Effectus theory

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Effectus theory

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Teaser, another reconstruction of QT:

an operational $\mathcal{T}$-effectus is (a subcategory of) Euclidean Jordan Algebras (finite-dimensional) with positive maps)

follows from [Webering, arXiv:1801.05798]
People

Jacobs

Cho

A. Westerbaan

B. W.
People

Jacobs

Cho

A. Westerbaan

B. W.

Tull

connection with OPTs

Wetering

Reconstruction

Adams

Type theory
Oxford CQM

f.d. Hilbert spaces

$C^2 \text{ (qubit)}, \ C^3 \text{ (qutrit)}, ...$

Effectus theory

von Neumann algebras

$C^2 \text{ (bit)}, \ M_2 \text{ (qubit)}, ...$
Oxford CQM

f.d. Hilbert spaces
\( \mathbb{C}^2 \) (qubit), \( \mathbb{C}^3 \) (qutrit), ...

operators
\( \mathbb{H}, \mathcal{A} \), ...

Effectus theory

von Neumann algebras
\( \mathbb{C}^2 \) (bit), \( M_2 \) (qubit), ...

contractive normal c.p. maps
\( a \mapsto \sum_i b_i^* a b_i \), ...

Oxford CQM

- f.d. Hilbert spaces
  - $C^2$ (qubit), $C^3$ (qutrit), ...

- Operators
  - $\hat{\mathbf{H}}$, $\hat{S}$, ...

- Parallel composition
  - $C^2 \otimes C^2$: two qubits

Effectus theory

- von Neumann algebras
  - $C^2$ (bit), $M_2$ (qubit), ...

- Contractive normal c.p. maps
  - $a \mapsto \sum_i b_i^* a b_i$, ...

- Probabilistic disjunction
  - $C^2 \oplus M_2$: bit or qubit
Oxford CQM

f.d. Hilbert spaces
$C^2$ (qubit), $C^3$ (qutrit),...

Operators
$\mathbb{H}$, $\mathbb{A}$, ...

Parallel composition
$C^2 \otimes C^2$: two qubits

Expressive calculus for ‘circuits’
Works best finite-dimensionally

Effectus theory

von Neumann algebras
$C^2$ (bit), $M_2$ (qubit), ...

Contractive normal c.p. maps
$a \mapsto \sum_i b_i^* a b_i$, ...

+ Probabilistic disjunction
$C^2 \oplus M_2$: bit or qubit

Hard to reason about circuits
Measurement, classical data and infinite dimensions built in.
Maps between $(\mu N)$-algebras go the opposite way

measure in sta. basis

$\mathbb{C}^2 \rightarrow M_2$

(bit ← qbit)

$(\lambda, \mu) \mapsto 2\lambda \chi \chi_{0} + \mu I X I$
Maps between (uN) algebras go the opposite way

measure in sta. basis

initialize as 0

\[ \mathbb{C}^2 \rightarrow M_2 \]

\[ \text{bit} \leftarrow \text{qbit} \]

\[ M_2 \rightarrow \mathbb{C} \]

\[ \text{qbit} \leftarrow 1 \]

\[ (\lambda, \mu) \rightarrow \mathcal{N}_{0x01} + \mu/1 \times 1 \]

\[ a \rightarrow \langle 01a10 \rangle \]
Maps between (vn) algebras go the opposite way

- Measure in standard basis
- Initialize as 0
- Discard qutrit

\[ \mathbb{C}^2 \rightarrow M_2 \] (bit ← qbit)
\[ M_2 \rightarrow \mathbb{C} \] (qbit ← 1)
\[ M_2 \rightarrow M_2 \otimes M_3 \] (qbit ← qbit ⊗ qutrit)

\[ (\lambda, \mu) \rightarrow \lambda 0 \otimes 0 + \mu 1 \otimes 1 \]

\[ a \rightarrow \langle 01 | a | 10 \rangle \]

\[ a \rightarrow a \otimes 1 \]
Maps between (vn) algebras go the opposite way

measure in sta. basis

- $\mathbb{C}^2 \rightarrow M_2$
  - (bit ← qbit)

initialize as 0

- $M_2 \rightarrow \mathbb{C}$
  - (qbit ← 1)

discard qutrit

- $M_2 \rightarrow M_2 \otimes M_3$
  - (qbit ← qbit⊗qutrit)

CPTP

$\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$

$p \mapsto \sum_i b_i^* p b_i$

CPU-map

$\mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$

$a \mapsto \sum_i b_i^* a b_i$
Overview

operational

t-effectus

is a

t-effectus
Overview

operational
\( t \)-effectus

is a

\( t \)-effectus

is an

\( f \)-effectus
Overview

operational
O-effectus

is a

t-effectus

is an

f-effectus

is a

O-effectus
Overview

operational

\( t \text{-effectus} \)

is a

\( t \text{-effectus} \)

is an

\( f \text{-effectus} \)

is a

\( \diamond \text{-effectus} \)

is an

effectus
Overview

Operational  

$\triangleright$-effectus  
is a  

$\triangleright$-effectus  
is an  

$\triangledown$-effectus  
is a  

$\triangledown$-effectus  
is an  

Effectus
An Effectus has objects \( A, B, X, Y, \ldots \) representing data types/systems.
An Effectus has

- objects A, B, X, Y, … representing data types/systems
- arrows f, g, … between them, representing maps/operations
An Effectus has

- objects $A, B, X, Y, \ldots$ representing data types/systems
- arrows $f, g, \ldots$ between them, representing maps/operations
- final object $1$ representing the system with one state

for any $A$, there is a unique $!_A : A \longrightarrow 1$
An Effectus has

. objects $A, B, X, Y, \ldots$ representing data types/systems
. arrows $f, g, \ldots$ between them, representing maps/operations
. final object $1$ representing the system with one state

for any $A$, there is a unique $!_A : A \rightarrow 1$

. Coproduct $A + B$ representing (probabilistic) disjunction

\[ A \xrightarrow{k_1} A + B \quad \xleftarrow{k_2} B \]

\[ \begin{array}{c}
A \\
\downarrow \, f
\end{array} \quad \begin{array}{c}
\vdash \quad \{f, g\} \\
\downarrow \, \tilde{C} \quad \downarrow \, g \\
B
\end{array} \]

\[ A \rightarrow \tilde{C} \rightarrow B \]
An Effectus has

- objects $A, B, X, Y, ...$ representing data types/systems
- arrows $f, g, ...$ between them, representing maps/operations
- final object $1$ representing the system with one state

For any $A$, there is a unique $!_A : A \to 1$

- Coproduct $A + B$ representing (probabilistic) disjunction

\[
\begin{array}{ccc}
A & \xrightarrow{K_1} & A + B & \xleftarrow{K_2} & B \\
& \searrow^{f} & \downarrow^{[f,g]} & \nearrow^{g} & \\
& & \hat{C} & &
\end{array}
\]

\[
\text{swap} = [k_2, k_1] : A + A \to A + A
\]

\[
\nu = [\text{id}, \text{id}] : A + A \to A
\]
Predicates and partial maps

\[ p : X \rightarrow 1+1 \]  

predicate on \( X \)
Predicates and partial maps

\[ p : X \rightarrow 1+1 \]

\[
\begin{array}{ccc}
X & \xrightarrow{1} & 1 \\
\downarrow & & \downarrow \kappa_1 \\
2 & \xrightarrow{1+1} & \end{array}
\]

predicate on \( X \)

the truth predicate on \( X \)
Predicates and partial maps

$p : X \to \{1,1\}$

- Predicate on $X$
- Truth predicate on $X$
- Negation/orthosupplement of $p$
Predicates and partial maps

\[ p : X \rightarrow 1 + 1 \]

\[ X \xrightarrow{\mathcal{L}} 1 \xrightarrow{x_1} 1 + 1 \]

\[ X \xrightarrow{\mathcal{M} \ p} 1 + 1 \]

\[ p^\perp \]

\[ X \xrightarrow{f} y + 1 \]

\[ \text{predicate on } X \]

\[ \text{the truth predicate on } X \]

\[ \text{the negation/orthosupplement of } p \]

\[ \text{a partial map } X \rightarrow y \]
Predicates and partial maps

\[ p : X \to 1+1 \]

\[ X \xrightarrow{!} 1 \xrightarrow{\mu_1} 1+1 \]

\[ X \xrightarrow{p} 1+1 \]

\[ X \xrightarrow{p^\perp} 1+1 \]

\[ X \xrightarrow{f} Y+1 \]

\[ X \xrightarrow{f} Y+1 \xrightarrow{[g,k_2]} Z+1 \]

\[ g \circ f \]

\( \mu \), the truth predicate on \( X \)

the negation/orthosuplement of \( p \)

a partial map \( X \to Y \)

composition of partial maps
Predicates and partial maps

\[ p : X \to \mathbb{1} + \mathbb{1} \]

\[ X \xrightarrow{1} \mathbb{1} \xrightarrow{k_1} \mathbb{1} + \mathbb{1} \]

\[ X \xrightarrow{\neg} \mathbb{1} + \mathbb{1} \]

\[ \mu \text{, the truth predicate on } X \]

\[ \text{the negation/orthosupplement of } p \]

\[ X \xrightarrow{f} y + 1 \]

\[ X \xrightarrow{f} y + 1 \xrightarrow{[g,k_2]} z + 1 \]

\[ g \circ f \]

\[ X \xrightarrow{f} y \xrightarrow{k_1} y + 1 \]

\[ \text{a partial map } X \to Y \]

\[ \text{composition of partial maps} \]

\[ \text{a map as a (total) partial map} \]
Predicates and partial maps

\( p : X \rightarrow 1+1 \)
- \( X \xrightarrow{p} 1+1 \)
- \( X \xrightarrow{p^\perp} 1+1 \)

Predicate on \( X \)
- the truth predicate on \( X \)
- the negation/orthosupplement of \( p \)

\( X \xrightarrow{f} y+1 \)
- \( X \xrightarrow{f} y+1 \xrightarrow{\partial_0 f} Z+1 \)
- \( X \xrightarrow{f} y \xrightarrow{\partial_1 f} y+1 \)
- \( X \xrightarrow{f} y+1 \xrightarrow{!+id} 1+1 \)

Partial map \( x \rightarrow y \)
- composition of partial maps
- a map as a (total) partial map
- \( 1 \otimes f \) is a measure of partiality of \( f \)
Predicates and partial maps

\[ p : X \rightarrow 1+1 \]

- \( p \) is a predicate on \( X \)
- \( \mu \), the truth predicate on \( X \)
- \( p^+ \), the negation/orthosupplement of \( p \)

\[ X \xrightarrow{f} y+1 \]

- \( f \) is a partial map \( X \rightarrow y \)
- \( g \circ f \) is the composition of partial maps
- \( 1 \circ f \) is a map as a (total) partial map

\[ 1 \circ f \text{ is a measure of partiality of } f \]

\[ 1 \circ f = 1 \Rightarrow f \text{ total} \]
\[ 1 \circ f = 0 \Rightarrow f = \kappa_{20} \]
Scalars and states

$\lambda: 1 \rightarrow 1+1$  
a scalar (so partial map $1 \rightarrow 1$)
Scalars and states

\[ \lambda: 1 \rightarrow 1+1 \]  
\[ \lambda \circ \mu = \lambda \circ \mu \]  

a scalar (so partial map \( 1 \rightarrow 1 \))  

product is composition as partial maps
Scalars and states

\( \lambda : 1 \to 1+1 \) is a scalar (so partial map \( 1 \to 1 \))

\( \rho \circ \mu = \rho \circ \mu \) product is composition as partial maps

Similarly \( \lambda \cdot p \equiv \lambda \circ p \) for scalar \( \lambda : 1 \to 1 \)

predicate \( p : X \to 1 \)
Scalars and states

\( \lambda: 1 \to 1+1 \) \hspace{1cm} \text{a scalar (so partial map} \ 1 \to 1) \\
\lambda \circ \mu = \lambda \circ \mu \) \text{ product is composition as partial maps}

Similarly \( \lambda \cdot p = \lambda \circ p \) \text{ for scalar } \lambda: 1 \to 1 \\
\text{predicate } p: X \to 1

\omega: 1 \to X \hspace{1cm} \text{a state on } X
Scalars and states

\( \lambda : 1 \to 1 + 1 \) a scalar (so partial map \( 1 \to 1 \))

\( \lambda \circ \mu = \lambda \circ \mu \) product is composition as partial maps

Similarly \( \lambda \cdot p = \lambda \circ p \) for scalar \( \lambda : 1 \to 1 \)

Predicate \( p : X \to 1 \)

\( \omega : 1 \to X \)

\( \omega \circ \lambda + \gamma \) convex combination of states

\( \lambda \omega \circ \lambda + \gamma \)
Scalars and states

\[ \lambda : 1 \to 1 + 1 \] a scalar (so partial map \( 1 \to 1 \))

\[ \lambda \circ \mu = \lambda \circ \mu \] product is composition as partial maps

Similarly \( \lambda \cdot p = \lambda \circ p \) for scalar \( \lambda : 1 \to 1 \)

Predicate \( p : X \to 1 \)

A state on \( X \)

Convex combination of states

Validity of a predicate in a state
Partial sum of predicates

If for predicates $p, q$ on $X$, there is a $b$ with

$$\begin{array}{c}
\begin{array}{c}
p \quad X \quad q \\
\downarrow \quad \downarrow \\
1+1 \leftarrow 1+1+1 \rightarrow 1+1
\end{array}
\end{array}$$

then $p$ and $q$ are summable (in symbols: $p \uparrow q$) and their sum $p \circ q$ is given by

$$\begin{array}{c}
\begin{array}{c}
p \circ q \\
\downarrow \quad \downarrow \\
1+1+1 \leftarrow 1+1+1 \rightarrow 1+1
\end{array}
\end{array}$$
Definition an **Effectus** is a category $C$ with

- finite coproducts and final object $1$,

where all diagrams of the form

$$
\begin{align*}
X + Y & \xrightarrow{id+!} X + I \\
! + id & \downarrow \quad \downarrow ! + id \\
1 + Y & \xrightarrow{id+!} 1 + I
\end{align*}
$$

are pullback squares and

- the following arrows are jointly manic

$$
\begin{align*}
1 + 1 + 1 & \xrightarrow{} 1 + 1 \\
\downarrow & \\
\times & \xrightarrow{}
\end{align*}
$$
**Definition** an **Effectus** is a category $C$ with

- finite coproducts and final object $1$,
- where all diagrams of the form

$$
\begin{align*}
X + Y & \xrightarrow{id + !} X + 1 \\
1 + Y & \xrightarrow{id + !} 1 + 1 \\
X + Y & \xrightarrow{! + id} X + 1 \\
1 + Y & \xrightarrow{! + id} 1 + 1
\end{align*}
$$

are pullback squares and

- the following arrows are jointly manic

$$
\begin{align*}
1 + 1 + 1 & \xrightarrow{! V} 1 + 1 \\
& \xrightarrow{X}
\end{align*}
$$

**Examples:** $\nu N^\op$, $\text{Set}$, $\text{CRng}^\op$, $\text{KLCD}$, any topos, $\text{EA}^\op$, ...
Structure in an effectus

\( \mathcal{M} \), the set of scalars is an effect monoid, that is: an effect algebra with biadditive product for which 1 is a unit.
Structure in an effectus

$M$, the set of scalars is an effect monoid, that is: an effect algebra with biadditive product for which 1 is a unit.

$\text{Pred}X$, the set of predicates on an object $X$ is an $M$-effect module, that is: an effect algebra with an action of $M$. 
Structure in an effectus

\[ M, \text{ the set of scalars is an effect monoid, that is: an effect algebra with biadditive product for which } 1 \text{ is a unit.} \]

\[ \text{Pred}X, \text{ the set of predicates on an object } X \text{ is an } M\text{-effect module, that is: an effect algebra with an action of } M. \]

\[ \text{Stat} X, \text{ the set of states on } X \text{ is an (abstract) } M^0\text{-convex set.} \]
Structure in an effectus

$M$, the set of scalars is an effect monoid, that is: an effect algebra with biadditive product for which 1 is a unit.

$\text{Pred} X$, the set of predicates on an object $X$ is an $M$-effect module, that is: an effect algebra with an action of $M$.

$\text{Stat} X$, the set of states on $X$ is an (abstract) $M^\text{op}$-convex set.
Definition: A category $C$ is an Effectus in partial form if

1. $C$ is a fin PAC – that is
   a. $C$ has coproducts
   b. $C$ is PCM-enriched, i.e.
      a. every $\text{Hom}(X, y) \times \text{Hom}(y, y)$ has partial binary operation $\otimes$ and distinguished map $\circ$ that turn it into a PCM
      b. $f \cdot g \Rightarrow [\text{hof} \otimes \text{hog} \quad (\text{hof}) \otimes (\text{hog}) = \text{ho}(f \circ g)$
         \hspace{1cm} \text{fok} \otimes \text{gok} \quad (\text{fok}) \otimes (\text{gok}) = (\text{fog}) \circ k$
   c. $D, 0b \perp D, 20b$ for any $b: X \to y + y$, where $D, \equiv \{\text{id}, 0\}: y + y \to y$ and $D_2 \equiv \{0, 1o\}$
   d. $f \cdot g \Rightarrow k_1 \cdot f \circ k_2 \cdot g$

2. $C$ “has effects” – that is: there is an object $I$ such that
   a. the PCM $\text{Hom}(X, I) \equiv \text{Pred} X$ is an effect algebra $\mathcal{A} \times X$
   b. $10f \perp 10g \Rightarrow f \perp g$
   c. $10f = 0 \Rightarrow f = 0$

A map $f$ is called total iff $10f = 1$.
Definition a category C is an Effectus in partial form if

1. C is a fin PAC — that is
   a. C has coproducts.
   b. C is PCM-
      \( \alpha \) every \( H \)
      and are
   c. \( \mathcal{D}, \mathcal{O} \leq \mathcal{D}_{\omega} \)
      where \( \mathcal{D}, \mathcal{O} \)
   d. \( f \perp g \Rightarrow k_{1} \mathcal{O} f \perp k_{2} \mathcal{O} g \)

2. C "has effects" — that is: there is an object I such that
   a. the PCM \( \text{Hom}(X, I) = \text{Pred} X \) is an effect algebra \( \mathcal{F} X \).
   b. \( 1 \mathcal{O} f \perp 1 \mathcal{O} g \Rightarrow f \perp g \)
   c. \( 1 \mathcal{O} f = 0 \Rightarrow f = 0 \).

A map \( f \) is called total iff \( 1 \mathcal{O} f = 1 \)
Definition: A category $C$ is an Effectus in partial form if

1. $C$ is a fin PAC — that is
   a. $C$ has coproducts.
   b. $C$ is PCM-
      a. every $\mathcal{K}$ and $\mathcal{G}$
      b. $f \perp g \Rightarrow$
   c. $D, ob \perp D_{20}$
      where $D, =$
   d. $f \perp g \Rightarrow k_1 o f \perp k_2 o g$

2. $C$ is an Effect Algebra iff
   a. for every $x$, there is a unique $x^\perp$ such that
      $x^\perp o x = 1 \equiv 0^\perp$
   b. $x \perp 1 \Rightarrow x = 0$

A PCM is an Effect Algebra iff

$$\mathcal{M}, 0, 0 \Rightarrow (M, 0, 0) \text{ with } 0 : M^2 \rightarrow M,$$

$0 \in M$ is a PCM iff

- $x \perp y \Rightarrow [y \perp x]
  - x o y = y o x$
- $0 \perp x$ and $0 o x = x$
- $x \perp y$ and $x o y \perp z$
  $$\Rightarrow [y \perp z, x \perp y \perp z]$$
- $(k o y) o z = x o (y o z)$

permutation $\Theta$

into a PCM

$\Rightarrow k o (f o g)$

$= (f o g) o k$

$\Theta = [0, i \alpha]$. object I such that

an effect algebra $\forall X$. 
They're the “same”

If $C$ is an effectus, then $\text{Pan } C$, its category of partial maps, is an effectus in partial form.

If $D$ is an effectus in partial form, then $\text{Tot } D$, its subcategory of total maps is an effectus.

Furthermore $\text{Pan } \text{Tot } D \cong D$ and $\text{Tot } \text{Pan } C \cong C$. 
-effectus
An effectus (in partial form) has quotients if:

For every predicate $p : X \to 1$, there is an obj. $X_{p^\perp}$ and (partial) map $\xi_{p^\perp} : X \to X_{p^\perp}$ with $1 \cdot \xi_{p^\perp} \leq p$, such that for any other $f : X \to Y$ with $1 \cdot f \leq p$, there is a unique $g$ with

\[
\begin{array}{ccc}
X & \xrightarrow{\xi_{p^\perp}} & X_{p^\perp} \\
\downarrow & & \downarrow \\
f & \searrow & g \\
\end{array}
\]
\(\Box\)-effectus preparation: quotients

An effectus (in partial form) has quotients if:

For every predicate \(p : X \rightarrow 1\), there is an obj. \(X_{p^+}\) and (partial) map \(\xi_{p^+} : X \rightarrow X_{p^+}\) with \(10 \xi_{p^+} \leq p\), such that for any other \(f : X \rightarrow Y\) with \(10f \leq p\), there is a unique \(g\) with

\[
\begin{array}{ccc}
X & \xrightarrow{\xi_{p^+}} & X_{p^+} \\
\downarrow \text{quotient map} & & \\
X/p^+ & \xrightarrow{g} & Y
\end{array}
\]
An effectus (in partial form) has quotients if:

For every predicate $p : X \to 1$, there is an obj $X_{p^\perp}$ and (partial) map $\xi_{p^\perp} : X \to X_{p^\perp}$ with $10 \xi_{p^\perp} \leq P$, such that for any other $f : X \to Y$ with $10f \leq P$, there is a unique $g$ with

$$
\begin{array}{ccc}
X & \xrightarrow{\xi_{p^\perp}} & X_{p^\perp} \\
\downarrow f & & \downarrow g \\
Y & & Y
\end{array}
$$

(In $\mathbb{N}$, $\xi : \xi_{p} \to \xi_{p^\perp}$ given by $\xi(a) = \sqrt{p}a\sqrt{p}$)
$\mathsf{\ Diamond}$-effectus preparation: comprehension

An effectus (in partial form) has comprehension if

For every predicate $p : X \to 1$, there is an obj. $\{X | p\}$ and (partial) map $\pi_p : \{X | p\} \to X$ with $p_0 \pi_p = 1_0 \pi_p$ such that for any other $f : Y \to X$ with $p_0 f = 1_0 f$, there is a unique $g$ with

$\begin{array}{ccc}
\{X | p\} & \xrightarrow{\pi_p} & X \\
g & \downarrow & \\
Y & \xrightarrow{f} & \\
\end{array}$
An effectus (in partial form) has comprehension if

For every predicate \( p : X \to 1 \), there is an obj. \( \{X|p\} \) and (partial) map \( \pi_p : \{X|p\} \to X \) with \( p \circ \pi_p = 1 \circ \pi_p \) such that for any other \( f : Y \to X \) with \( p \circ f = 1 \circ f \), there is a unique \( g \) with

\[
\begin{array}{c}
\{X|p\} \\
g
\end{array} \xleftarrow{\text{comp. map}} \xrightarrow{\pi_p} X
\]

and

\[
\begin{array}{c}
Y \\
f
\end{array}
\]
\( \Delta \)-effectus preparation: comprehension

An effectus (in partial form) has comprehension if

For every predicate \( p : X \to 1 \), there is an obj. \( \{ X \mid p \} \) and (partial) map \( \pi_p : \{ X \mid p \} \to X \) with \( p \circ \pi_p = 1 \circ \pi_p \) such that for any other \( f : Y \to X \) with \( p \circ f = 1 \circ f \), there is a unique \( g \) with

\[
\begin{array}{ccc}
\{ X \mid p \} & \xrightarrow{\pi_p} & X \\
g & \downarrow & \downarrow f \\
Y & \xRightarrow{\text{comp. map}} & X
\end{array}
\]

\( \text{(In } \nu N, \nu \chi: A \to LpL\nu Lp) \) given by \( \pi (a) = Lp\nu Lp \)
Aside: why these names?

For effectus $C$, define category $\mathcal{S}_{\text{Pred}_C}$ by

- objects: pairs $(X, p)$, $X$ object in $C$, $p \in \text{Pred}_X$
- an arrow $(X, p) \rightarrow (Y, q)$ is a map $f: X \rightarrow Y$ in $\text{Par}_C$ with $p \leq (q \circ f)^1$

There is an obvious $U: \mathcal{S}_{\text{Pred}_C} \rightarrow \text{Par}_C$, $(X, p) \mapsto X$ with adjoints $X \mapsto (X, 0)$ and $X \mapsto (X, 1)$.
Aside: why these names?

For effectus $C$, define category $\mathcal{S}^{\text{Pred}_{C}}$ by

- objects: pairs $(X, p)$, $X$ object in $C$, $p \in \text{Pred}_{C}X$
- an arrow $(X, p) \to (Y, q)$ is a map $f: X \to Y$ in $\text{Par}_C$ with $p \leq (q \circ f)^{-1}$

There is an obvious $U: \mathcal{S}^{\text{Pred}_{C}} \to \text{Par}_C$, $(X, p) \mapsto X$

with adjoints $X \mapsto (X, 0)$ and $X \mapsto (X, 1)$

Exists iff $C$ has quotients
Aside: why these names?

For effectus $C$, define category $\mathcal{S} \text{Pred}_\sqsubseteq$ by

- objects: pairs $(X, p)$, $X$ object in $C$, $p \in \text{Pred} X$
- an arrow $(X, p) \to (Y, q)$ is a map $f: X \to Y$ in $\text{Par}(C)$ with $p \leq (q \circ f)$

There is an obvious $\mathcal{U}: \mathcal{S} \text{Pred}_\sqsubseteq \to \text{Par}(C)$, $(X, p) \mapsto X$ with adjoints $X \mapsto (X, 0)$ and $X \mapsto (X, 1)$.

\[\text{Exists iff } C \text{ has quotients} \quad \mathcal{S} \text{Pred}_\sqsubseteq \leftarrow \text{Par}(C) \leftarrow \text{Par} C \leftarrow\]

\[\text{Exists iff } C \text{ has comprehension} \]
\(\Delta\)-effectus preparation: images & sharp predicates

An effectus (in partial form) has images if:

for every map \(f: X \rightarrow Y\)

there is a least predicate \(\text{im} f\) on \(Y\)

with \((\text{im} f) \circ f = 1 \circ f\).
Δ-effectus preparation: images & sharp predicates

An effectus (in partial form) has images if:

- for every map $f: X \rightarrow Y$
- there is a least predicate $\text{im} f$ on $Y$ with $(\text{im} f) o f = 1 o f$.

A map $f$ is faithful if $\text{im} f = 1$. 
An effectus (in partial form) has images if:

for every map $f: \times \rightarrow \mathcal{Y}$
there is a least predicate $\text{Im} f$ on $\mathcal{Y}$
with $(\text{Im} f) \circ f = 1 \circ f$.

A map $f$ is faithful if $\text{Im} f = 1$.

(Equiv.: $p \circ f = 0 \implies p = 0$.)
\textit{effectus preparation:} images $\&$ sharp predicates

An effectus (in partial form) has images if:

for every map $f: X \rightarrow Y$

there is a least predicate $\text{im} f$ on $Y$

with $(\text{im} f) \circ f = 1 \circ f$.

A map $f$ is \textbf{faithful} if $\text{im} f = 1$.

(Equiv.: $p \circ f = 0 \Rightarrow p = 0$.)

A predicate $p$ is \textbf{sharp} if $p \equiv \text{im} f$ for some $f$. 
A -effectus preparation: images & sharp predicates

An effectus (in partial form) has images if:

for every map $f: X \to Y$
there is a least predicate $\Im f$ on $Y$
with $(\Im f) \circ f = 1 \circ f$.

A map $f$ is faithful if $\Im f = 1$.
(Equiv.: $\rho f = 0 \Rightarrow p = 0$.)

A predicate $p$ is sharp if $p \equiv \Im f$ for some $f$.
(In vN: sharp iff projection)
Dfn. a $\bigodot$-effectus is an effectus with quotients, comprehension and images with: $s$ sharp $\Rightarrow s^\sharp$ sharp.
Proposition. In a C-effectus the sharp predicates $\text{SPred}_X$ on $X$ are a sub-effect algebra of $\text{Pred}_X$ and an orthomodular lattice.
Factorization in $\mathcal{O}$-effects

\[ x \xrightarrow{f} y \]
Factorization in $\Theta$-effectus

\[ X \xrightarrow{\xi} \xi_{(10f)^T} \rightarrow X/_{(10f)^T} \rightarrow \{ y | \text{im} f \} \]
Factorization in $\mathcal{O}$-effectus

\[ X \xrightarrow{f} Y \]
\[ \downarrow \pi (1of)^\perp \]
\[ X_{(1of)^\perp} \rightarrow \{y \mid \text{lim } f\} \]

$\pi'$ is always total ($1of' = 1$) and faithful.
Factorization in $\textbf{0}$-effectus

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{\xi_{(1of)^\perp}} & & \uparrow_{\Pi_{\text{im } f}} \\
X/(1of)^\perp & \longrightarrow & \{y | \text{im } f\}
\end{array}
$$

$f'$ is always total ($1of'=1$) and faithful. $f$ is pure if $f'$ is an isomorphism.
Factorization in 0-effectus

\[
\begin{array}{c}
X \xrightarrow{\xi} Y \\
\downarrow \xi_{(1of)^{-1}} \quad \uparrow \Pi_{\text{im} f} \\
X_{(1of)^{-1}} \longrightarrow \{y \mid \text{im} f\}
\end{array}
\]

\(f\) is always total \((1of' = 1)\) and faithful.

\(f\) is **pure** if \(f'\) is an isomorphism.

\(f\) pure \& \(1of = 1 \Rightarrow f\) comprehension map
Factorization in 0-effectus

\[ X \xrightarrow{g} Y \]
\[ X / (1of) \longrightarrow \{ y \mid imf \} \]

\[ f \]

\[ f' \]

\[ f' \] is always total \((1of' = 1)\) and faithful.

\( f \) is **pure** if \( f' \) is an isomorphism.

\(-f \) pure \& \( 1of = 1 \) \( \Rightarrow \) \( f \) comprehension map

\(-f \) pure \& \( imf = 1 \) \( \Rightarrow \) \( f \) quotient map
Pure maps in $\mathbb{VN}$

The pure maps $B(H) \to B(K)$ are precisely those of the form $T \mapsto V^*TV$.
Pure maps in $\mathcal{VN}$

The pure maps $B(H) \rightarrow B(K)$ are precisely those of the form $T \mapsto V^*TV$

**Theorem.** An ncp-map $\gamma: A \rightarrow B$ with Paschke/Stinespring dilation

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\downarrow \rho & & \downarrow \rho \\
\rho' & \xrightarrow{h} & \rho''
\end{array}
\]

is pure if and only if $\rho$ is surjective.
Ceiling \( \Gamma_p \) and floor \( \Lambda_p \)

Define \( \Lambda_p \equiv \text{im } \pi_p \) \((\pi_p \text{ comprh. for } p)\)
\[
\Gamma_p \equiv \Lambda_p^\perp \perp
\]
Ceiling $\lceil p \rceil$ and floor $\lfloor p \rfloor$

Define $\lfloor p \rfloor \equiv \text{im } \pi_p$ (\(\pi_p\) comprh. for \(p\))

$\lceil p \rceil \equiv L_{p+1} \downarrow$

(In vN: $\lceil p \rceil$ least projection above \(p\))
Ceiling $\Gamma_p$ and floor $L_p$

Define $L_p = \text{im } \pi_p$ (where $\pi_p$ is the projection for $p$)

$\Gamma_p = L_p \dashv \bot$

(In vN: $\Gamma_p$ least projection above $p$)

Proposition. In a $0$-effectus

- $L_p \leq p$
- $p \leq q \Rightarrow \bot \leq L_p$
- $L_p \vdash L_p$
Ceiling $\lceil p \rceil$ and floor $\lfloor p \rfloor$

Define $\lfloor p \rfloor = \text{im } \pi_p \quad (\pi_p \text{ comprh. for } p)$

$\Gamma_p \equiv \lceil p \rceil^\perp$

(In vN: $\Gamma_p$ least projection above $p$)

Proposition. In a 0-effectus

- $\lceil p \rceil \leq p$
- $\lceil \Gamma_p \rceil = \lceil p \rceil$
- $\Gamma_p \leq p = \lceil p \rceil$
- $p \leq q \Rightarrow \lceil p \rceil \leq \lceil q \rceil$
- $\Gamma_p \leq \Gamma_p \circ f$
Ceiling $\Gamma_p$ and floor $\ell_p$

Define $\ell_p = \text{im} \pi_p$ (where $\pi_p$ comprh. for $p$)

$\Gamma_p \equiv \ell_p^\perp$

(In vN: $\Gamma_p$ least projection above $p$)

Proposition. In a $0$-effectus

- $\ell_p \leq p$
- $\ell_p \downarrow = \ell_p$
- $\exists \ell_p \downarrow$
- $\Gamma_p \downarrow \Gamma_p$

- $\Gamma_p \otimes \ell \equiv \Gamma_p$
Ceiling \( \Gamma_p \) and floor \( \Lambda_p \)

Define \( \Lambda_p \equiv \text{im} \, \pi_p \) (\( \pi_p \) comprh. for \( p \))

\[
\Gamma_p \equiv \Lambda_p \downarrow
\]

(In vN: \( \Gamma_p \) least projection above \( p \))

Proposition. In a \( \emptyset \)-effectus

\[
\begin{align*}
\cdot \quad & \Lambda_p \leq p \\
\cdot \quad & \bot \Lambda_p = \Lambda_p \\
\cdot \quad & \Gamma_p \downarrow = \Lambda_p \\
\cdot \quad & \Gamma_p \downarrow \downarrow \leq \Gamma_p \downarrow \\
\cdot \quad & \Gamma_p \downarrow \downarrow \downarrow = \Gamma_p \downarrow
\end{align*}
\]

(In vN: \( \Gamma_f(a)^7 = \Gamma_f(ra^7)^7 \) useful rule)
For $f: X \rightarrow Y$ in a $\Delta$-effectus, define

$$S^e \text{Pred}_X \leftrightarrow S^e \text{Pred}_Y$$

by

$$f^e(s) = \Gamma_{S^e f} I$$

and

$$f^e(s) = \text{im}(f \circ \pi_s)$$
the possibilistic restriction

For \( f: X \rightarrow Y \) in a \( \Theta \)-effectus, define

\[
SPreax X \leftrightarrow SPrea Y
\]

by

\[
f^\Theta(s) \equiv \Gamma_{s\circ f}^\Theta
\]

\[
f^\Theta(s) \equiv \text{im}(f \circ \Pi_s)
\]

De Morgan duals:

\[
f^\Delta(s) \equiv f^\Theta(s^+)^\perp
\]

\[
f^\Theta(s) \equiv f^\Delta(s^+)\perp
\]
For \( f: X \rightarrow Y \) in a \( \Theta \)-effectus, define
\[
\text{SPred} X \leftrightarrow \text{SPred} Y
\]
by
\[
\begin{align*}
\text{SPred}(s) &= \Gamma_{\text{sof}} f \\
\text{SPred}(s) &= \text{im}(f \circ \pi_s)
\end{align*}
\]
In \( \nu \mathcal{N} \):
\[
f^\square = g^\square \iff \text{for every normal state } w \text{ and effect } a, \text{ we have } \omega(f(a)) = 0 \implies \omega(g(a))
\]
De Morgan duals:
\[
\begin{align*}
f^\square(s) &= f^\square(s^\perp) \\
f^\square(s) &= f^\square(s^\perp)
\end{align*}
\]
Proposition. In a $\Omega$-effectus:

- $f^\circ(s) \leq t^\perp$ iff $f_\circ(t) \leq s^\perp$
Proposition. In a \( \Delta \)-effectus:

\[ \cdot \, f^*(\mathcal{E}) \leq \mathcal{E}^\perp \text{ iff } f_*(\mathcal{E}) \leq \mathcal{S}^\perp \]

Equivalently: \( f^* \perp f_* \)
Proposition. In a \( \Downarrow \)-effectus:

- \( f \circ (s) \leq \downarrow \) iff \( f \circ (e) \leq \downarrow \)
  
  Equivalently: \( f \circ -1 f \)

- \( f \circ f \circ f \circ f = f \)
  
  \( f \circ f \circ f \circ f \circ f = f \)
Proposition. In a $\Diamond$-effectus:

- $f^\Diamond(x) \leq e^\Downarrow$ iff $f^\Diamond(e) \leq s^\perp$
  
  Equivalently: $f \circ 1 \circ f^\Diamond$

- $f^\Diamond \circ f^\Diamond \circ f^\Diamond = f^\Diamond$
  $f^\Diamond \circ f^\Diamond \circ f^\Diamond = f^\Diamond$

- $(f \circ g) \circ = f \circ (g \circ)$
  $(f \circ g) \circ = g \circ f^\Diamond$
Proposition. In a $\Diamond$-effectus:

- $f \circ (s) \leq e^1$ iff $f \circ (e) \leq s^1$
  
  Equivalently: $f \circ - \circ f$

- $f \circ f \circ f \circ f = f \circ f \circ f = f \circ f$

- $(f \circ g) \circ g = f \circ (g \circ g) \circ (f \circ g) \circ f = g \circ f$

- $(\pi_s) \circ (\pi_s \circ (e)) = s \wedge e$
Proposition. In a $\Diamond$-effectus:

- $f^{\Diamond}(s) \leq \mathsf{\top}$ iff $f^{\Diamond}(t) \leq \mathsf{\top}$
  
  Equivalently: $f^{\Diamond} \vdash f^{\Diamond}$

- $f^{\Diamond} \circ f^{\Diamond} \circ f^{\Diamond} = f^{\Diamond}$
  
- $(f^{\Diamond} \circ g)^{\Diamond} = f^{\Diamond} \circ g^{\Diamond}$
  
- $(\mathsf{t})^{\Diamond} \circ (\mathsf{t}_s^{\Diamond}(t)) = s \land t$

- $\mathsf{im} f = f^{\Diamond}(1)$
  
- $f \circ f^{\top} = f^{\Diamond}(1)$
Definitions

f : X ⇀ Y : g are ◦-adjoint iff f ◦ = g ◦
\( \Diamond \)-definitions

\[ f : x \Rightarrow y \text{ are } \Diamond \text{-adjoint iff } f \circ \Diamond = \Diamond \circ g \]

\[ f : x \rightarrow x \text{ is } \Diamond \text{-self adjoint iff } f \circ \Diamond = \Diamond \circ f \]
$\mathcal{O}$-definitions

- $f : x \to y$ and $g$ are $\mathcal{O}$-adjoint iff $f \circ g = \mathcal{O}$
- $f : x \to x$ is $\mathcal{O}$-self adjoint iff $f \circ f = \mathcal{O}$

- $f$ is $\mathcal{O}$-positive iff
  - $f$ is pure
  - $f = g \circ g$ for some $\mathcal{O}$-self adjoint $g$. 
\[\text{Definitions}\]
\[f: x \rightarrow y: g \text{ are } \bigtriangledown\text{-adjoint iff } f \circ g = g \circ f\]
\[f: x \rightarrow x \text{ is } \bigtriangledown\text{-self adjoint iff } f \circ f = f \circ f\]
\[f \text{ is } \bigtriangledown\text{-positive iff}\]
\[\begin{align*}
&\text{f is pure} \\
&f = g \circ g \text{ for some } \bigtriangledown\text{-self adjoint } g
\end{align*}\]

\[\text{Theorem. In } \mathcal{V}N, \text{ the } \bigtriangledown\text{-positive maps}\]
\[\text{are precisely } a \rightarrow \sqrt{b}a\sqrt{b}.\]
&-effectus
Dfn. an $\&$-effectus is $\hat{*}$-effectus where

- for every predicate $p$ on $X$, there is a unique $\hat{*}$-positive map $\text{asrt}_p : X \to X$ with $1_0 \text{asrt}_p = p$.

- $\xi \circ \pi$ is pure for any comprehension $\pi$ and quotient map $\xi$. 
Dfn. an $\&$-effectus is $\circ$-effectus where

- for every predicate $p$ on $X$, there is a unique $\circ$-positive map $asrt_p : X \to X$ with $1o asrt_p = p$.
- $\xi o \pi$ is pure for any comprehension $\pi$ and quotient map $\xi$.

Write $p \& q \equiv q o asrt_p$, and $p^2 \equiv p \& p$. 
Proposition. In an $\&$-effectus TFAE

- $p$ is sharp
- $p \& p = p$
- $\text{asrt}_p \circ \text{asrt}_p = \text{asrt}_p$
Proposition. In an $\&$-effectus TFAE
  1. $p$ is sharp
  2. $p \& p = p$
  3. $\text{asrt}_p \circ \text{asrt}_p = \text{asrt}_p$

Proposition. In an $\&$-effectus

\[ \text{im } f \leq s \iff \text{asrt}_s \circ f = f \]

\[ 1 \circ f \leq \epsilon \iff f \circ \text{asrt}_\epsilon = f \]
Polar decomposition

In an $A$-effectus, any pure map $f: X \to Y$ factors as follows:

$$
X \xrightarrow{-} X \xrightarrow{\xi} \frac{X}{\mathcal{F}_0} \xrightarrow{\pi} \{y \mid \text{im} f \} \xrightarrow{\pi} Y
$$

- Pure map $h$ with $\mathcal{F}_0$ sharp
- Partial isometry
Dfn. a $\dag$-effectus $\mathcal{C}$ is an $\&$-effectus, where the subcat. of pure maps is a $\dag$-category with:

- $\text{asrt}_p^\dag = \text{asrt}_p$
- $f$ is $\bowtie$-adjoint to $f^\dag$
- For every $\dag$-positive $f$, there is a unique $\dag$-positive $g$ with $f = g \circ g$.
- $\bowtie$-positive maps are $\dag$-positive.
Dfn. a $\dagger$-effectus $C$ is an $\&$-effectus, where
the subcat. of pure maps is a $\dagger$-category with:

- $\text{asr} \mathcal{E}_p = \text{asr} \mathcal{E}_p$
- $f$ is $\&$-adjoint to $f^\dagger$
- for every $\dagger$-positive $f$, there is a unique $\dagger$-positive $g$ with $f = g \circ g$
- $\&$-positive maps are $\dagger$-positive

It follows $10f^\dagger = \text{im} f$ and $\text{im} f^\dagger = 10f^\dagger$, 
Dfn. a $\dagger$-effectus $\mathcal{C}$ is an $\&$-effectus, where the subcat. of pure maps is a $\dagger$-category with:

- $\text{asrt}_p^\dagger = \text{asrt}_p$
- $f$ is $\&$-adjoint to $f^\dagger$
- for every $\dagger$-positive $f$, there is a unique $\dagger$-positive $g$ with $f = g \circ g$.
- $\&$-positive maps are $\dagger$-positive.

It follows $\text{im} f^\dagger = \text{im} f$ and $\text{im} f^\dagger = \text{im} f^\dagger$, (so with slight abuse of notation):

$$\pi_5^\dagger = \xi_5 \perp \text{ and } \xi_5^\dagger = \pi_5 \text{ (sharps $s$)}$$
Theorem. An \&-effectus is a \&-effectus iff

1. for every predicate \( p \), there is a unique predicate \( q \) with \( q \& q = p \)
2. \( \text{asrt}^2_{p \& q} = \text{asrt}_p \circ \text{asrt}^2_q \circ \text{asrt}_p \)
3. \( \xi s \) is sharp for all sharp \( s, t \).
Theorem. An $\&$-effectus is a $\dagger$-effectus iff

1. for every predicate $p$, there is a unique predicate $q$ with $q \& q = p$
2. $\text{asrt}_p^2 p \& q = \text{asrt}_p \circ \text{asrt}_q^2 \circ \text{asrt}_p$

Cf. Fundamental Id. of Quad. Jordan Algs.

3. to $\xi$, is sharp for all sharp $s, t$. 

Dfn. An effectus is operational iff

- the scalars are isomorphic to $[0, 1]$,
- the predicates are jointly monic,
- $p \leq q \Rightarrow \forall \omega. p \omega \leq q \omega$,
- every object $X$ is ‘finite-dimensional’: that is: $\text{Stat } X$ ‘is’ a closed convex subset of a finite-dimensional vector space.
Theorem. (Wetering)

• The category $EJA$ of Euclidean Jordan Algebras with positive maps is a $t$-effectus.

• Any operational $t$-effectus is equivalent to a subcategory of $EJA$. 
Theorem. (Wetering)

• The category $\text{EJA}$ of Euclidean Jordan Algebras with positive maps is a $t$-effectus.

• Any operational $t$-effectus is equivalent to a subcategory of $\text{EJA}$.

($\text{vN}^\text{op}$ is $t$-effectus, but not operational)
Theorem. Every *t*-effectus is a homological category in the sense of Martin Grandis.
Theorem. Every t-effectus is a homological category in the sense of Martin Grandis.

Corollary. Grandis’ Snake Lemma holds for von Neumann algebras...
Grandis’ Snake Lemma. If we have a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0 \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & \\
0 & \rightarrow & A' & \rightarrow & B' & \rightarrow & C'
\end{array}
\]

in a $t$-effectus such that

- $\text{im } f = \Gamma_{10g7}^\bot$
- $\text{im } h = \Gamma_{10k7}^\bot$
- $g$ quotient with $10g$ sharp
- $h$ comprehension

then...

- $b^0(b_o(\text{im } f)) = \Gamma_{10b7}^\bot \text{v } \text{im } f$
- $b_o(b^0(\text{im } h)) = (\text{im } h) \wedge \text{im } b$
- $k^0(k_o(\text{im } b)) = (\text{im } h) \text{v } \text{im } b$
- $f_o(f^0(b_o(0))) = \Gamma_{10b7}^\bot \text{v } \text{im } f$
\[ \{A \parallel (10a)^{\perp}\} \xrightarrow{\tilde{\xi}} \{B \parallel (10b)^{\perp}\} \xrightarrow{\tilde{g}} \{C \parallel (10c)^{\perp}\} \]

\[ \downarrow \pi_{(10a)^{\perp}} \quad \downarrow \pi_{(10b)^{\perp}} \quad \downarrow \pi_{(10c)^{\perp}} \]

\[ A \xrightarrow{f} B \xrightarrow{g} C \rightarrow O \]

\[ \downarrow a \quad \downarrow b \quad \downarrow c \]

\[ 0 \rightarrow A' \xrightarrow{h} B' \xrightarrow{k} C' \]

\[ \downarrow \xi_{ima} \quad \downarrow \xi_{imc} \quad \downarrow \xi_{imb} \]

\[ A' \parallel ima \xrightarrow{k} B' \parallel imb \xrightarrow{k} C' \parallel imc \]

with \( \text{im} f = \Gamma_{(001)^{\perp}}, \text{im} \tilde{g} = \Gamma_{(10d)^{\perp}}, \ldots \)
Take away

- QT reconstructions don't need dilations/purifications or parallel composition $\otimes$.
- $\text{compVh} \circ \text{iso} \circ \text{qnot.}$ is the right kind of pure.
- $(\ )^\circ$, $\Delta$-adjoint and $\Delta$-positive.

Further reading

- "Dagger and dilations", BW arXiv 1803.01911.
- "Introduction to Effectus theory"
  Cho, Jacobs, Westerbaan, BW arXiv 1512.05813
- "Reconstruction of Quantum Theory from univ. filters"
  Wetering arXiv 1801.05798