Endomorphism Semialgebras in Categorical Quantum Mechanics

Kevin Dunne

University of Strathclyde

November 2016

Symmetric Monoidal Categories

Definition

A strict symmetric monoidal category $(\mathcal{A}, \otimes, I)$ consists of

- Objects $A, B, C, \ldots$
- Morphisms $f : A \to B$
- Monoidal product $\otimes$

$$f : A \to B \qquad\qquad g : C \to D$$

$$f \otimes g : A \otimes B \to C \otimes D$$

- Monoidal unit I

$$I \otimes A \;\cong\; A \;\cong\; A \otimes I$$

†-Symmetric Monoidal Categories

Definition

A *dagger* on $(\mathcal{A}, \otimes, I)$ consists of an involutive symmetric monoidal functor

$$\dagger : \mathcal{A}^{\mathsf{op}} \to \mathcal{A}$$

i.e. every morphism has an adjoint

$$A \xrightarrow{\quad f \quad} B \qquad\qquad B \xrightarrow{\quad f^{\dagger} \quad} A$$

$$f^{\dagger\dagger} = f$$

†-Biproducts

A †-category *has finite †-biproducts* it has:

- a zero object 0
- for each X and Y, an object $X \oplus Y$, which is a product and coproduct such that $\pi = \kappa^\dagger$

For $f, g : X \to Y$ define the *biproduct convolution* $f + g : X \to Y$

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

Examples

The category **Hilb** has:

- Objects: Hilbert spaces
- Morphisms $f : H_1 \to H_2$ bounded linear maps
- Monoidal product: tensor product of Hilbert spaces
- Monoidal unit : $\mathbb{C}$
- Dagger: Hermitian adjoint
- Biproducts: direct sum of Hilbert spaces
- Biproduct convolution: pointwise sum of operators

Examples

The category **Rel** has:

- Objects: sets
- Morphisms $f : A \to B$ relations
- Monoidal product: Cartesian product of sets
- Monoidal unit : singleton $\{*\}$
- Dagger: $a \sim_{f^\dagger} b$ iff $b \sim_f a$
- Biproducts: disjoint union of sets
- Biproduct convolution: union of relations

Frobenius Algebras

Definition

A †-special commutative Frobenius algebra in $(\mathcal{A}, \otimes, I)$ consists of:

- An object X
- An algebra $\mu : X \otimes X \to X \quad \eta : I \to X$
- Its adjoint, a coalgebra $\mu^\dagger : X \to X \otimes X, \quad \eta^\dagger : X \to I$

Satisfying some equations...

Example - **fdHilb**

Let $\{ |e_i\rangle \}_{i \in I}$ be an orthonormal basis. Define the Frobenius algebra:

$$H \otimes H \xrightarrow{\mu} H \qquad\qquad H \xrightarrow{\mu^\dagger} H \otimes H$$

$$|e_i\rangle \otimes |e_j\rangle \longmapsto \delta_{ij}\ |e_i\rangle \qquad\qquad |e_i\rangle \longmapsto |e_i\rangle \otimes |e_i\rangle$$

Example - **fdHilb**

Let $\{\, |e_i\rangle \,\}_{i \in I}$ be an orthonormal basis. Define the Frobenius algebra:

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\quad \mu \quad} & H \\
|e_i\rangle \otimes |e_j\rangle & \longmapsto & \delta_{ij}\, |e_i\rangle
\end{array}
\qquad\qquad
\begin{array}{ccc}
H & \xrightarrow{\quad \mu^\dagger \quad} & H \otimes H \\
|e_i\rangle & \longmapsto & |e_i\rangle \otimes |e_i\rangle
\end{array}
$$

A new description of orthogonal bases

Bob Coecke, Dusko Pavlovic and Jamie Vicary
Oxford University Computing Laboratory

Theorem (Coecke, Pavlovic, Vicary)

*Every Frobenius algebra in **fdHilb** is of this form.*

Observables in Monoidal Categories

"Hence orthogonal and orthonormal bases can be axiomatised in terms of composition of operations and tensor product only, without any explicit reference to the underlying vector spaces."

Measurement Outcomes

Definition

Let $\mu^\dagger : X \to X \otimes X$ be a Frobenius algebra

Elements $x \in X$ are called *set-like* if

$$\mu^\dagger(x) = x \otimes x$$

Frobenius algebras in **Rel**

Theorem (Pavlovic)

The Frobenius algebras in $(\textbf{\textit{Rel}}, \times, \{*\})$ *are exactly the abelian groupoids.*

$$\mu : X \times X \to X$$

$$\text{i.e. } X \cong \bigsqcup_{i \in I} A_i \text{ for abelian groups } A_i$$

Set-like elements $\alpha : \{*\} \to X$ are the subsets $A_i \subset X$

The Topos Approach

For H a Hilbert space, $\mathrm{Hom}(H, H)$ is a C^*-algebra

Definition

The category **Hilb-Alg**(H) has

- Objects: commutative C^*-subalgebras

$$\mathbf{C} \subset \mathrm{Hom}(H, H)$$

- Morphisms: inclusions of subalgebras

The subcategory $\textbf{Hilb-Alg}_{\mathrm{vN}}(H) \hookrightarrow \textbf{Hilb-Alg}(H)$ of commutative von Neumann subalgebras

The Gelfand Spectrum

$$\mathsf{Spec} : \mathbf{Hilb\text{-}Alg}(H)^{\mathsf{op}} \to \mathbf{Set}$$

$$\begin{aligned} \mathsf{Spec}(\mathbf{C}) &= \{\ \rho : \mathbf{C} \to \mathbb{C} \mid \rho \text{ a unital } C^*\text{-algebra homomorphism } \} \\ &= \text{ the Gelfand Spectrum of } \mathbf{C} \end{aligned}$$

The Structure of Finite Dimensional C^*-Algebras

For $H \in \mathbf{fdHilb}$, and $\mathbf{C} \in \mathbf{Hilb\text{-}Alg}_{\mathrm{vN}}(H)$

$$\mathbf{C} \cong \bigoplus_{p_i} p_i \mathbf{C}$$

for self-adjoint orthgonal idempotents p_i

$$p_i p_i = p_i = p_i^{\dagger} \qquad p_i p_j = 0 \text{ if } i \neq j$$

$$\sum_i p_i = \mathrm{id} : H \to H$$

$$\mathrm{Spec}(\mathbf{C}) \cong \{p_i\}_i$$

Connection in Finite Dimension

For $H \in \mathbf{fdHilb}$ there is a correspondence

$$\begin{array}{ccc} \text{Frobenius algebra} & & \\ \mu : H \otimes H \to H & \longrightarrow & \mathbf{C} \in \mathbf{Hilb\text{-}Alg}_{\mathrm{vN}}(H) \end{array}$$

$$\begin{array}{ccc} \text{Set-like elements} & & \\ \text{of } \mu & \longrightarrow & \mathrm{Spec}(\mathbf{C}) \end{array}$$

for each $|x\rangle \in H$

$$\begin{array}{ccc} H & \xrightarrow{\mu_{|x\rangle}} & H \\ |y\rangle & \longmapsto & \mu\big(|x\rangle \otimes |y\rangle\big) \end{array}$$

$$\mathbf{C} = \{\ \mu_{|x\rangle} : H \to H \mid \text{for each } |x\rangle \in H\ \}$$

Semialgebras

A Ring R	A Semiring S
An R-Module	An S-Semimodule
An Associative R-Algebra	An Associative S-Semialgebra

Semirings

Definition

A *semiring* S consists of a set S together with a commutative monoid structure:

$$+ : S \times S \to S$$

and a monoid structure:

$$\cdot : S \times S \to S$$

such that multiplication distributes over addition.

A $*$-*semiring* is one equipped with an involution

$$S \xrightarrow{(-)^*} S$$

Semimodules

Let S be a commutative semiring

Definition

An S-semimodule consists of a commutative monoid $(M, +_M, 0_M)$ together with *scalar multiplication*

$$S \times M \to M$$

Semialgebras

Let S be a commutative semiring.

Definition

An associative *S-semialgebra* consists of a semimodule $(M, +_M, 0_M)$ equipped with a monoid structure

$$\cdot_M : M \times M \to M$$

such that $(M, +_M, 0_M, \cdot_M, 1_M)$ forms a semiring.

For S a $*$-semiring, an S^*-semialgebra is one such that $(M, +_M, 0_M, \cdot_M, 1_M)$ is a $*$-semiring.

Examples

Let $(\mathcal{A}, \otimes, I)$ be a †-symmetric monoidal category with †-biproducts.

Hom(I, I) has the structure of a commutative $*$-semiring with

- Multiplication given by morphism composition
- Involution given by †
- Addition given by biproduct convolution

$$I \xrightarrow{\Delta} I \oplus I \xrightarrow{s \oplus r} I \oplus I \xrightarrow{\nabla} I$$

Examples

Let $(\mathcal{A}, \otimes, I)$ be a †-symmetric monoidal category with †-biproducts.

For each X and Y in $\mathcal{A}$, $\mathrm{Hom}(X, Y)$ is an S-semimodule, for $S = \mathrm{Hom}(I, I)$.

Addition given by biproduct convolution

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

Scalar multiplication given by

$$X \xrightarrow{\sim} X \otimes I \xrightarrow{f \otimes s} Y \otimes I \xrightarrow{\sim} Y$$

Examples

Let $(\mathcal{A}, \otimes, I)$ be a $\dagger$-symmetric monoidal category with $\dagger$-biproducts.

For each object X in $\mathcal{A}$, $\mathrm{Hom}(X, X)$ is an S^*-semialgebra.

Multiplication given by morphism composition.

Categories of Endomorphism S^*-Semialgebras

For $(\mathcal{A}, \otimes, I)$ and $X \in \mathcal{A}$ define the category

$$\mathcal{A}\text{-}\mathbf{Alg}(X)$$

- Objects: commutative $*$-subsemialgebras of $\mathrm{Hom}(X, X)$
- Morphisms: inclusions of subsemialgebras

Define the subcategory

$$\mathcal{A}\text{-}\mathbf{Alg}_{\mathrm{vN}}(X) \hookrightarrow \mathcal{A}\text{-}\mathbf{Alg}(X)$$

Semialgebras satisfying $\mathbf{A}'' = \mathbf{A}$ where

$$\mathbf{A}' = \{ \ f : A \to A \mid fg = gf \text{ for all } g \in \mathbf{A} \ \}$$

The Abstract Spectral Presheaf

$(\mathcal{A}, \otimes, I)$ with scalars $S = \mathsf{Hom}(I, I)$

Define the functor

$$\mathsf{Spec} : \mathcal{A}\text{-}\mathbf{Alg}(X)^{\mathsf{op}} \to \mathbf{Set}$$

$$\mathsf{Spec}(\mathbf{A}) = \{\ \rho : \mathbf{A} \to S \mid \rho \text{ a unital } *\text{-algebra homomorphism}\ \}$$

Endomorphism Semialgebras of Relations

Now we consider $(\mathcal{A}, \otimes, I) = (\mathbf{Rel}, \times, \{*\})$

$$\mathbf{Rel\text{-}Alg}_{\mathsf{vN}}(A) \hookrightarrow \mathbf{Rel\text{-}Alg}(A)$$

This is a proper subcategory, even for finite A.

Endomorphism Semialgebras of Relations

Questions:

What do the semialgebras $\mathbf{A} \in \mathbf{Rel\text{-}Alg}_{\mathsf{vN}}(A)$ look like?

For a semialgebra $\mathbf{A} \in \mathbf{Rel\text{-}Alg}_{\mathsf{vN}}(A)$, what does $\mathrm{Spec}(\mathbf{A})$ look like?

What do the Frobenius algebras $\mu : A \times A \to A$ have to do with the semialgebras $\mathbf{A} \in \mathbf{Rel\text{-}Alg}_{\mathsf{vN}}(A)$?

A Structure Theorem for **Rel**

Theorem

For $\boldsymbol{A} \in \boldsymbol{Rel\text{-}Alg}_{vN}(A)$

$$\boldsymbol{A} \cong \bigoplus_i \; p_i \boldsymbol{A}$$

for self-adjoint orthgonal idempotents p_i

$$p_i p_i = p_i = p_i^\dagger \qquad\qquad p_i p_j = 0 \text{ if } i \neq j$$

$$\sum_i p_i = \mathrm{id} : A \to A$$

$$Spec(\boldsymbol{A}) \cong \{p_i\}_i$$

We Already Saw...

For $H \in \mathbf{fdHilb}$ there is a correspondence

Frobenius Algebra $\mu : H \otimes H \to H$	$\longrightarrow$	$\mathbf{C} \in \mathbf{Hilb\text{-}Alg}_{\mathsf{vN}}(H)$
Set-like elements of μ	$\longrightarrow$	$\mathrm{Spec}(\mathbf{C})$

Frobenius Algebras and von Neumann Semialgebras

For $A \in \mathbf{Rel}$ there is a correspondence

Frobenius Algebra $\mu : A \times A \to A$	$\longrightarrow$	$\mathbf{A} \in \mathbf{Rel\text{-}Alg}_{\mathsf{vN}}(A)$
Set-like elements of μ	$\longrightarrow$	$\mathrm{Spec}(\mathbf{A})$

Frobenius Algebras and von Neumann Semialgebras

Theorem

Each Frobenius algebra $\mu : A \times A \to A$ determines a semialgebra

$$\boldsymbol{A} \in \boldsymbol{Rel\text{-}Alg}_{vN}(A)$$

for each $B \subseteq A$, define

$$A \xrightarrow{\quad \mu_B \quad} A$$

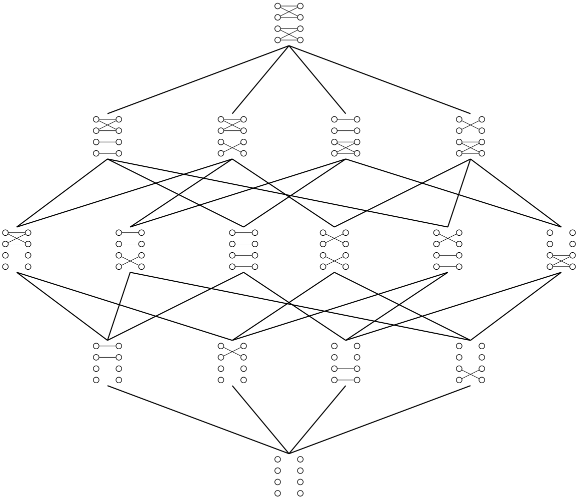
$$a \sim_{\mu_B} c \quad \text{iff} \quad \exists b \in B \ \text{ such that } \ (a, b) \sim_\mu c$$

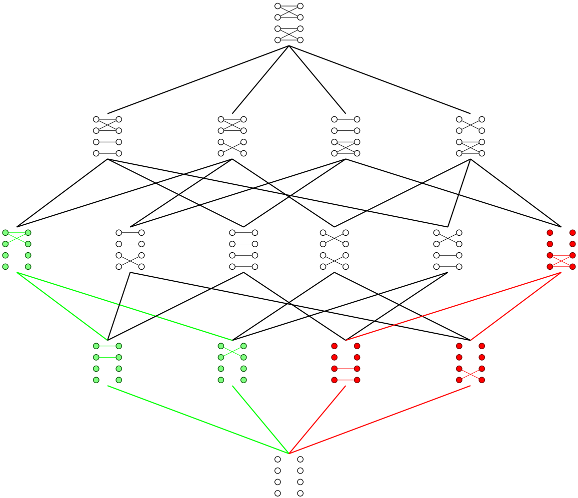
$$\mathbf{A} = \{\ \mu_B : A \to A \mid B \subseteq A\ \}$$

Example

Let $A = \{a, b, c, d\}$, and consider the Frobenius algebra $A \cong \mathbb{Z}_2 \sqcup \mathbb{Z}_2$

μ	a	b	c	d
a	a	b	$\emptyset$	$\emptyset$
b	b	a	$\emptyset$	$\emptyset$
c	$\emptyset$	$\emptyset$	c	d
d	$\emptyset$	$\emptyset$	d	c





Observationally Equivalent Frobenius Algebras

Definition

Two Frobenius algebras in $(\mathcal{A}, \otimes, I)$

$$\mu : X \otimes X \to X \qquad \nu : X \otimes X \to X$$

are said to be *observationally equivalent* if they are isomorphic and have the same set-like elements.

Observationally Equivalent Frobenius Algebras

$A = \{a, b\}$ can be endowed with the group structure $\mathbb{Z}_2$ in two ways

μ	a	b
a	a	b
b	b	a

ν	a	b
a	b	a
b	a	b

Frobenius Algebras and von Neumann Semialgebras

Theorem
*Observationally equivalent Frobenius algebras in **Rel** determine identical endomorphism S^*-semialgebras.*

Future Work

- How much of what holds for **fdHilb** and **Rel** holds for $\mathcal{A}$?
- Can we bring phase groups, complementarity, etc. into the endomorphism semialgebra picture?
- Does the spectral presheaf for **Rel** have any global sections?