Effectus theory

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Teaser, another reconstruction of QT:

an operational f-effectus (a subcategory of) is (finite-dimensional) Euclidian Jordan Algebras (with positive maps)

follows from [Wetering, arXiv:1801.05798]

People

Jacobs

Cho

A. Westerbaan

$(\neq)$

B. W.

People

Jacobs

Cho

A. Westerbaan

$(\neq)$

B. W.

Tull

connection with OPTs

Wetering

Reconstruction

Adams

Type theory

Oxford CQM	Effectus theory
f.d. Hilbert spaces	von Neumann algebras
$\mathbb{C}^2$ (qubit), $\mathbb{C}^3$ (qutrit), ...	$\mathbb{C}^2$ (bit), M_2 (qubit), ...

Oxford CQM

f.d. Hilbert spaces

$\mathbb{C}^2$ (qubit), $\mathbb{C}^3$ (qutrit), …

operators

$\boxed{H}$, , …

Effectus theory

von Neumann algebras

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contractive normal c.p. maps

$a \mapsto \sum_i b_i^* a b_i$, …

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$\otimes$ parallel composition

$\mathbb{C}^2 \otimes \mathbb{C}^2$: two qubits

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$+$ probabilistic disjunction

$\mathbb{C}^2 \oplus M_2$: bit or qubit

Oxford CQM

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, , ...

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$\mathbb{C}^2 \otimes \mathbb{C}^2$: two qubits

Expressive calculus for 'circuits'

Works best finite-dimensionally

Effectus theory

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$\mathbb{C}^2 \oplus M_2$: bit or qubit

Hard to reason about circuits

Measurement, classical data and infinite dimensions built in.

Maps between (vN) algebras go the opposite way

measure in std. basis

$$\mathbb{C}^2 \longrightarrow M_2$$

(bit $\longleftarrow$ qbit)

$$(\lambda, \mu) \longmapsto \lambda|0\rangle\langle 0| + \mu|1\rangle\langle 1|$$

Maps between (vN) algebras go the opposite way

measure in std. basis	$\mathbb{C}^2 \longrightarrow M_2$ (bit $\longleftarrow$ qbit)	$(\lambda, \mu) \longmapsto \lambda\|0\rangle\langle 0\| + \mu\|1\rangle\langle 1\|$
initialize as 0	$M_2 \longrightarrow \mathbb{C}$ (qbit $\longleftarrow$ 1)	$a \longmapsto \langle 0\|a\|0\rangle$

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CPTP

$$B(\mathcal{H}) \to B(\mathcal{K})$$

$$\rho \longmapsto \sum_i b_i^* \rho\, b_i$$

cpu-map

$$B(\mathcal{K}) \longrightarrow B(\mathcal{H})$$

$$a \longmapsto \sum_i b_i\, a\, b_i^*$$

Overview

operational
†-effectus

is a

†-effectus

Overview

operational
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&-effectus

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◇-effectus

Overview

operational

$+$-effectus

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$+$-effectus

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&-effectus

is a

$\Diamond$-effectus

is an

effectus

Overview

operational
$+$-effectus

is a

$+$-effectus — adds $+$

is an

$\&$-effectus — adds $\&$

is a

$\diamond$-effectus — adds $(\ )^{\diamond}$, $(\ )_{\diamond}$

is an

effectus

An Effectus has

- objects $A, B, X, Y, \ldots$ representing data types / systems

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- Coproduct $A+B$ representing (probabilistic) disjunction

$$
\begin{tikzcd}
A \arrow[r, "\kappa_1"] \arrow[dr, bend right, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\kappa_2"'] \arrow[dl, bend left, "g"] \\
& C &
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$\mathrm{swap} \equiv [\kappa_2, \kappa_1] : A+A \longrightarrow A+A$

$\nabla \equiv [id, id] : A+A \longrightarrow A$

Predicates and partial maps

$p : X \to 1+1$ predicate on X

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the truth predicate on X

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composition of partial maps

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a map as a (total) partial map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\hat{f}"'] & Y \arrow[r, "\kappa_1"] & Y+1 \end{tikzcd}$$

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$$\boxed{\begin{aligned} 1 \hat{\circ} f = 1 &\implies f \text{ total} \\ 1 \hat{\circ} f = 0 &\implies f = \kappa_2 \circ ! \end{aligned}}$$

Scalars and states

$\lambda : 1 \longrightarrow 1+1$ a scalar (so partial map $1 \longrightarrow 1$)

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$$\begin{tikzcd} 1 \arrow[r, "\lambda"] \arrow[rr, bend right, "\lambda\omega \ovee \lambda^{\perp}\psi"'] & 1+1 \arrow[r, "{[\omega,\psi]}"] & X \end{tikzcd}$$

convex combination of states

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convex combination of states

$$\begin{tikzcd} 1 \arrow[r, "\omega"] \arrow[rr, bend right, "p \circ \omega"'] & X \arrow[r, "p"] & 1+1 \end{tikzcd}$$

validity of a predicate in a state

Partial sum of predicates

If for predicates p, q on X, there is a b with

$$
\begin{tikzcd}
 & X \arrow[dl, bend right, "p"'] \arrow[d, "b"] \arrow[dr, bend left, "q"] & \\
1+1 & 1+1+1 \arrow[l, "{[\kappa_1,\kappa_2,\kappa_2]}"] \arrow[r, "{[\kappa_2,\kappa_1,\kappa_2]}"'] & 1+1
\end{tikzcd}
$$

then p and q are summable (in symbols: $p \perp q$)
and their sum $p \ovee q$ is given by

$$
\begin{tikzcd}
X \arrow[d, "b"] \arrow[dd, bend right=60, "p \ovee q"'] \\
1+1+1 \arrow[d, "{[\kappa_1,\kappa_1,\kappa_2]}"] \\
1+1
\end{tikzcd}
$$

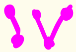
Definition an *Effectus* is a category $\mathcal{C}$ with

- finite coproducts and final object 1,
- where all diagrams of the form

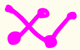
$$\begin{tikzcd}
X+Y \arrow[r, "id+!"] \arrow[d, "!+id"'] & X+1 \arrow[d, "!+id"] \\
1+Y \arrow[r, "id+!"'] & 1+1
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "!"] \arrow[d, "\kappa_1"'] & 1 \arrow[d, "\kappa_1"] \\
X+Y \arrow[r, "!+!"'] & 1+1
\end{tikzcd}$$

are pullback squares and

- the following arrows are jointly monic



$$\begin{tikzcd}
1+1+1 \arrow[r, shift left] \arrow[r, shift right] & 1+1
\end{tikzcd}$$



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$$\begin{tikzcd} X+Y \arrow[r, "id+!"] \arrow[d, "!+id"'] & X+1 \arrow[d, "!+id"] \\ 1+Y \arrow[r, "id+!"'] & 1+1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\kappa_1"'] & 1 \arrow[d, "\kappa_1"] \\ X+Y \arrow[r, "!+!"'] & 1+1 \end{tikzcd}$$

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$$\begin{tikzcd} 1+1+1 \arrow[r, shift left] \arrow[r, shift right] & 1+1 \end{tikzcd}$$

Examples: vN^{op}, $\mathbf{Set}$, $\mathbf{CRng}^{op}$, $\mathcal{K}\ell(\mathcal{D})$, any topos, $\mathbf{EA}^{op}$, ...

Structure in an effectus

M, the set of scalars is an effect monoid,
that is: an effect algebra with biadditive product
for which 1 is a unit.

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$$\begin{tikzcd}
\mathrm{Conv}_{M^{op}} & & \mathrm{EMod}_M^{op} \\
& \mathcal{C} \arrow[ul, "\mathrm{Stat}"] \arrow[ur, "\mathrm{Pred}"'] &
\end{tikzcd}$$

Definition a category C is an Effectus in partial form if

1. C is a finPAC — that is
   a. C has coproducts
   b. C is PCM-enriched, i.e.:
      α. every $\mathrm{Hom}(X,Y)$ has partial binary operation $\ovee$ and distinguished map 0 that turn it into a PCM
      β. $f \perp g \Rightarrow \left[\begin{array}{ll} h\circ f \perp h\circ g & (h\circ f)\ovee(h\circ g) = h\circ(f\ovee g) \\ f\circ k \perp g\circ k & (f\circ k)\ovee(g\circ k) = (f\ovee g)\circ k \end{array}\right.$
   c. $\rhd_1 \circ b \perp \rhd_2 \circ b$ for any $b: X \to Y+Y$,
      where $\rhd_1 \equiv [id, 0] : Y+Y \to Y$ and $\rhd_2 \equiv [0, id]$.
   d. $f \perp g \Rightarrow \kappa_1 \circ f \perp \kappa_2 \circ g$
2. C "has effects" — that is: there is an object I such that
   a. the PCM $\mathrm{Hom}(X, I) \equiv \mathrm{Pred}\, X$ is an effect algebra $\forall X$.
   b. $1\circ f \perp 1\circ g \Rightarrow f \perp g$
   c. $1\circ f = 0 \Rightarrow f = 0$.

A map f is called total iff $1\circ f = 1$

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     - α. every Ho... and ar...
     - β. $f \perp g \Rightarrow$ ... $) = h \circ (f \circledcirc g) = (f \circledcirc g) \circ k$
   - c. $\rhd_1 \circ b \perp \rhd_2 \circ$ ... where $\rhd_1 \equiv$ ... $\rhd_2 \equiv [0, id]$.
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A map f is called **total** iff $1 \circ f = 1$

> $(M, \circledcirc, 0)$ with $\circledcirc : M^2 \to M$, $0 \in M$ is a **PCM** iff
> - $x \perp y \Rightarrow \begin{cases} y \perp x \\ x \circledcirc y = y \circledcirc x \end{cases}$
> - $0 \perp x$ and $0 \circledcirc x = x$
> - $x \perp y$ and $x \circledcirc y \perp z \Rightarrow \begin{cases} y \perp z, \quad x \perp y \circledcirc z \\ (x \circledcirc y) \circledcirc z = x \circledcirc (y \circledcirc z) \end{cases}$

Definition a category C is an **Effectus in partial form** if

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   b. C is PCM- ... peration $\oslash$ into a PCM

   α. every Hc ... and a ... $) = h \circ (f \oslash g)$ $= (f \oslash g) \circ k$

   β. $f \perp g \Rightarrow$ ...

   c. $\triangleright_1 \circ b \perp \triangleright_2 \circ$ ... where $\triangleright_1 \equiv$ ... $\triangleright_2 \equiv [0, id]$.

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$(M, \oslash, 0)$ with $\oslash : M^2 \to M$, $0 \in M$ is a **PCM** iff

- $x \perp y \Rightarrow \begin{cases} y \perp x \\ x \oslash y = y \oslash x \end{cases}$
- $0 \perp x$ and $0 \oslash x = x$
- $x \perp y$ and $x \oslash y \perp z \Rightarrow \begin{cases} y \perp z, \; x \perp y \oslash z \\ (x \oslash y) \oslash z = x \oslash (y \oslash z) \end{cases}$

2. C ... object I such that ... an effect algebra $\forall X$.

   a.

   b.

   c.

A map f

A PCM is an **Effect Algebra** iff

- for every x, there is a unique $x^\perp$ such that
  $$x^\perp \oslash x = 1 \equiv 0^\perp$$
- $x \perp 1 \Rightarrow x = 0$

They're the "same"

If C is an effectus, then $\mathrm{Par}\, C$, its category of partial maps, is an effectus in partial form.
If D is an effectus in partial form, then $\mathrm{Tot}\, D$, its subcategory of total maps is an effectus.

Furthermore $\mathrm{Par}\,\mathrm{Tot}\, D \cong D$
and $\mathrm{Tot}\,\mathrm{Par}\, C \cong C$.

◊-effectus

$\Diamond$-effectus preparation: quotients

An effectus (in partial form) has **quotients** if:

For every predicate $p : X \to 1$, there is an obj. $X/p^\perp$ and (partial) map $\xi_{p^\perp} : X \to X/p^\perp$ with $1 \circ \xi_{p^\perp} \le p$, such that for any other $f : X \to Y$ with $1 \circ f \le p$, there is a unique g with

$$\begin{tikzcd} X \arrow[r, "\xi_{p^\perp}"] \arrow[dr, "f"'] & X/p^\perp \arrow[d, dashed, "g"] \\ & Y \end{tikzcd}$$

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(In vN, $\xi : \lceil p \rceil A \lceil p \rceil \to A$ given by $\xi(a) = \sqrt{p}\, a \sqrt{p}$)

◇-effectus preparation: comprehension

An effectus (in partial form) has comprehension if

For every predicate $p : X \to 1$, there is an obj. $\{X|p\}$ and (partial) map $\pi_p : \{X|p\} \to X$ with $p \circ \pi_p = 1 \circ \pi_p$ such that for any other $f : Y \to X$ with $p \circ f = 1 \circ f$, there is a unique g with

$$\begin{tikzcd}
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Y \arrow[u, dashed, "g"] \arrow[ur, "f"'] &
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(In vN, $\pi : A \to \lfloor p \rfloor A \lfloor p \rfloor$ given by $\pi(a) = \lfloor p \rfloor a \lfloor p \rfloor$)

Aside: why these names?

For effectus C, define category $\int \mathrm{Pred}_\Box$ by

- objects: pairs (X, p), X object in C, $p \in \mathrm{Pred}\,X$
- an arrow $(X,p) \longrightarrow (Y,q)$ is a map $f : X \longrightarrow Y$ in $\mathrm{Par}\,C$ with $p \leq (q^{\perp} \circ f)^{\perp}$

There is an obvious $u : \int \mathrm{Pred}_\Box \to \mathrm{Par}\,C$, $(X,p) \mapsto X$ with adjoints $X \mapsto (X, 0)$ and $X \mapsto (X, 1)$

$$\begin{tikzcd}
\int \mathrm{Pred}_\Box \arrow[d, "u"] \\
\mathrm{Par}\,C \arrow[u, bend left=70, "0"] \arrow[u, bend right=70, "1"']
\end{tikzcd}$$

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Exists iff C has quotients

$$\begin{tikzcd}[row sep=huge]
\int \mathrm{Pred}_\Box \arrow[d, bend right=80, red, ""{name=R, right}] \arrow[d, "u"'{name=U}] \\
\mathrm{Par}\, C \arrow[u, bend left=40, ""{name=A}] \arrow[u, bend right=40, ""{name=B}]
\arrow[from=R, to=A, phantom, "\dashv", red]
\arrow[from=A, to=U, phantom, "\dashv"]
\arrow[from=U, to=B, phantom, "\dashv"]
\end{tikzcd}$$

Aside: why these names?

For effectus C, define category $\int \mathrm{Pred}_\square$ by

- objects: pairs (X, p), X object in C, $p \in \mathrm{Pred}\, X$
- an arrow $(X, p) \longrightarrow (Y, q)$
  is a map $f : X \longrightarrow Y$ in $\mathrm{Par}\, C$ with $p \le (q^\perp \circ f)^\perp$

There is an obvious $u : \int \mathrm{Pred}_\square \to \mathrm{Par}\, C$, $(X, p) \mapsto X$
with adjoints $X \mapsto (X, 0)$ and $X \mapsto (X, 1)$

Exists iff C has quotients

$$\begin{tikzcd}
\int \mathrm{Pred}_\square \arrow[d, bend right=80] \arrow[d, "u"] \arrow[d, bend left=80] \\
\mathrm{Par}\, C \arrow[u, bend left=40] \arrow[u, bend right=40]
\end{tikzcd}$$

Exists iff C has comprehension

$\Diamond$-effectus preparation: images & sharp predicates

An effectus (in partial form) has images if:

for every map $f: X \to Y$
there is a least predicate $\mathrm{im}\, f$ on Y
with $(\mathrm{im}\, f) \circ f = 1 \circ f$.

$\Diamond$-effectus preparation: images & sharp predicates

An effectus (in partial form) has *images* if:

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A map f is *faithful* if $\mathrm{im}\, f = 1$.

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(Equiv.: $p \circ f = 0 \implies p = 0$.)

$\lozenge$-effectus preparation: images & sharp predicates

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A predicate p is sharp if $p \equiv \mathrm{im}\, f$ for some f.

$\Diamond$-effectus preparation: images & sharp predicates

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A predicate p is sharp if $p \equiv \mathrm{im}\, f$ for some f.

(In vN: sharp iff projection)

Dfn. a $\diamond$-effectus is an effectus
with quotients,
comprehension
and images
with: s sharp $\Rightarrow$ $s^\perp$ sharp.

Proposition. In a $\Diamond$-effectus the sharp predicates $\mathrm{SPred}\, X$ on X are a sub-effect algebra of $\mathrm{Pred}\, X$ and an orthomodular lattice.

Factorization in $\mathcal{B}$-effectus

$$X \xrightarrow{f} Y$$

Factorization in $\Updownarrow$-effectus

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\xi_{(1 \circ f)^\perp}"'] & Y \\
X/(1 \circ f)^\perp \arrow[r, dashed, "f'"'] & \{Y \mid \operatorname{im} f\} \arrow[u, "\pi_{\operatorname{im} f}"']
\end{tikzcd}
$$

Factorization in $\Updownarrow$-effectus

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f' is always total $(1\circ f' = 1)$ and faithful.

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f is **pure** if f' is an isomorphism.

f pure & $1 \circ f = 1 \implies f$ comprehension map

f pure & $\mathrm{im} f = 1 \implies f$ quotient map

Pure maps in vN

The pure maps $B(H) \to B(K)$ are precisely those of the form $T \longmapsto V^* T V$

Pure maps in vN

The pure maps $B(H) \to B(K)$ are precisely those of the form $T \longmapsto V^* T V$

Theorem. an ncp-map $\varphi : \mathcal{A} \to \mathcal{B}$ with Paschke / Stinespring dilation

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & \mathcal{B} \\ & \mathcal{P} \arrow[ur, "h"'] & \end{tikzcd}$$

is pure if and only if ρ is surjective.

Ceiling $\lceil p \rceil$ and floor $\lfloor p \rfloor$

Define $\lfloor p \rfloor \equiv \operatorname{im} \pi_p$ (π_p comprh. for p)

$\lceil p \rceil \equiv \lfloor p^\perp \rfloor^\perp$

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(In vN: $\lceil p \rceil$ least projection above p)

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Proposition. In a $\diamond$-effectus

- $\lfloor p \rfloor \leq p$
- $p \leq q \Rightarrow \lfloor p \rfloor \leq \lfloor q \rfloor$
- $\lfloor \lfloor p \rfloor \rfloor = \lfloor p \rfloor$

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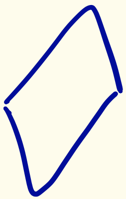
- $\lceil \lceil p \rceil \circ f \rceil = \lceil p \circ f \rceil$

(In vN: $\lceil f(a) \rceil = \lceil f(\lceil a \rceil) \rceil$ useful rule)



$\Diamond$, the possibilistic restriction

For $f : X \to Y$ in a $\Diamond$-effectus, define

$$\begin{tikzcd} \mathrm{SPred}\, X \arrow[r, bend left, "f_\Diamond"] & \mathrm{SPred}\, Y \arrow[l, bend left, "f^\Diamond"] \end{tikzcd}$$

by

$$f^\Diamond(s) \equiv \ulcorner s \circ f \urcorner$$

$$f_\Diamond(s) \equiv \mathrm{im}(f \circ \pi_s)$$

$\Diamond$, the possibilistic restriction

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$$\mathrm{SPred}\,X \underset{f^{\Diamond}}{\overset{f_{\Diamond}}{\rightleftarrows}} \mathrm{SPred}\,Y$$

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De Morgan duals:

$$f^{\Box}(s) \equiv f^{\Diamond}(s^{\perp})^{\perp}$$

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In vN:

$f^{\diamond} = g^{\diamond}$ iff for every normal state ω and effect a, we have

$$\omega(f(a)) = 0 \rightleftarrows \omega(g(a))$$

Proposition. In a $\Diamond$-effectus:

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- $(f \circ g)_{\diamond} = f_{\diamond} \circ g_{\diamond}$ $\quad$ $(f \circ g)^{\diamond} = g^{\diamond} \circ f^{\diamond}$

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- $(\pi_s)_{\Diamond}(\pi_s^{\Box}(t)) = s \wedge t$

- $\operatorname{im} f = f_{\Diamond}(1)$ $\qquad$ $\ulcorner 1 \circ f \urcorner = f^{\Diamond}(1)$

$\diamond$-definitions

$f : X \rightleftarrows Y : g$ are $\diamond$-adjoint iff $f_\diamond = g^\diamond$

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f is $\diamond$-positive iff

$$\left[\begin{array}{l} f \text{ is pure} \\ f = g \circ g \text{ for some } \diamond\text{-self adjoint } g. \end{array}\right.$$

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$f : X \rightleftarrows Y : g$ are $\diamond$-adjoint iff $f_\diamond = g^\diamond$

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$$\left[\begin{array}{l} f \text{ is pure} \\ f = g \circ g \text{ for some } \diamond\text{-self adjoint } g. \end{array}\right.$$

Theorem. In vN, the $\diamond$-positive maps are precisely $a \mapsto \sqrt{b}\, a \sqrt{b}$.

& effectus

Dfn. an &-effectus is $\diamond$-effectus where

- for every predicate p on X, there is a unique $\diamond$-positive map $\mathrm{asrt}_p : X \to X$ with $1 \circ \mathrm{asrt}_p = p$.
- $\xi \circ \pi$ is pure for any comprehension π and quotient map ξ.

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Write $p \& q \equiv q \circ \mathrm{asrt}_p$ and $p^2 \equiv p \& p$.

Proposition. In an &-effectus TFAE

- p is sharp
- $p \mathbin{\&} p = p$
- $\mathrm{asrt}_p \circ \mathrm{asrt}_p = \mathrm{asrt}_p$

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Proposition. In an &-effectus

$$\mathrm{im}\, f \leq s \iff \mathrm{asrt}_s \circ f = f$$

$$1 \circ f \leq t \iff f \circ \mathrm{asrt}_t = f$$

Polar decomposition

In an &-effectus, any pure map $f: X \to Y$ factors as follows:

$$X \xrightarrow{\mathrm{asrt}_{1 \circ f}} X \xrightarrow{\xi} X/_{\ulcorner 1 \circ f \urcorner^{\perp}} \xrightarrow{\vartheta} \{Y \mid \mathrm{im}\, f\} \xrightarrow{\pi} Y$$

$\uparrow$ $\Diamond$-positive

pure map h with $1 \circ h$ sharp
$\approx$ partial isometry

† -effectus

Dfn. a **$\dagger$-effectus** C is an $\&$-effectus, where the subcat. of pure maps is a $\dagger$-category with:

- $\mathrm{asrt}_p^\dagger = \mathrm{asrt}_p$
- f is $\diamond$-adjoint to $f^\dagger$
- for every $\dagger$-positive f, there is a unique $\dagger$-positive g with $f = g \circ g$.
- $\diamond$-positive maps are $\dagger$-positive

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- $\mathrm{asrt}^{\dagger}_p = \mathrm{asrt}_p$
- f is $\diamond$-adjoint to $f^{\dagger}$
- for every $\dagger$-positive f, there is a unique $\dagger$-positive g with $f = g \circ g$.
- $\diamond$-positive maps are $\dagger$-positive

It follows $\ulcorner 1 \circ f^{\dagger} \urcorner = \mathrm{im}\, f$ and $\mathrm{im}\, f^{\dagger} = \ulcorner 1 \circ f \urcorner$,

Dfn. a **†-effectus** C is an &-effectus, where the subcat. of pure maps is a †-category with:

- $\mathrm{asrt}^{\dagger}_{p} = \mathrm{asrt}_{p}$
- f is $\diamond$-adjoint to $f^{\dagger}$
- for every †-positive f, there is a unique †-positive g with $f = g \circ g$.
- $\diamond$-positive maps are †-positive

It follows $\ulcorner 1 \circ f^{\dagger} \urcorner = \mathrm{im}\, f$ and $\mathrm{im}\, f^{\dagger} = \ulcorner 1 \circ f \urcorner$,

(so with slight abuse of notation:)

$$\pi^{\dagger}_{s} = \xi_{s^{\perp}} \quad \text{and} \quad \xi^{\dagger}_{s^{\perp}} = \pi_{s} \qquad (\text{sharp } s)$$

Theorem. An &-effectus is a †-effectus iff

- for every predicate p, there is a unique predicate q with $q \,\&\, q = p$
- $\mathrm{asrt}^2_{p \& q} = \mathrm{asrt}_p \circ \mathrm{asrt}^2_q \circ \mathrm{asrt}_p$
- $t \circ \xi_s$ is sharp for all sharp s, t.

Theorem. An &-effectus is a †-effectus iff

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  Cf. Fundamental Id. of Quad. Jordan Algs.
- $t \circ \xi_s$ is sharp for all sharp s, t.

Dfn. An effectus is operational iff

- the scalars are isomorphic to $[0,1]$,
- the predicates are jointly monic,
- $p \leq q \iff \forall$ state ω. $p \circ \omega \leq q \circ \omega$,
- every object X is 'finite-dimensional': that is: $\mathcal{S}tat\ X$ 'is' a closed convex subset of a finite-dimensional vector space.

Theorem. (Wetering)

- The category $\mathbf{EJA}$ of Euclidean Jordan Algebras with positive maps is a †-effectus.
- Any operational †-effectus is equivalent to a subcategory of $\mathbf{EJA}$.

Theorem. (Wetering)

- The category **EJA** of Euclidean Jordan Algebras with positive maps is a †-effectus.
- Any operational †-effectus is equivalent to a subcategory of **EJA**.

(vN^{op} is †-effectus, but not operational)

Theorem. Every t-effectus is a homological category in the sense of Martin Grandis

Theorem. Every †-effectus is a *homological category* in the sense of Martin Grandis

Corollary. Grandis' *Snake Lemma* holds for von Neumann algebras...

Grandis' Snake Lemma. If we have a diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "h"'] & B' \arrow[r, "k"'] & C' & \end{tikzcd}$$

in a t-effectus such that

- $\mathrm{im}\, f = \ulcorner 1 \circ g \urcorner^{\perp}$
- $\mathrm{im}\, h = \ulcorner 1 \circ k \urcorner^{\perp}$
- g quotient with $1 \circ g$ sharp
- h comprehension
- $b^{\square}(b_{\diamond}(\mathrm{im} f)) = \ulcorner 1 \circ b \urcorner^{\perp} \vee \mathrm{im} f$
- $b_{\diamond}(b^{\square}(\mathrm{im}\, h)) = (\mathrm{im}\, h) \wedge \mathrm{im}\, b$
- $k^{\square}(k_{\diamond}(\mathrm{im}\, b)) = (\mathrm{im}\, h) \vee \mathrm{im}\, b$
- $f_{\diamond}(f^{\square}(b^{\square}(0))) = \ulcorner 1 \circ b \urcorner^{\perp} \wedge \mathrm{im} f$

then ...

$$
\begin{tikzcd}
 & \{A \mid (1 \circ a)^{\perp}\} \arrow[r, "\bar{f}"] \arrow[d, "\pi_{(1 \circ a)^{\perp}}"] & \{B \mid (1 \circ b)^{\perp}\} \arrow[r, "\bar{g}"] \arrow[d, "\pi_{(1 \circ b)^{\perp}}"] & \{C \mid (1 \circ c)^{\perp}\} \arrow[d, "\pi_{(1 \circ c)^{\perp}}"] \arrow[dddll, "\exists d"'] & \\
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "h"] \arrow[d, "\xi_{\mathrm{im}\, a}"] & B' \arrow[r, "k"] \arrow[d, "\xi_{\mathrm{im}\, b}"] & C' \arrow[d, "\xi_{\mathrm{im}\, c}"] & \\
 & A'/\mathrm{im}\, a \arrow[r, "\bar{h}"'] & B'/\mathrm{im}\, b \arrow[r, "\bar{k}"'] & C'/\mathrm{im}\, c &
\end{tikzcd}
$$

with $\mathrm{im}\, \bar{f} = \ulcorner 1 \circ \bar{g} \urcorner^{\perp}$, $\mathrm{im}\, \bar{g} = \ulcorner 1 \circ d \urcorner^{\perp}$, ...

Take away

- QT reconstructions don't need dilations/purifications or parallel composition $\otimes$.
- comprh $\circ$ iso $\circ$ quot. is the right kind of pure.
- $(\ )^{\diamond}$, $\diamond$-adjoint and $\diamond$-positive.

Further reading

- "Dagger and dilations", BW arXiv 1803.01911.
- "Introduction to Effectus theory"
  Cho, Jacobs, Westerbaan, BW arXiv 1512.05813
- "Reconstruction of Quantum Theory from univ. filters"
  Wetering arXiv 1801.05798