Infinite-dimensional Categorical Quantum Mechanics
A talk for CLAP Scotland

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5 Apr 2017
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We want to do (diagrammatic) CQM in $\infty$-dimensions, but...

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  - NO traces, cups or caps
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Can we recover all of this (using non-standard analysis)?

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Can we recover all of this (using non-standard analysis)? **YES, WE CAN.**

\(^1\)Although there is a characterisation of orthonormal bases in terms of $H^*$-algebras.
Non-standard analysis: an algebraic way to handle limit constructions\(^2\).

\(^2\)Regardless of topological convergence. The sceptics out there might prefer to think directly in terms of the ultraproduc construction: we work in spaces of sequences, quotiented by a notion of “asymptotic equality”, or “equality almost everywhere”, determined by some non-principal ultrafilter \(\mathcal{F}\) on \(\mathbb{N}\).
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(a) Natural numbers are unbounded, and hence:

(i) infinite non-standard natural numbers exist  
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(b) Algebraic manipulation of series (without taking limits):
   (i) consider a sequence of partial sums $a_n := \sum_{j=1}^{n} b_j$
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(c) Some genuinely new finite vectors arise in non-standard Hilbert spaces:

$$\frac{1}{\sqrt{\nu}} \sum_{n=1}^{\nu} |e_n\rangle,$$
where $\left\{|e_n\rangle \right\}_{\nu}$ form an orthonormal basis

$\nu$ is an infinite natural

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The heavy lifting in non-standard analysis is done by the following result.

**Theorem (Transfer Theorem)**

A sentence $\varphi$ holds in the standard model $M$ of some theory—with quantifiers ranging over standard elements, functions, relations and subsets—if and only if the sentence $\varphi$ holds in any/all non-standard models $\star M$ of the theory—with quantifiers ranging over internal non-standard elements, functions, relations and subsets.
Example (Natural predecessors)

Consider the sentence defining predecessors in the natural numbers:

$$\forall n \in \mathbb{N}. \left[ n \neq 0 \Rightarrow \exists m \in \mathbb{N}. n = m + 1 \right]$$

By TT, the following sentence holds in the non-standard model $*\mathbb{N}$:

$$\forall n \in *\mathbb{N}. \left[ n \neq 0 \Rightarrow \exists m \in *\mathbb{N}. n = m + 1 \right]$$

Hence all non-zero non-standard naturals have predecessors.
Example (Well-ordering of naturals)

Consider the sentence defining the well-order property for the natural numbers, i.e. saying that every non-empty subset of $\mathbb{N}$ has a minimum:

$$\forall A \subseteq \mathbb{N}. \left( A \neq \emptyset \Rightarrow \exists m \in A. \forall a \in A. m \leq a \right)$$

By TT, the following sentence holds in the non-standard model $\ast \mathbb{N}$:

$$\forall A \subseteq \ast \mathbb{N}. \left( \ast A \neq \emptyset \Rightarrow \exists m \in \ast A. \forall a \in \ast A. m \leq a \right)$$

Hence all non-empty internal subsets $A \subseteq \ast \mathbb{N}$ have a minimum. (The requirement that $A$ be internal is key here: e.g. the subset of all infinite non-standard naturals has no minimum, but it is also not internal.)
Example (Partial sums)

Consider the sentence defining the sequence \( s : \mathbb{N} \to \mathbb{R} \) of partial sums for every sequence \( f : \mathbb{N} \to \mathbb{R} \) in the standard model \( \mathbb{R} \):

\[
\forall f : \mathbb{N} \to \mathbb{R}. \exists s : \mathbb{N} \to \mathbb{R}.
\quad [s(0) = f(0) \land [\forall n \in \mathbb{N}. s(n + 1) = s(n) + f(n + 1)]]
\]

By TT, the following sentence holds in the non-standard model \( \ast \mathbb{R} \):

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\forall f : \ast \mathbb{N} \to \ast \mathbb{R}. \exists s : \ast \mathbb{N} \to \ast \mathbb{R}.
\quad [s(0) = f(0) \land [\forall n \in \ast \mathbb{N}. s(n + 1) = s(n) + f(n + 1)]]
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Hence every internal sequence \( f : \ast \mathbb{N} \to \ast \mathbb{R} \) admits a corresponding internal sequence of partial sums \( s : \ast \mathbb{N} \to \ast \mathbb{R} \), i.e. the notation \( \sum_{n=0}^{m} f(n) \) is legitimate for all \( m \in \ast \mathbb{N} \).
The category $^\ast$Hilb - objects

Objects are pairs $\mathcal{H} := (|\mathcal{H}|, P_\mathcal{H})$ specified by the following data:

(i) a non-standard Hilbert space $|\mathcal{H}|$ (the underlying Hilbert space);

(ii) an internal non-standard linear map $P_\mathcal{H} : |\mathcal{H}| \to |\mathcal{H}|$ such that:

- $P_\mathcal{H}$ is a self-adjoint idempotent (the truncating projector);

- there are a non-standard natural $D \in ^\ast \mathbb{N}$ and a family $(|e_d\rangle)_{d=1}^D$ of non-standard vectors in $|\mathcal{H}|$ (an orthonormal basis for $\mathcal{H}$) such that $P_\mathcal{H} = D \sum_{d=1}^D |e_d\rangle\langle e_d|$.

By Transfer Theorem we have that $D$ is unique, and we define the dimension of object $\mathcal{H}$ to be the non-standard natural $\text{dim} \mathcal{H} := D$. 

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- \( P_\mathcal{H} \) is a self-adjoint idempotent (the **truncating projector**);
- there are a non-standard natural \( D \in \mathbb{\ast \mathbb{N}} \) and a family \((|e_d\rangle)_{d=1}^{D}\) of non-standard vectors in \( |\mathcal{H}| \) (an **orthonormal basis** for \( \mathcal{H} \)) such that

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P_\mathcal{H} = \sum_{d=1}^{D} |e_d\rangle\langle e_d|
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By Transfer Theorem we have that $D$ is unique, and we define the dimension of object $\mathcal{H}$ to be the non-standard natural $\dim \mathcal{H} := D$. 
Morphisms $F : \mathcal{H} \rightarrow \mathcal{K}$ in $\ast\text{Hilb}$ are the those internal non-standard linear maps $F : |\mathcal{H}| \rightarrow |\mathcal{K}|$ such that:

$$P_{\mathcal{K}} \circ F \circ P_{\mathcal{H}} = F$$
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In particular, the identity for an object $\mathcal{H}$ is the truncating projector:

$$id_\mathcal{H} := P_\mathcal{H}$$
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This makes $\ast\text{Hilb}$ a full subcategory of the Karoubi envelope for the category of non-standard Hilbert spaces and $\ast\mathbb{C}$-linear maps.
Morphisms $F : \mathcal{H} \to \mathcal{K}$ in $\ast\text{Hilb}$ can be expressed as matrices with non-standard dimensions, using orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$:

$$F = \dim \mathcal{K} \dim \mathcal{H} \sum_{d' = 1}^{\dim \mathcal{K}} \sum_{d = 1}^{\dim \mathcal{H}} |e'_{d'}\rangle F_{d' d} \langle e_d|$$

In particular, the identity on $\mathcal{H}$ can be expressed as follows:

$$\text{id}_\mathcal{H} = \dim \mathcal{H} \sum_{d = 1}^{\dim \mathcal{H}} |e_d\rangle \langle e_d|$$

Equipped with Kronecker product, conjugate transpose, and the $\ast\text{C}$-linear structure of matrices, $\ast\text{Hilb}$ is an enriched $\dagger$-symmetric monoidal category, with $\ast\text{C}$ as its field of scalars.
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The category $\mathcal{Hilb} - \dagger$-symmetric monoidal structure

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Equipped with Kronecker product, conjugate transpose, and the $\mathbb{C}$-linear structure of matrices, $\mathcal{Hilb}$ is an enriched $\dagger$-symmetric monoidal category, with $\mathbb{C}$ as its field of scalars.
If $|e_d\rangle_{d=1}^{\dim \mathcal{H}}$ is an orthonormal basis for $\mathcal{H}$, the following comultiplication and counit define a unital special commutative $\dagger$-Frobenius algebra on $\mathcal{H}$:

$$
\text{comultiplication} := \sum_{d=1}^{\dim \mathcal{H}} |e_d\rangle \otimes |e_d\rangle \otimes \langle e_d|,
$$

$$
\text{counit} := \sum_{d=1}^{\dim \mathcal{H}} \langle e_d|.
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If $\ket{e_d}_{d=1}^\dim \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$, the following comultiplication and counit define a unital special commutative $\dagger$-Frobenius algebra on $\mathcal{H}$:

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\begin{align*}
\delta \colon & \quad \sum_{d=1}^{\dim \mathcal{H}} \ket{e_d} \otimes \ket{e_d} \otimes \bra{e_d} \\
\epsilon \colon & \quad \sum_{d=1}^{\dim \mathcal{H}} \bra{e_d}
\end{align*}
\]

When $\ket{e_d}_{d=1}^\dim \mathcal{H}$ is the non-standard extension of a standard complete orthonormal basis $\ket{e_d}_{d=1}^\infty$, the comultiplication is the non-standard extension of the standard isometry given by the $\mathbb{H}^*$-algebra associated with $\ket{e_d}_{d=1}^\infty$. In that case, the counit is the genuinely non-standard object.
The category $\mathcal{H}^*$-dagger compact structure

(i) Consider an object $\mathcal{H}$, and a decomposition $P_\mathcal{H} = \sum_{d=1}^{\dim \mathcal{H}} |e_d\rangle\langle e_d|$ of its truncating projector in terms of some orthonormal basis of $\mathcal{H}$.

(ii) Let $|\xi_d\rangle$ be the state in $|\mathcal{H}|^*$ corresponding to the effect $\langle e_d|$ in $\mathcal{H}$.

(iii) The dual object is defined by $\mathcal{H}^* = (|\mathcal{H}|^*, P_\mathcal{H}^*)$, where we let

$$P_\mathcal{H}^* = \frac{\dim \mathcal{H}}{\sum_{n=1}^\text{dim } \mathcal{H} |\xi_n\rangle \otimes |e_n\rangle}.$$ 

(iv) Cups and caps can then be defined as follows:

$$\dim \mathcal{H} \sum_{n=1}^{\text{dim } \mathcal{H}} |\xi_n\rangle \otimes |e_n\rangle = \dim \mathcal{H} \sum_{n=1}^{\text{dim } \mathcal{H}} \langle e_n| \otimes \langle \xi_n|.$$ 

(v) The category-theoretic dimension for $\mathcal{H}$ is $\text{Tr} P_\mathcal{H} = \dim \mathcal{H}$.

By Transfer Theorem, this definition is independent of the choice of basis.
Consider an object \( \mathcal{H} \), and a decomposition \( P_{\mathcal{H}} = \sum_{d=1}^{\dim \mathcal{H}} |e_d \rangle \langle e_d| \) of its truncating projector in terms of some orthonormal basis of \( \mathcal{H} \).

Let \( |\xi_d \rangle \) be the state in \( |\mathcal{H}|^* \) corresponding to the effect \( \langle e_d | \) in \( \mathcal{H} \).

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Consider an object $\mathcal{H}$, and a decomposition $P_\mathcal{H} = \sum_{d=1}^{\dim \mathcal{H}} |e_d\rangle\langle e_d|$ of its truncating projector in terms of some orthonormal basis of $\mathcal{H}$.

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The category $\text{*Hilb}$ - dagger compact structure

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$$P_{\mathcal{H}^*} := \sum_{d=1}^{\dim \mathcal{H}} |\xi_d\rangle\langle \xi_d|$$

(iv) Cups and caps can then be defined as follows:

$$\left( \sum_{n=1}^{\dim \mathcal{H}} |\xi_n\rangle \otimes |e_n\rangle \right) := \sum_{n=1}^{\dim \mathcal{H}} \langle e_n| \otimes \langle \xi_n|$$

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The category \(*\text{Hilb} - \text{dagger compact structure}\)

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(iv) Cups and caps can then be defined as follows:

\[
\begin{align*}
\cup & := \bigoplus_{n=1}^{\dim \mathcal{H}} |\xi_n\rangle \otimes |e_n\rangle \\
\cap & := \bigoplus_{n=1}^{\dim \mathcal{H}} \langle e_n| \otimes \langle \xi_n|
\end{align*}
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(v) The category-theoretic dimension for \(\mathcal{H}\) is \(\text{Tr} P_{\mathcal{H}} = \dim \mathcal{H}\).

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Wavefunctions in an $n$-dimensional box with periodic boundary conditions. 

(i) Underlying Hilbert space $\mathcal{L}^2[(\mathbb{R}/\mathbb{Z})^n]$.

(ii) Complete orthonormal basis of momentum eigenstates:

$$|\chi_k\rangle := x \rightarrow e^{-i2\pi k \cdot x}$$

(iii) Dimension $D := (2\omega + 1)^n$, where $\omega$ is some infinite natural.
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Classical structure corresponding to the momentum observable:

$$\sum_{k_1=-\omega}^{+\omega} \ldots \sum_{k_n=-\omega}^{+\omega} |\chi_k\rangle \otimes |\chi_k\rangle \otimes \langle \chi_k|$$
The following multiplication and unit define a unital quasi-special commutative $\dagger$-Frobenius algebra, with normalisation factor $(2\omega + 1)^n$:

$$
\begin{align*}
\cdots \sum_{k_1, h_1 = -\omega}^{+\omega} |\chi_{k+h}\rangle \otimes \langle \chi_k | \otimes \langle \chi_h | \\
\sum_{k_n, h_n = -\omega}^{+\omega} |\chi_{k+h}\rangle \otimes \langle \chi_k | \otimes \langle \chi_h |
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The addition used here is that of the abelian group $*\mathbb{Z}_{2\omega+1}^n$:

- from the point of view of $*\mathbb{Z}^n$, it is cyclic on $\{-\omega, \ldots, +\omega\}^n$;
- from the point of view of $\mathbb{Z}^n$, it cycles “beyond infinity”.

In particular, it contains $\mathbb{Z}^n$ as a proper subgroup.
The classical states for $\dot{x}$ are those in the following form, where $x$ takes the form $x = \frac{1}{2\omega + 1} q$ for some $q \in \mathbb{Z}^n_{2\omega + 1}$ (i.e. we have $x \in \mathbb{R}^n_{\omega + 1}$):

$$|\delta_x\rangle := \sum_{k_1 = -\omega}^{+\omega} \cdots \sum_{k_n = -\omega}^{+\omega} \chi_k(x)^* |\chi_k\rangle$$
The classical states for a wavefunction with periodic boundary conditions take the form $x = \frac{1}{2\omega + 1} q$ for some $q \in \mathbb{Z}^n_{2\omega + 1}$ (i.e. we have $x \in \frac{1}{2\omega + 1} \mathbb{Z}^n_{2\omega + 1}$):

$$|\delta_x\rangle := \sum_{k_1=-\omega}^{+\omega} \cdots \sum_{k_n=-\omega}^{+\omega} \chi_k(x)^* |\chi_k\rangle$$

The classical states for $\delta_x$ behave as Dirac deltas:

$$\langle \delta_{x_0} | f \rangle \simeq f(x_0),$$

for near-standard smooth $f$ and near-standard $x_0$.

We call them the **position eigenstates**, and $\delta_x$ the **position observable**.
The requirement that $x \in \frac{1}{2\omega+1} \mathbb{Z}_{2\omega+1}^n$ for position eigenstates $|\delta_x\rangle$ is a consequence of the fact that the functions $\chi_k$ are multiplicative characters of $\mathbb{Z}^n$, but not necessarily of $\mathbb{Z}_{2\omega+1}^n$. From the non-standard point of view, $\frac{1}{2\omega+1} \mathbb{Z}_{2\omega+1}^n$ is a periodic lattice of infinitesimal mesh $\frac{1}{2\omega+1}$ in the non-standard torus $\mathbb{R}/\mathbb{Z}^n$. From the standard point of view, $\frac{1}{2\omega+1} \mathbb{Z}_{2\omega+1}^n$ approximates all elements of the standard torus ($\mathbb{R}/\mathbb{Z}^n$) up to infinitesimal equivalence.
The requirement that $x \in \frac{1}{2\omega + 1} * \mathbb{Z}_{2\omega + 1}^n$ for position eigenstates $|\delta_x\rangle$ is a consequence of the fact that the functions $\chi_k$ are multiplicative characters of $\mathbb{Z}_n$, but not necessarily of $* \mathbb{Z}_{2\omega + 1}^n$.

An undesirable extra phase $e^{i2\pi(2\omega + 1)s \cdot x}$ (for generic $s_j \in \{-1, 0, +1\}$) appears when equation $|\delta_x\rangle \otimes |\delta_x\rangle$ is expanded, and this phase cancels out in general if and only if $x \in \frac{1}{2\omega + 1} * \mathbb{Z}_{2\omega + 1}^n$. 
Interlude - approximating tori by periodic lattices

- The requirement that \( x \in \frac{1}{2\omega + 1} \mathbb{Z}^n_{2\omega + 1} \) for position eigenstates \( |\delta_x\rangle \) is a consequence of the fact that the functions \( \chi_k \) are multiplicative characters of \( \mathbb{Z}^n \), but not necessarily of \( \mathbb{Z}^n_{2\omega + 1} \).

- An undesirable extra phase \( e^{i2\pi(2\omega + 1)s \cdot x} \) (for generic \( s_j \in \{-1, 0, +1\} \)) appears when equation \( \bigotimes_{\sigma} |\delta_x\rangle = |\delta_x\rangle \otimes |\delta_x\rangle \) is expanded, and this phase cancels out in general if and only if \( x \in \frac{1}{2\omega + 1} \mathbb{Z}^n_{2\omega + 1} \).

- From the non-standard point of view, \( \frac{1}{2\omega + 1} \mathbb{Z}^n_{2\omega + 1} \) is a periodic lattice of infinitesimal mesh \( \frac{1}{2\omega + 1} \) in the non-standard torus \( *(\mathbb{R}/\mathbb{Z})^n \).
The requirement that \( x \in \frac{1}{2\omega+1} \ast \mathbb{Z}_{2\omega+1}^n \) for position eigenstates \( |\delta_x\rangle \) is a consequence of the fact that the functions \( \chi_k \) are multiplicative characters of \( \mathbb{Z}^n \), but not necessarily of \( \ast \mathbb{Z}_{2\omega+1}^n \).

An undesirable extra phase \( e^{i2\pi(2\omega+1)s \cdot x} \) (for generic \( s_j \in \{-1, 0, +1\} \)) appears when equation \( \bigotimes |\delta_x\rangle = |\delta_x\rangle \otimes |\delta_x\rangle \) is expanded, and this phase cancels out in general if and only if \( x \in \frac{1}{2\omega+1} \ast \mathbb{Z}_{2\omega+1}^n \).

From the non-standard point of view, \( \frac{1}{2\omega+1} \ast \mathbb{Z}_{2\omega+1}^n \) is a periodic lattice of infinitesimal mesh \( \frac{1}{2\omega+1} \) in the non-standard torus \( \ast (\mathbb{R}/\mathbb{Z})^n \).

From the standard point of view, \( \frac{1}{2\omega+1} \ast \mathbb{Z}_{2\omega+1}^n \) approximates all elements of the standard torus \( (\mathbb{R}/\mathbb{Z})^n \) up to infinitesimal equivalence.
Case study - wavefunctions with periodic boundary

The position and momentum observables are strongly complementary, a manifestation of the Weyl Canonical Commutation Relations.

\[
\delta x \chi_k = \chi_k \delta x \quad \delta x \chi_k T_x T_x B_k B_k \chi_k^* (x)
\]
Case study - wavefunctions with periodic boundary

The position and momentum observables are strongly complementary, a manifestation of the Weyl Canonical Commutation Relations.

- Position observable defined by the group algebra for boosts $B_k$.
- Momentum observable acts as the group algebra for translations $T_x$:

$$\left\{ |\delta_x\rangle \mid x \in \frac{1}{2\omega+1} \mathbb{Z}^n_{2\omega+1} \right\} \approx \left( \frac{1}{2\omega+1} \mathbb{Z}^n_{2\omega+1}, +, 0 \right)$$
Case study - wavefunctions with periodic boundary

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The Weyl Canonical Commutation Relations in graphical form:
Wavefunctions on an $n$-dimensional lattice $\mathbb{Z}^n$.

(i) Underlying Hilbert space $L^2[\mathbb{Z}^n]$.

(ii) Complete orthonormal basis of position eigenstates:

$$|\delta_k\rangle := h \mapsto \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise} \end{cases}$$

(iii) Dimension $D := (2\omega + 1)^n$, where $\omega$ is some infinite natural.
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Classical structure corresponding to the position observable:

$$\begin{array}{c}
\sum_{k_1=-\omega}^{+\omega} \ldots \sum_{k_n=-\omega}^{+\omega} |\delta_k\rangle \otimes |\delta_k\rangle \otimes \langle \delta_k| \\
\sum_{k_1=-\omega}^{+\omega} \ldots \sum_{k_n=-\omega}^{+\omega} \langle \delta_k|
\end{array}$$
The following multiplication and unit define a unital quasi-special commutative $\dagger$-Frobenius algebra, with normalisation factor $(2\omega + 1)^n$:

\[ : = \sum_{k_1, h_1 = -\omega}^{+\omega} \cdots \sum_{k_n, h_n = -\omega}^{+\omega} \ket{\delta_{k+h}} \otimes \bra{\delta_k} \otimes \bra{\delta_h} \]

\[ \circ : = \ket{\delta_0} \]

Its classical states are those in the following form, for $x \in \frac{1}{2}(2\omega + 1) \star \mathbb{Z}$:

\[ \ket{\chi_x} : = \sum_{k_1, h_1 = -\omega}^{+\omega} \cdots \sum_{k_n, h_n = -\omega}^{+\omega} \ket{\delta_k} e^{-i\frac{\pi}{\omega} k \cdot x} \otimes \bra{\delta_h} \]

We call them the momentum eigenstates (they are self-evidently plane-waves), and the momentum observable. Once again, position and momentum observables are strongly complementary.
Case study - wavefunctions on lattices

The following multiplication and unit define a unital quasi-special commutative \( ^\dagger \)-Frobenius algebra, with normalisation factor \((2\omega + 1)^n\):

\[
\otimes_{k_1, h_1 = -\omega}^{+\omega} \ldots \otimes_{k_n, h_n = -\omega}^{+\omega} |\delta_{k+h}\rangle \otimes |\delta_k\rangle \otimes |\delta_h\rangle
\]

\[
\circ := |\delta_0\rangle
\]

Its classical states are those in the following form, for \( x \in \frac{1}{2\omega + 1} \star \mathbb{Z}_{2\omega + 1}^n \):

\[
|\chi_x\rangle := \sum_{k_1 = -\omega}^{+\omega} \ldots \sum_{k_n = -\omega}^{+\omega} e^{-i2\pi k \cdot x} |\delta_k\rangle
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\[
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\text{\bigcirc} & := |\delta_0\rangle
\end{align*}
\]

Its classical states are those in the following form, for \( x \in \frac{1}{2\omega+1} \star \mathbb{Z}^n_{2\omega+1} \):

\[
|\chi_x\rangle := \sum_{k_1=-\omega}^{+\omega} \cdots \sum_{k_n=-\omega}^{+\omega} e^{-i2\pi k \cdot x} |\delta_k\rangle
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We call them the **momentum eigenstates** (they are self-evidently plane-waves), and \( \circ \) the **momentum observable**. Once again, position and momentum observables are strongly complementary.
A common trick in non-standard analysis sees standard real space approximated by non-standard lattices of infinitesimal mesh.

\textsuperscript{4}For the sceptics out there: an odd non-standard natural $\kappa \in \ast\mathbb{N}$ is an equivalence class $\kappa = [(k_i)_{i \in \mathbb{N}}]$ of sequences the elements of which are “asymptotically odd”, or “odd almost everywhere”, according to the chosen non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.
A common trick in non-standard analysis sees standard real space approximated by non-standard lattices of infinitesimal mesh.

(i) Fix two odd $4\omega_{uv}, \omega_{ir} \in \star \mathbb{N}$.

For the sceptics out there: an odd non-standard natural $\kappa \in \star \mathbb{N}$ is an equivalence class $\kappa = [(k_i)_{i \in \mathbb{N}}]$ of sequences the elements of which are “asymptotically odd”, or “odd almost everywhere”, according to the chosen non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.
Interlude - approximating real space by lattices

A common trick in non-standard analysis sees standard real space approximated by non-standard lattices of infinitesimal mesh.

(i) Fix two odd\(^4\) infinite naturals \(\omega_{uv}, \omega_{ir} \in \mathbf{\star \mathbb{N}}\).

(ii) Write \(\omega_{uv}\omega_{ir} = 2\omega + 1\) for some (unique) infinite natural \(\omega \in \mathbf{\star \mathbb{N}}\).

---

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(iii) Consider the periodic lattice \(\frac{1}{\omega_{uv}} \mathbb{Z}^n_{2\omega+1}\) of infinitesimal mesh in the non-standard torus \((\star \mathbb{R} / \omega_{ir} \mathbb{Z})^n\).

\(^4\)For the sceptics out there: an odd non-standard natural \(\kappa \in \star \mathbb{N}\) is an equivalence class \(\kappa = [(k_i)_{i \in \mathbb{N}}]\) of sequences the elements of which are “asymptotically odd”, or “odd almost everywhere”, according to the chosen non-principal ultrafilter \(\mathcal{F}\) on \(\mathbb{N}\).
A common trick in non-standard analysis sees standard real space approximated by non-standard lattices of infinitesimal mesh.

(i) Fix two odd\(^4 \) infinite naturals \( \omega_{uv}, \omega_{ir} \in \#\mathbb{N} \).

(ii) Write \( \omega_{uv}\omega_{ir} = 2\omega + 1 \) for some (unique) infinite natural \( \omega \in \#\mathbb{N} \).

(iii) Consider the periodic lattice \( \frac{1}{\omega_{uv}} \#\mathbb{Z}_{2\omega+1}^n \) of infinitesimal mesh in the non-standard torus \( \#\mathbb{R}/\omega_{ir}\#\mathbb{Z} \).

(iv) The standard reals \( \mathbb{R} \) are recovered by restricting to the (aperiodic) sub-lattice of finite elements \( \frac{1}{\omega_{uv}} \#\mathbb{Z}_{2\omega+1}^n \cap \#\mathbb{R}/\omega_{ir}\#\mathbb{Z} \), and then quotienting by infinitesimal equivalence \( \simeq \):

\[
\mathbb{R} \simeq \left( \frac{1}{\omega_{uv}} \#\mathbb{Z}_{2\omega+1}^n \cap \#\mathbb{R}/\omega_{ir}\#\mathbb{Z} \right) / \simeq
\]

\(^4\)For the sceptics out there: an odd non-standard natural \( \kappa \in \#\mathbb{N} \) is an equivalence class \( \kappa = [(k_i)_{i \in \mathbb{N}}] \) of sequences the elements of which are “asymptotically odd”, or “odd almost everywhere”, according to the chosen non-principal ultrafilter \( \mathcal{F} \) on \( \mathbb{N} \).
Wavefunctions in $n$-dimensional real space $\mathbb{R}^n$.

(i) Underlying Hilbert space $L^2[\mathbb{R}^n]$.

(ii) Orthonormal set of non-standard momentum eigenstates:

$$|\chi_p\rangle := x \mapsto \frac{1}{\sqrt{\omega_{uv}}} e^{-i2\pi (p \cdot x)}, \text{ for all } p \in \frac{1}{\omega_{uv}} \mathbb{Z}_{2\omega+1}^n$$

(iii) Dimension $D := (2\omega + 1)^n$, where $2\omega + 1 = \omega_{uv}\omega_{ir}$. 

S Gogioso, F Genovese (Oxford) Infinite-dimensional CQM CLAP Scotland 21 / 24
Wavefunctions in $n$-dimensional real space $\mathbb{R}^n$.

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$$|\chi_p\rangle := \mathbf{x} \mapsto \frac{1}{\sqrt{\omega_{uv}}} e^{-i2\pi (p \cdot \mathbf{x})}, \text{ for all } p \in \frac{1}{\omega_{uv}} * \mathbb{Z}^{n}_{2\omega+1}$$

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Classical structure corresponding to the **momentum observable**:

$$\begin{array}{c}
\sum_{p_1=-\omega_{ir}}^{+\omega_{ir}} \ldots \sum_{p_n=-\omega_{ir}}^{+\omega_{ir}} |\chi_p\rangle \otimes |\chi_p\rangle \otimes \langle \chi_p| \\
\sum_{p_1=-\omega_{ir}}^{+\omega_{ir}} \ldots \sum_{p_n=-\omega_{ir}}^{+\omega_{ir}} \langle \chi_p|
\end{array}$$
The following multiplication and unit define a unital quasi-special commutative $\dagger$-Frobenius algebra, with normalisation factor $(2\omega + 1)^n$: 

$$
\begin{align*}
&:= \sum_{p_1,q_1=-\omega_i}^{+\omega_i} \cdots \sum_{p_n,q_n=-\omega_i}^{+\omega_i} |\chi_{p+q}\rangle \otimes \langle \chi_p| \otimes \langle \chi_q| \\
&= |\chi_0\rangle
\end{align*}
$$
Case study - wavefunctions in real space

The following multiplication and unit define a unital quasi-special commutative †-Frobenius algebra, with normalisation factor \((2\omega + 1)^n\):

\[
\begin{align*}
\prod_{\pm \omega} \ &= \sum_{p_1, q_1 = -\omega}^{+\omega} \cdots \sum_{p_n, q_n = -\omega}^{+\omega} |\chi_{p+q}\rangle \otimes \langle \chi_p| \otimes \langle \chi_q| \\
\sum_{\pm \omega} \ &= \ |\chi_0\rangle
\end{align*}
\]

Its classical states are those in the following form, for \(x \in \frac{1}{\omega \pi} \ast \mathbb{Z}_{2\omega+1}^n\):

\[
|\delta_x\rangle := \sum_{p_1 = -\omega}^{+\omega} \cdots \sum_{p_n = -\omega}^{+\omega} \chi_p(x)^* |\chi_p\rangle
\]

Once again, the classical states for \(\bullet\) behave as Dirac deltas, so we call them the **position eigenstates**, and \(\bullet\) the **position observable**.
The following multiplication and unit define a unital quasi-special commutative $\dagger$-Frobenius algebra, with normalisation factor $(2\omega + 1)^n$:

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\bullet &:= \sum_{p_1, q_1 = -\omega}^{+\omega} \cdots \sum_{p_n, q_n = -\omega}^{+\omega} |\chi_{p+q}\rangle \otimes \langle \chi_p| \otimes \langle \chi_q|
\end{align*}$$

Its classical states are those in the following form, for $x \in \frac{1}{\omega} \ast \mathbb{Z}_{2\omega+1}$:

$$|\delta_x\rangle := \sum_{p_1 = -\omega}^{+\omega} \cdots \sum_{p_n = -\omega}^{+\omega} \chi_p(x)^* |\chi_p\rangle$$

Once again, the classical states for $\bullet$ behave as Dirac deltas, so we call them the position eigenstates, and $\bullet$ the position observable. And once again the position and momentum observables are strongly complementary.
The framework already covers a lot more material:

- quantum fields on infinite lattices (non-separable);
- quantum fields in real spaces (non-separable);
- quantum algorithm for the Hidden Subgroup Problem on $\mathbb{Z}^n$;
- Mermin-type non-locality arguments for infinite-dimensional systems.
More stuff out there, and a lot more to come

The framework already covers a lot more material:

- quantum fields on infinite lattices (non-separable);
- quantum fields in real spaces (non-separable);
- quantum algorithm for the Hidden Subgroup Problem on $\mathbb{Z}^n$;
- Mermin-type non-locality arguments for infinite-dimensional systems.

And even more material is currently being worked out:

- position/momentum duality, quantum symmetries and dynamics;
- applications to other quantum protocols (e.g. RFI quantum teleport’n);
- wavefunctions/fields over general locally compact abelian Lie groups;
- wavefunctions/fields over Minkowski space;
- connections with Feynman diagrams.
Thank You!

Thanks for Your Attention!

Any Questions?

S Abramsky, C Heunen. *H*-algebras and nonunital FAs*. arXiv:1011.6123

CQM := “Categorical Quantum Mechanics”
FA := “Frobenius algebra”

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5 This is a revised and extended version, and will be out by the end of the week.