Dagger category theory: monads and limits

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Structure of the talk

1. Brief intro

2. Dagger monads

3. Dagger limits

4. The question of evil
Introduction

- Dagger category is a category equipped with a dagger: a functorial way of reversing the direction of arrows:

\[
\begin{align*}
& A \xrightarrow{f = f^{\dagger\dagger}} B \\
& B \xrightarrow{f^{\dagger}} A
\end{align*}
\]

- Any groupoid $G$ has a dagger given by $f^{\dagger} := f^{-1}$

- The category $\text{Rel}$ of sets and relations.

- The category $\text{FHilb}$ of finite-dimensional Hilbert spaces and linear maps.

- The category $\text{Prob}$ having finite sets as objects, doubly stochastic matrices as maps.
The way of the dagger

- Dagger isomorphism, henceforward a unitary, is an isomorphism $f$ such that $f^{-1} = f^\dagger$.

- A dagger projection is an endomorphism $p$ satisfying $p = p^2 = p^\dagger$.

- A dagger functor satisfies $F(f^\dagger) = (Ff)^\dagger$.

- Note: no need to define “dagger natural transformation”: if $F, G$ are dagger functors and $\sigma : F \rightarrow G$, then $\sigma^\dagger : G \rightarrow F$.

- monoidal dagger categories, compact dagger categories...
Three questions

- But what are dagger monads?
- Or dagger limits?
- If this is not trivially trivial, why not?
Two tentative answers

- Dagger categories are EVIL

- **DagCat**, the category of dagger categories, dagger functors and natural transformations is not just a 2-category, it is a *dagger 2-category*.

- I.e. 2-cells have a dagger, so one should require unitary 2-cells etc.

- A vague but handy principle: If the statement $P$ implies $Q$ for categories, then $P^\dagger+$ (maybe some equations) implies $Q^\dagger+$ (maybe some equations) for dagger categories.
Dagger monads

- Wish: dagger monads should be to dagger adjunctions as monads are to adjunctions (say this, but have a pic instead)

- A dagger adjunction is an adjunction in \textbf{DagCat}. Note that there is no distinction between left and right.

- The underlying endofunctor of a dagger monad should at least be a dagger functor. But then it induces a comonad.

- Maybe the monad and the comonad should be required to interact in the right way?
Dagger monads

We argue that the right way is given by the Frobenius law

\[ \mu_T \circ T \mu^\dagger = T \mu \circ \mu^\dagger T. \]

Example: \(- \otimes M\) for a dagger Frobenius algebra.

Lemma

Dagger adjunctions induce dagger Frobenius monads

Lemma

\[
\begin{array}{c}
(A \xrightarrow{f} T(B)) \mapsto (B \xrightarrow{\eta} T(B) \xrightarrow{\mu^\dagger} T^2(B) \xrightarrow{T(f^\dagger)} T(A))
\end{array}
\]

is a dagger on \(\text{Kl}(T)\) commuting with the functors \(\mathbf{C} \to \text{Kl}(T)\) and \(\text{Kl}(T) \to \mathbf{C}\)
Dagger monads

Definition
Let $T$ be a monad on a dagger category $\mathbf{C}$. A Frobenius-Eilenberg-Moore algebra, or FEM-algebra for short, is an Eilenberg-Moore algebra $a: T(A) \rightarrow A$ that makes the following diagram commute.

\[
\begin{array}{ccc}
T(A) & \xrightarrow{T(a)^\dagger} & T^2(A) \\
\mu^\dagger \downarrow & & \downarrow \mu \\
T^2(A) & \xrightarrow{T(a)} & T(A)
\end{array}
\]

Denote the category of FEM-algebras $(A, a)$ and algebra homomorphisms by $\text{FEM}(T)$. 
Dagger monads

For $T = - \otimes M$ this becomes

\[ (1) \]

\[\begin{align*}
\text{Theorem} \\
FEM\text{-algebras form the largest full subcategory of } C^T \text{ containing } C_T \text{ that carries a dagger commuting with the forgetful functor } C^T \to C. \\
\text{There are EM-algebras that are not FEM.}
\end{align*}\]
Dagger monads

Theorem
Let $F$ and $G$ be dagger adjoints, and write $T = G \circ F$ for the induced dagger Frobenius monad. There are unique dagger functors $K$ and $J$ making the following diagram commute.

Moreover, $J$ is full, $K$ is full and faithful, and $J \circ K$ is the canonical inclusion.
On the proof

Lemma

Let $T$ be a dagger Frobenius monad. An EM-algebra $(A, a)$ is FEM if and only if $a^\dagger$ is a homomorphism $(A, a) \to (TA, \mu_A)$.

Proof.
Everything else is easy, just need to prove that $J$ lands us in $\text{FEM}(T)$. Let $(A, a)$ be in the image. Since $J \circ K$ equals the canonical inclusion and $(A, a)$ is associative, the homomorphism $a: (TA, \mu_A) \to (A, a)$ is in the image as well. Hence it’s dagger is in the image too, so by the lemma $(A, a)$ is Frobenius. \qed
What are dagger limits?

Desiderata:

▶ Unique up to unique \textit{unitary}

▶ Defined canonically for arbitrary diagrams

▶ Definition shouldn’t depend on additional structure (e.g. enrichment)

▶ Generalizes dagger biproducts and dagger equalizers

▶ Connections to dagger adjunctions and dagger Kan extensions
Unique up to unitary

Let \((L, l_A)\) and \((M, m_A)\) be two limits of the same diagram, and let \(f : L \to M\) to be the unique isomorphism of limits. Then \(f^{-1}\) is an iso of limits \(f : M \to L\) and \(f^\dagger\) is an iso of colimits. \((M, m_A^\dagger) \to (L, l_A^\dagger)\). Thus \(f\) is unitary iff it is simultaneously a map of limits and a map of colimits.

Lemma

Two limits are unitarily isomorphic iff the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & L \\
\downarrow & & \downarrow \\
M & \longrightarrow & B
\end{array}
\]

commutes for all \(A, B\)

So finding the right notion of a limit is a matter of fixing the maps \(A \to L \to B\).
Dagger-shaped limits

This is easy in the special case when the diagram is a dagger functor:

**Definition**

Let $C$ be a dagger category with zero morphisms. Let $J$ be a small dagger category and $D: J \to C$ be a dagger functor. Then the *dagger limit* of $D$ is a limit $(L, \{l_A\}_{A \in J})$ (in the ordinary sense) of diagram $D: J \to (C, \dagger)$ such that

(i) For each $A \in J$ the map $l_A \circ l_A^\dagger$ is a dagger projection.

(ii) $l_B \circ l_A = 0$ whenever there are no maps $A \to B$ in $J$.

This definition is unique up to unitary.

**Theorem**

Let $C$ and $J$ be dagger categories. $C$ has all $J$-shaped limits iff the diagonal functor $\Delta: C \to [J, C]$ has a dagger adjoint $L$ such that $\epsilon \circ \epsilon^\dagger$ is idempotent, where $\epsilon: \Delta \circ L \to \text{id}$ is the counit.
What about the general case?

- Admittedly, one wants limits that aren’t dagger-shaped as well.

- But what would this mean for loops? Consider e.g.

\[
\begin{array}{c}
C \\ \downarrow^{1/4} \\
C \\
\end{array} & \xrightarrow{2} & \begin{array}{c}
C \\
\downarrow^{2} \\
C \\
\end{array}
\]

- Or infinite chains?

\[
\begin{array}{c}
\cdots \\ \xrightarrow{2} \\
C \\
\xrightarrow{2} \\
C \\
\xrightarrow{2} \\
C \\
\xrightarrow{2} \\
\cdots 
\end{array}
\]
Dagger categories are EVIL...

▶ Yes: Consider the forgetful functor $\text{FHilb} \to \text{FVect}$. There is no dagger on $\text{FVect}$ that is respected by it.

▶ Proof: Equip a vector space $V$ with two different inner products, and consider the map $v \mapsto v$. It is not unitary in $\text{FHilb}$, but it maps to identity in $\text{FVect}$

▶ But maybe there are some qualifications to such evil behavior...
... but they ain’t all that bad

Definition
A dagger equivalence is an equivalence \((F, G, \epsilon, \eta)\) in \textbf{DagCat} such that \(\epsilon\) and \(\eta\) are unitary.

Now, if \((\mathcal{C}, \dagger)\) is a dagger category and \(F: \mathcal{C} \leftrightarrow \mathcal{D}: G\) is an equivalence in \textbf{Cat}, with \(\eta: \text{id}_\mathcal{C} \to GF\) and \(\epsilon: FG \to \text{id}_\mathcal{D}\), when does \((F, G, \epsilon, \eta)\) lift to a dagger equivalence? Obviously it is necessary that \(\eta\) and \(G\epsilon\) are unitary.

Theorem
This is sufficient.

Theorem
As long there is a unitary isomorphism \(GFA \to A\) for each \(A\), one can always replace \(F\) and \(G\) with isomorphic functors and lift that to a dagger equivalence.
Conclusion

▶ **DagCat** is not just a 2-category and thus dagger category theory is nontrivial.

▶ Dagger monads are those that satisfy the Frobenius law.

▶ A nice theory of dagger-shaped limits, although the general case is still in the works

▶ Restrictions on how evil dagger categories are