Dagger category theory: monads and limits

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Structure of the talk

1. Brief intro
2. Dagger monads
3. Dagger limits
4. The question of evil

Introduction

- Dagger category is a category equipped with a dagger: a functorial way of reversing the direction of arrows:

$$A \xrightarrow{f = f^{\dagger\dagger}} B$$

$$B \xrightarrow[f^{\dagger}]{} A$$

- Any groupoid **G** has a dagger given by $f^{\dagger} := f^{-1}$
- The category **Rel** of sets and relations.
- The category **FHilb** of finite-dimensional Hilbert spaces and linear maps.
- The category **Prob** having finite sets as objects, doubly stochastic matrices as maps.

The way of the dagger

- Dagger isomorphism, henceforward a unitary, is an isomorphism f such that $f^{-1} = f^\dagger$.
- A dagger projection is an endomorphism p satisfying $p = p^2 = p^\dagger$
- A dagger functor satisfies $F(f^\dagger) = (Ff)^\dagger$
- Note: no need to define "dagger natural transformation": if F, G are dagger functors and $\sigma\colon F \to G$, then $\sigma^\dagger\colon G \to F$.
- monoidal dagger categories, compact dagger categories...

Three questions

- But what are dagger monads?
- Or dagger limits?
- If this is not trivially trivial, why not?

Two tentative answers

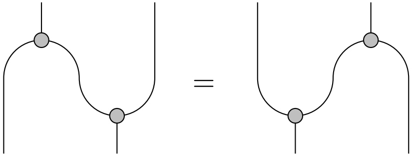
- Dagger categories are EVIL
- **DagCat**, the category of dagger categories, dagger functors and natural transformations is not just a 2-category, it is a *dagger* 2-category.
- I.e. 2-cells have a dagger, so one should require unitary 2-cells etc.
- A vague but handy principle: If the statement P implies Q for categories, then $P^\dagger$+(maybe some equations) implies $Q^\dagger$+(maybe some equations) for dagger categories.

Dagger monads

- Wish: dagger monads should be to dagger adjunctions as monads are to adjunctions (say this, but have a pic instead)
- A dagger adjunction is an adjunction in **DagCat**. Note that there is no distinction between left and right.
- The underlying endofunctor of a dagger monad should at least be a dagger functor. But then it induces a comonad.
- Maybe the monad and the comonad should be required to interact in the right way?

Dagger monads

We argue that the right way is given by the Frobenius law



i.e. $\mu T \circ T\mu^\dagger = T\mu \circ \mu^\dagger T$. Example: $- \otimes M$ for a dagger Frobenius algebra.

Lemma

Dagger adjunctions induce dagger Frobenius monads

Lemma

$$(A \xrightarrow{f} T(B)) \;\mapsto\; (B \xrightarrow{\eta} T(B) \xrightarrow{\mu^\dagger} T^2(B) \xrightarrow{T(f^\dagger)} T(A))$$

is a dagger on $\mathrm{Kl}(T)$ *commuting with the functors* $\mathbf{C} \to \mathrm{Kl}(T)$ *and* $\mathrm{Kl}(T) \to \mathbf{C}$

Dagger monads

Definition

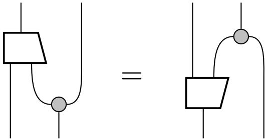
Let T be a monad on a dagger category $\mathbf{C}$. A *Frobenius-Eilenberg-Moore algebra*, or *FEM-algebra* for short, is an Eilenberg-Moore algebra $a\colon T(A) \to A$ that makes the following diagram commute.

$$\begin{array}{ccc} T(A) & \xrightarrow{\;T(a)^\dagger\;} & T^2(A) \\ {\scriptstyle \mu^\dagger}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2(A) & \xrightarrow[\;T(a)\;]{} & T(A) \end{array}$$

Denote the category of FEM-algebras (A, a) and algebra homomorphisms by $\mathrm{FEM}(T)$.

Dagger monads

For $T = - \otimes M$ this becomes



(1)

Theorem

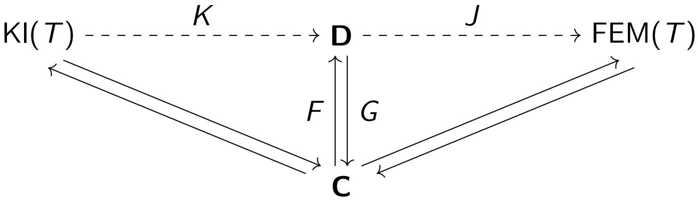
FEM-algebras form the largest full subcategory of $\mathbf{C}^T$ *containing* $\mathbf{C}_T$ *that carries a dagger commuting with the forgetful functor* $\mathbf{C}^T \to \mathbf{C}$.

There are EM-algebras that are not FEM.

Dagger monads

Theorem

Let F and G be dagger adjoints, and write $T = G \circ F$ for the induced dagger Frobenius monad. There are unique dagger functors K and J making the following diagram commute.



Moreover, J is full, K is full and faithful, and $J \circ K$ is the canonical inclusion.

On the proof

Lemma

Let T be a dagger Frobenius monad. An EM-algebra (A, a) is FEM if and only if $a^\dagger$ is a homomorphism $(A, a) \to (TA, \mu_A)$.

Proof.

Everything else is easy, just need to prove that J lands us in $\mathrm{FEM}(T)$. Let (A, a) be in the image. Since $J \circ K$ equals the canonical inclusion and (A, a) is associative, the homomorphism $a\colon (TA, \mu_A) \to (A, a)$ is in the image as well. Hence it's dagger is in the image too, so by the lemma (A, a) is Frobenius. $\square$

What are dagger limits?

Desiderata:

- Unique up to unique *unitary*
- Defined canonically for arbitrary diagrams
- Definition shouldn't depend on additional structure (e.g. enrichment)
- Generalizes dagger bipdroducts and dagger equalizers
- Connections to dagger adjunctions and dagger Kan extensions

Unique up to unitary

Let (L, l_A) and (M, m_A) be two limits of the same diagram, and let $f \colon L \to M$ to be the unique isomorphism of limits. Then f^{-1} is an iso of limits $f \colon M \to L$ and $f^{\dagger}$ is an iso of *colimits*. $(M, m_A^{\dagger}) \to (L, l_A^{\dagger})$. Thus f is unitary iff it is simultaneously a map of limits and a map of colimits.

Lemma

Two limits are unitarily isomorphic iff the diagram

$$\begin{array}{ccc} A & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

commutes for all A, B

So finding the right notion of a limit is a matter of fixing the maps $A \to L \to B$.

Dagger-shaped limits

This is easy in the special case when the diagram is a dagger functor:

Definition

Let $\mathbf{C}$ be a dagger category with zero morphisms. Let $\mathbf{J}$ be a small dagger category and $D\colon \mathbf{J} \to \mathbf{C}$ be a dagger functor. Then the *dagger limit* of D is a limit $(L, \{l_A\}_{A\in\mathbf{J}})$ (in the ordinary sense) of diagram $D\colon \mathbf{J} \to (\mathbf{C}, \dagger)$ such that

(i) For each $A \in \mathbf{J}$ the map $l_A \circ l_A^\dagger$ is a dagger projection.

(ii) $l_B \circ l_A = 0$ whenever there are no maps $A \to B$ in $\mathbf{J}$.

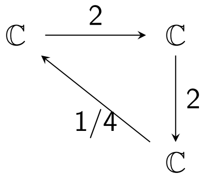
This definition is unique up to unitary.

Theorem

Let $\mathbf{C}$ and $\mathbf{J}$ be dagger categories. $\mathbf{C}$ has all $\mathbf{J}$-shaped limits iff the diagonal functor $\Delta\colon \mathbf{C} \to [\mathbf{J}, \mathbf{C}]$ has a dagger adjoint L such that $\epsilon \circ \epsilon^\dagger$ is idempotent, where $\epsilon\colon \Delta \circ L \to \mathrm{id}$ is the counit.

What about the general case?

- Admittedly, one wants limits that aren't dagger-shaped as well
- But what would this mean for loops? Consider e.g.



- Or infinite chains?

$$\cdots \xrightarrow{2} \mathbb{C} \xrightarrow{2} \mathbb{C} \xrightarrow{2} \mathbb{C} \xrightarrow{2} \cdots$$

Dagger categories are EVIL...

- Yes: Consider the forgetful functor **FHilb** $\to$ **FVect**. There is no dagger on **FVect** that is respected by it.

- Proof: Equip a vector space V with two different inner products, and consider the map $v \mapsto v$. It is not unitary in **FHilb**, but it maps to identity in **FVect**

- But maybe there are some qualifications to such evil behavior...

... but they ain't all that bad

Definition

A dagger equivalence is an equivalence (F, G, ϵ, η) in **DagCat** such that ϵ and η are unitary.

Now, if $(\mathbf{C}, \dagger)$ is a dagger category and $F\colon \mathbf{C} \leftrightarrows \mathbf{D} \colon G$ is an equivalence in **Cat**, with $\eta\colon \mathrm{id}_{\mathbf{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathbf{D}}$, when does (F, G, ϵ, η) lift to a dagger equivalence? Obviously it is necessary that η and $G\epsilon$ are unitary.

Theorem

This is sufficient.

Theorem

As long there is a unitary isomorphism $GFA \to A$ for each A, one can always replace F and G with isomorphic functors and lift that to a dagger equivalence.

Conclusion

- **DagCat** is not just a 2-category and thus dagger category theory is nontrivial.
- Dagger monads are those that satisfy the Frobenius law.
- A nice theory of dagger-shaped limits, although the general case is still in the works
- Restrictions on how evil dagger categories are