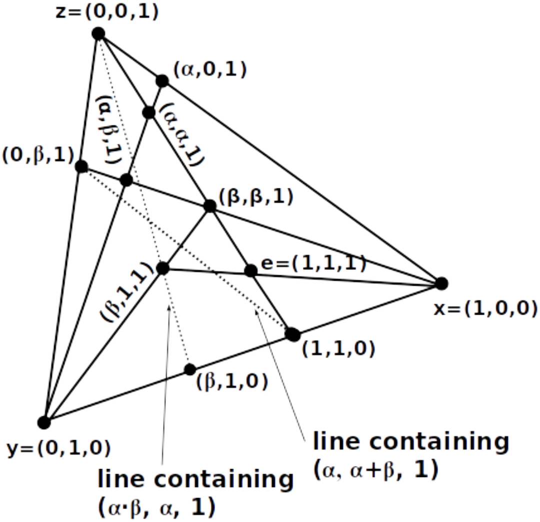
Theorem (Birkhoff, Frink, Jónsson)

Let L be a geomodular lattice of dimension $\geqslant 4$.
Then there is a linear space E over a division ring K such that L is isomorphic to $\mathcal{L}(E)$.
K is (up to isomorphism) uniquely determined.

Coordinatising geomodular lattices



Pairing a linear space with its dual

Definition

Let E be a linear space over a division ring K.
Assume that K possesses an antiautomorphism *.
Then a mapping $(-,-)\colon E \times E \to K$ is called
an anisotropic quadratic form if:

- $x \mapsto (-,x)$ is an antilinear mapping from E to its dual $E^\star$;

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- $x \mapsto (-,x)$ is an antilinear mapping from E to its dual $E^\star$;
- $(x,x) \neq 0$ if $x \neq 0$;
- $(y,x) = 0$ iff $(x,y) = 0$.

Hermitean spaces

Theorem

By "rescaling", an anisotropic quadratic form becomes hermitean:

- * is involutorial;
- $(x, y) = (y, x)^\star$,

A hermitean space is a linear space
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For a subspace A, we put

$$A^\perp \;=\; \{x \in E \colon (x, a) = 0 \text{ for all } a \in A\}.$$

The set of closed subspaces of E is

$$\mathcal{C}(E) = \{A \in \mathcal{L}(E) \colon A^{\perp\perp} = A\}.$$

Ortholattices

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A lattice L is called AC if L is atomistic and fulfils the covering property.

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Let E be a hermitean space.
Then $\mathcal{C}(E)$ is an irreducible, complete, AC ortholattice.

Hermitean spaces and ortholattices

Theorem (Birkhoff - von Neumann)

Let E be a linear space of dimension $4 \leqslant n < \omega$ over a division ring K. Assume that ${}^\perp\colon \mathcal{L}(E) \to \mathcal{L}(E)$ is an orthocomplementation.

Then there exists an involutorial antiautomorphism * of K and a hermitean form on E inducing ${}^\perp$.

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Theorem (cf. Maeda-Maeda)

An irreducible, complete, AC ortholattice of dimension $\geqslant 4$ is isomorphic to $\mathcal{C}(E)$ for some hermitean space E.

Orthomodular spaces

Definition

An orthomodular space is a hermitean space such that all closed subspaces are splitting:

$$E = A + A^{\perp}$$

for any $A \in \mathcal{C}(E)$.

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An irreducible, complete, AC orthomodular lattice
of dimension $\geqslant 4$ is isomorphic to $\mathcal{C}(E)$
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Theorem (Solèr)

Let E be an orthomodular space over the division ring K. Assume that E contains an infinite orthonormal sequence. Then K is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

We say that a lattice fulfils Pappus's Theorem if:

for any atoms $a_0, a_1, a_2, b_0, b_1, b_2$ belonging to a plane,

$(a_1 \vee b_0) \wedge (a_0 \vee b_1) \leqslant ((a_2 \vee b_0) \wedge (b_2 \vee a_0)) \vee ((a_2 \vee b_1) \wedge (a_1 \vee b_2))$.

We say that a lattice fulfils the Square Root Axiom if:

for any four distinct atoms a, b, c, d such that $a \vee b = c \vee d$, there are atoms y and z such that
$y \nleq b \vee z$, $y \nleq c \vee d$, $z \nleq c \vee d$, $z \nleq a \vee y$,
a, b, c, d, y, z belong to a plane, and

$$(a \vee y) \wedge (z \vee d) \leqslant ((b \vee z) \wedge (c \vee y)) \vee ((a \vee b) \wedge (z \vee y)).$$

The complex Hilbert spaces and lattices

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Theorem (Wilbur)

An irreducible, complete AC orthomodular lattice of infinite dimension and fulfilling Pappus's Theorem and the Square Root Axiom is isomorphic to $\mathcal{C}(E)$ for a complex Hilbert space E.

Review of the lattice-theoretic conditions

For an $\aleph_0$-dimensional complex Hilbert space, $(\mathcal{C}(E); \cap, \vee, ^\perp, \{0\}, E)$ is an irreducible, complete AC OML fulfilling Pappus's Theorem and the Square Root Axiom.

Review of the lattice-theoretic conditions

For an $\aleph_0$-dimensional complex Hilbert space, $(\mathcal{C}(E); \cap, \vee, ^{\perp}, \{0\}, E)$ is an irreducible, complete AC OML fulfilling Pappus's Theorem and the Square Root Axiom.

$\mathcal{C}(E)$	physical interpretability?
lattice operations	for compatible propositions only
orthomodularity	yes
atomisticity	for finite-state systems
covering property	not obvious
completeness	not obvious
irreducibility	reasonable
Pappus's Theorem	not obvious
Square Root Axiom	not obvious

Correspondence with the algebraic environment

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The lattice-theoretic reconstruction is struggling with the most common notions/constructions regarding Hilbert spaces:

$\to$: (general) linear maps;

$\oplus$: direct sums;

$\otimes$: tensor products.

But – we have a correspondence for

$\rightleftarrows$: automorphisms.

Modifications and extensions

Can we do better?

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Strategy:

- We no longer insist on "purely algebraic" conditions; we allow statements concerning **automorphisms** instead.

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- Conditions/operations that are physically hard to interpret and/or technically horribly involved, should be discarded.
- We should overcome the restriction to the infinite-dimensional case.

Strategy:

- We no longer insist on "purely algebraic" conditions; we allow statements concerning **automorphisms** instead.
- We replace ortholattices with **orthogonality spaces**.

The power of postulates regarding automorphisms

Let E be an orthogonality space over the division ring K.

Let $C_1(K) \; = \; \{r \in K \colon r\, r^\star = 1$ and r is in the centre of $K\}$.

If $A, A^\perp \in \mathcal{C}(E)$ have dimension $\geqslant 2$, the group of automorphisms fixing A and $A^\perp$ is $C_1(K)$.

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Theorem (Mayet)

Let L be an irreducible, complete, AC orthomodular lattice. Let $a, b, c \in L$ be pairwise orthogonal and of height $\geqslant 3$, and assume that there is an automorphism φ of L such that

- $\varphi(c) < c$,
- $\varphi(x) = x$ if $x \leqslant a$ or $x \leqslant b$,
- φ restricted to $[0, a \vee b]$ is not involutive.

Then L is isomorphic to $\mathcal{C}(H)$ for a complex Hilbert space H.

Orthogonality spaces

Definition (Foulis)

An orthogonality space is a set X endowed with a symmetric, irreflexive binary relation $\perp$.

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Standard example

Let X be the set of unit vectors of a Hilbert space. Put $x \perp y$ if $(x, y) = 0$. Then X is an orthogonality space.

Associated ortholattice

Let X be an orthogonality space.

For $A \subseteq X$, we put again

$$A^{\perp} \;=\; \{x \in X \colon x \perp a \text{ for all } a \in A\}.$$

The set of all orthoclosed subsets is

$$\mathcal{C}(X, \perp) \;=\; \{A \subseteq X \colon A^{\perp\perp} = A\}.$$

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Theorem

$\mathcal{C}(X, \perp)$ is a complete, atomistic ortholattice.

Automorphisms of orthogonality spaces

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Observation

Each automorphism of $(X, \perp)$ induces an ortholattice automorphisms of $\mathcal{C}(X, \perp)$.

Postulates regarding automorphisms

For an orthogonality space, $(X, \perp)$ we consider the properties:

(F1) For any $A \subseteq X$ and $e \in X$ be such that $e \notin A^{\perp\perp}$, there is an automorphism $\varphi\colon X \to X$ such that

- $\varphi(e) \perp A$,
- $\varphi(x) = x$ for any $x \in X$ such that $x \perp A, e$ or $x \perp A, \varphi(e)$.

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(F2) Let $A \subseteq X$ contain more than one element. For any automorphism φ fixing $A^{\perp\perp}$, and for any $n \geqslant 2$, there is an automorphism ψ such that

- $\psi^n = \varphi$,
- ψ fixes A as well.

Consequences of (F1)

Lemma

Let $(X, \perp)$ be an orthogonality space fulfilling (F1).
Then $\mathcal{C}(X, \perp)$ is atomistic and fulfils the covering property, that is, is AC.

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Theorem (Dacey)

$\mathcal{C}(X, \perp)$ is orthomodular if and only if:

- For a maximal orthogonal subset D of $A \in \mathcal{C}(X, \perp)$, we have $D^{\perp\perp} = A$.

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Lemma

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Then $\mathcal{C}(X, \perp)$ is an orthomodular lattice.

Rank of an orthogonality space

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Lemma

Let $(X, \perp)$ fulfil (F1).
The rank of $(X, \perp)$ is the height of $\mathcal{C}(X, \perp)$.

Irreducibility of an orthogonality space

Definition

$(X, \perp)$ is reducible if there is an $A \subseteq X$ such that
$A^\perp$ and $A^{\perp\perp}$ are both non-empty and
each $x \in X$ is contained in exactly one of $A^\perp$ or $A^{\perp\perp}$.

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Otherwise, we will call $(X, \perp)$ irreducible.

Lemma

If $(X, \perp)$ is irreducible, $\mathcal{C}(X, \perp)$ is directly irreducible.

Theorem

Let the orthogonality space $(X, \perp)$ be
of rank $\geqslant 4$ and irreducible, and let X fulfil (F1).

Then there is an orthogonality space E such that $\mathcal{C}(X, \perp)$ is isomorphic to $\mathcal{C}(E)$.

Description of the $\aleph_0$-dimensional complex Hilbert space

Theorem

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Theorem

Let the orthogonality space $(X, \perp)$ be of rank $\aleph_0$ and irreducible, and let X fulfil (F1) and (F2).

Then $\mathcal{C}(X, \perp)$ is isomorphic to $\mathcal{C}(H)$ for an $\aleph_0$-dimensional complex Hilbert space H.

The finite-dimensional case

A modified automorphism axiom

For an orthogonality space, $(X, \perp)$ we consider the property:

(F1') Let $e, f \in X$ be distinct. Then there is an automorphism $\varphi\colon X \to X$ such that

- $\varphi(e) \perp f$,
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- for any $x \perp f$, we have $x \perp e$ iff $x \perp \varphi(e)$, in which case $\varphi(x) = x$.

Theorem

Let the orthogonality space $(X, \perp)$ be of rank $4 \leqslant n < \omega$ and irreducible, and let X fulfil (F1').

Then there is an n-dimensional orthogonality space E such that $\mathcal{C}(X, \perp)$ is isomorphic to $\mathcal{C}(E)$.

Conclusion so far

- The infinite-dimensional complex Hilbert space can be described as an orthogonality space – essentially by means of postulates concerning automorphisms.
- The finite-dimensional orthogonal spaces can be described similarly.

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The remaining challenge

In the finite-dimensional case,
how can we characterise the complex numbers??

Let us compile some facts and ideas ...

Let us recall:

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Main proof steps:

(1) The fixed field $F = \{\alpha \in K : \alpha^\star = \alpha\}$ is totally ordered, the positive cone being $\{(x, x) : x \in E\}$.

(2) $(F; \leqslant)$ is an archimedean totally ordered group.

(3) The order of F is complete.

Review of the crucial part

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(2) $(F; \leqslant)$ is an archimedean totally ordered group.

(3) The order of F is complete.

Arguments of (2) and (3) rely essentially on the infinite dimensionality. No hope.

Interestingly, however, **topology** has an effect similar to infinite-dimensionality:

Theorem (Pontrjagin)

Let K be a topological division ring.
Assume that K is locally compact, connected, and Hausdorff.
Then K is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

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Theorem (Salzmann, Löwen)

A compact, connected topological projective plane whose automorphism group is transitive is classical.

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Question

How can we use these facts to characterise the right "topological orthogonality spaces"?

Delimiting the division ring by other means

Let $(X, \perp)$ be an orthogonality space
that is irreducible, of finite rank $\geqslant 4$, and fulfils (F1').
Let E be the representing orthogonal space over K.

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Theorem (Eckmann, Zabey)

Let V be a linear space over $\mathrm{GF}(p^d)$ of finite dimension $\geqslant 3$.
Then $\mathcal{L}(V)$ does not possess an orthocomplementation.

Delimiting the division ring by other means

Let $(X, \perp)$ be an orthogonality space
that is irreducible, of finite rank $\geqslant 4$, and fulfils (F1').
Let E be the representing orthogonal space over K.

Theorem (Eckmann, Zabey)

Let V be a linear space over $\mathrm{GF}(p^d)$ of finite dimension $\geqslant 3$.
Then $\mathcal{L}(V)$ does not possess an orthocomplementation.

Corollary

The division ring K is infinite.

Characteristic 0

Theorem (Jones)

Let V be an orthogonal space of dimension $\geqslant 4$ such that:

- For any $e, f \in V$ there is an automorphism φ such that $\varphi(e) = f$ and $\varphi(x) = x$ whenever $x \perp e, f$.

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K has characteristic 0.

Commutativity

Consider the property:

(F3) Let $u, v \in X$ such that $u \perp v$ and let $w \in \{u, v\}^{\perp\perp}$ be distinct from u, v. Let φ be an automorphism fixing u, v, w, and any $x \perp u, v$. Then φ fixes also any $x \in \{u, v\}^{\perp\perp}$.

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Lemma

If $(X, \perp)$ fulfils (F3), K is commutative.

Recall:

(F2) For any automorphism φ fixing some at least 2-dimensional $A \in \mathcal{C}(X, \perp)$, and for any $n \geqslant 2$, there is an automorphism ψ such that $\psi^n = \varphi$ and ψ fixes A as well.

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Lemma

If $(X, \perp)$ fulfils (F1'), (F2), and (F3), then $K = F(i)$ for some $i \in K$ such that $i^2 = 1$.

Automorphism circles

Definition

Let $\mathbb{T}$ be the group of unit complex numbers.
An automorphism circle is a homomorphism κ from $\mathbb{T}$ to the group of automorphisms of $(X, \perp)$ such that $\kappa(\alpha) = \mathrm{id}$ iff $\alpha = \pm 1$.

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Axioms for automorphism circles

(A1) Let $u, v \in X$ be such that $u \perp v$ and let φ be an automorphism of $(X, \perp)$ such that $\varphi(u) = v$ and $\varphi(x) = x$ if $x \perp u, v$. Then there is an automorphism circle κ fixing the same elements as φ and such that $\kappa(i) = \varphi$.

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(A2) Let $u, v \in X$ be such that $u \perp v$. Let κ be an automorphism circle such that $\kappa(i)(u) = v$ and $\kappa(i)(x) = x$ for $x \perp u, v$. Then any other automorphism circle with these properties is of the form $\varphi^{-1} \circ \kappa \circ \varphi$, where φ is an automorphism fixing u, v, and any $x \perp u, v$.

Automorphism circles

Axioms for automorphism circles, ctd.

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Theorem

Let $(X, \perp)$ fulfil (F1')–(F3) and (A1)–(A3). Then K is generated by $\{\alpha \in K : \alpha\,\alpha^\star = 1\}$, and this multiplicative subgroup is isomorphic to $\mathbb{T}$.

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- Orthogonality spaces together with postulates regarding automorphisms have proved to be a not too bad possibility.
- Easy axiomatics does not mean easy structure – what about concrete representations?

Lattices and equivalence relations

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The OMLs of interest arise in a similar fashion.

Theorem (Harding)

Let $\Pi(X)$ be the lattice of equivalence relations on a set X, partially ordered by inclusion.
Then $0 = \{\{x\} \colon x \in X\}$ and $1 = \{X\}$. Let

$$\begin{aligned} \mathrm{Fact}(X) &= \\ &\{(R,S) \in \Pi(X)^2 \colon R \text{ and } S \text{ commute},\ R \cap S = 0,\ R \vee S = 1\}, \end{aligned}$$

and define

$$(R_1, S_1) \leqslant (R_2, S_2) \quad \text{if} \quad R_1 \subseteq R_2,\ S_2 \subseteq S_1,\ R_1 \text{ and } S_2 \text{ commute}.$$

Then $\mathrm{Fact}(X)$ is an orthomodular poset.

Vision

Observation (Harding)

Let E be an orthogonal space.
The ortholattice $\mathcal{C}(E)$ is embeddable into $\mathrm{Fact}(E)$:
Each $A \in \mathcal{C}(E)$ gives rise to a product decomposition of E.

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Question

How do the results presented here relate to representations by means of set decompositions?

Theorem

Let the orthogonality space $(X, \perp)$ be of rank $\aleph_0$ and irreducible, and let X fulfil (F1) and (F2).

Then $\mathcal{C}(X, \perp)$ is isomorphic to $\mathcal{C}(H)$
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- Is there a proof of the coordinatisation of (eligible) ortholattices that takes into account the orthostructure?
- What about orthogonality spaces and common constructions?
- Do the postulates on automorphisms have a "good" interpretation?
- Proceed analogously for other structures than orthogonality spaces: e.g., test spaces, orthoalgebras, partial Boolean algebras.