A categorical semantics for causal structure

Aleks Kissinger and Sander Uijlen

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Process theory

:=

Symmetric monoidal category

+ interpretation of morphisms as processes
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Symmetric monoidal categories

\[ f : A \to B := \begin{array}{c} f \\ \downarrow \\ A \end{array} \]

\[ g \circ f := \begin{array}{c} g \\ \downarrow \\ f \\ \downarrow \\ A \end{array} \]

\[ f \otimes g := \begin{array}{c} f \\ \downarrow \\ A \\ \downarrow \\ B \end{array} \begin{array}{c} g \\ \downarrow \\ B \end{array} \]

\[ 1_A := \begin{array}{c} A \\ \downarrow \end{array} \]

\[ 1_I := \begin{array}{c} A \\ \downarrow \end{array} \]

\[ \sigma_{A,B} := \begin{array}{c} B \\ \downarrow \end{array} \begin{array}{c} A \\ \downarrow \end{array} \begin{array}{c} A \\ \downarrow \end{array} \begin{array}{c} B \\ \downarrow \end{array} \]
States, effects, numbers

Morphisms in/out of the monoidal unit get special names:

\[
\text{state} \,:= \quad \triangleleft \rho \\
\text{effect} \,:= \quad \triangle \pi \\
\text{number} \,:= \quad \lambda
\]
Interpretation: discarding + causality

Consider a special family of discarding effects:

\[ \overline{A}, \overline{A \otimes B} := \overline{A} \overline{B}, \overline{I} := 1 \]

This enables us to say when a process is causal:

\[ \overline{B} = \overline{A} \]

“If the output of a process is discarded, it doesn’t matter which process happened.”
The classical case

\( \text{Mat}(\mathbb{R}_+) \) is the category whose objects are natural numbers and morphisms are *matrices of positive numbers*. Then:

\[
\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array}
\end{array}
\sum_i \rho^i = 1
\]

Causal states = probability distributions
Causal processes = stochastic maps
The quantum case

**CPM** is the category whose objects are Hilbert spaces and morphisms are *completely positive maps*. Then:

\[
\begin{align*}
\mathbb{1} & = \text{Tr}(\cdot) \\
\rho & = \text{Tr}(\rho) = 1
\end{align*}
\]

Causal states = density operators
Causal processes = CPTPs
A causal structure on $\Phi$ associates input/output pairs with a set of ordered events:

$$\mathcal{G} := \begin{cases} 
(A, A') & \leftrightarrow & A \\
(B, B') & \leftrightarrow & B \\
(C, C') & \leftrightarrow & C \\
(D, D') & \leftrightarrow & D \\
(E, E') & \leftrightarrow & E
\end{cases}$$
Causal structure of a process

**Definition**

Φ admits causal structure $\mathcal{G}$, written $\Phi \vdash \mathcal{G}$ if the output of each event only depends on the inputs of itself and its causal ancestors.
Example: one-way signalling

\[
P(A' | AB) = P(A' | A)
\]
Example: non-signalling

\[
P(A'|AB) = P(A'|A) \quad P(B'|AB) = P(B'|B)
\]
An acyclic diagram comes with a canonical choice of causal structure:

\[
\begin{align*}
& & & e & \\
& & b & & d \\
& a & & c & \\
& & & & \\
\end{align*}
\]

\[\preceq\]

\[
\begin{align*}
& & & E & \\
& & B & & D \\
& A & & C & \\
& & & & \\
\end{align*}
\]

**Theorem**

*All acyclic diagrams of processes admit their associated causal structure if and only if all processes are causal.*
Higher-order causal structure

We can also define (super-)processes with *higher-order causal structure*: 

\[ w = \Phi_1 \Phi_2 \]

These can introduce definite, or *indefinite* causal structure:

\[ s = \Phi_1 \Phi_2 \]

E.g. Quantum Switch, OCB $W$-matrix, ...
The questions

**Q1:** Can we define a category whose *types* express causal structure?

**Q2:** Can we define a category whose *types* express higher-order causal structure?

It turns out answering **Q2** gives the answer to **Q1**.
Compact closed categories

An easy way to get higher-order processes is to use compact closed categories:

**Definition**
An SMC $C$ is *compact closed* if every object $A$ has a *dual* object $A^*$, i.e. there exists $\eta_A : I \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to I$, satisfying:

\[
(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \quad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}
\]
Higher-order processes

Processes send states to states:

$\rho \mapsto f$

In compact closed categories, everything is a state, thanks to *process-state duality*:

$f : A \rightarrow B \leftrightarrow \rho_f : A^* \otimes B$

$\Rightarrow$ higher order processes are the same as first-order processes:

$\left( \begin{array}{c} f \\ \rho \end{array} \right) \mapsto \left( \begin{array}{c} w \\ f \end{array} \right) : (A \rightarrow B) \circ (C \rightarrow D)$
Some handy notation

We can treat *everything* as a state, and write states in any shape we like:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D \\
\hline \\
C \\
\hline \\
B \\
\hline \\
A
\end{array}
\end{array}
\end{array}
\quad := 
\begin{array}{c}
\begin{array}{c}
A^* \\
\hline \\
B \\
\hline \\
C^* \\
\hline \\
D
\end{array}
\end{array}
\]

Then plugging shapes together means composing the appropriate caps:
Some handy notation

It looks like we can now freely work with higher-order causal processes:

\[ A \to (B \to C) \to D \]

…but there's a problem.
The compact collapse

In a compact closed category:

\[(A \otimes B)^* = A^* \otimes B^*\]

Which gives:

\[
(A \rightarrow B) \rightarrow C \quad \Rightarrow \quad (A \rightarrow B)^* \otimes C
\]

\[
\Rightarrow (A^* \otimes B)^* \otimes C
\]

\[
\Rightarrow A \otimes B^* \otimes C
\]

\[
\Rightarrow B^* \otimes A \otimes C
\]

\[
\Rightarrow B \rightarrow A \otimes C
\]

\(\Rightarrow\) everything collapses to first order!
The compact collapse

But first-order causal \( \neq \) second-order causal:

\[
\forall \Phi \text{ causal} . \quad \begin{array}{c}
\Phi \\
\downarrow \\
\Phi
\end{array} = \quad \begin{array}{c}
\Phi \\
\downarrow \\
\Phi
\end{array}
\]

So, causal types are richer than compact-closed types. In particular:

\[
A \rightarrow B := (A \otimes B^*)^* \not\cong A^* \otimes B
\]

If we drop this iso from the definition of compact closed, we get a *-autonomous category.
Definition

A \textit{*-autonomous category} is a symmetric monoidal category equipped with a full and faithful functor \((-)^* : C^{\text{op}} \to C\) such that, by letting:

\[ A \to B := (A \otimes B^*)^* \]  

(1)

there exists a natural isomorphism:

\[ C(A \otimes B, C) \cong C(A, B \to C) \]  

(2)
The recipe

Precausal category $\mathcal{C}$ $\mapsto$ Caus$[\mathcal{C}]$

- compact closed category of ‘raw materials’
- $\ast$-autonomous category capturing ‘logic of causality’

$\text{Mat}(\mathbb{R}_+)$ $\mapsto$ higher-order stochastic maps
CPM $\mapsto$ higher-order quantum channels
Precausal categories

Precausal categories give ‘good’ raw materials, i.e. discarding behaves well w.r.t. the categorical structure. The standard examples are $\text{Mat}(\mathbb{R}_+)$ and CPM.

Definition

A precausal category is a compact closed category $\mathcal{C}$ such that:

(C1) $\mathcal{C}$ has discarding processes for every system

(C2) For every (non-zero) system $A$, the dimension of $A$:

$$d_A := \begin{array}{c} \downarrow \varepsilon A \\ \downarrow \subseteq \end{array}$$

is an invertible scalar.

(C3) $\mathcal{C}$ has enough causal states

(C4) Second-order causal processes factorise
Enough causal states

\[
\left( \forall \rho \text{ causal}. \quad \begin{array}{c}
\rho \\
\downarrow \\
\rho
\end{array} \quad f = g
\right) \implies \begin{array}{c}
\rho \\
\downarrow \\
\rho
\end{array} \quad f = g
\]
Second-order causal processes factorise

\[(\forall \Phi \text{ causal .}) \quad \Rightarrow \quad (\exists \Phi_1, \Phi_2 \text{ causal .})\]
Theorem

In a pre-causal category, one-way signalling processes factorise:

\[
\begin{aligned}
\exists \Phi' \text{ causal } . \quad \Phi = \Phi' \\
\end{aligned}
\]

\[
\implies
\begin{aligned}
\exists \Phi_1, \Phi_2 \text{ causal } . \quad \Phi = \Phi_1 \Phi_2 \\
\end{aligned}
\]
**Proof.** Treat $\Phi$ as a second-order process by bending wires. Then for any causal $\Psi$, we have:

\[
\Phi \Psi = \Psi \Phi' = \Phi' \Psi = \Phi
\]

So $\Phi$ is second-order causal. By (C4):

\[
\Phi = \Phi_2 \Phi_1 = \Phi_1 \Phi_2
\]
Theorem (No time-travel)

No non-trivial system $A$ in a precausal category $C$ admits time travel. That is, if there exist systems $B$ and $C$ such that:

$\Phi_{A \rightarrow B} \text{ causal} \implies \Phi_{A \rightarrow C} \text{ causal}$

then $A \cong I$. 
**Proof.** For any causal process $\Psi$ and causal state $\downarrow$:

$$\Phi : A 
\downarrow B := \Psi \downarrow C \downarrow A$$

is causal. So:

$$A \begin{array}{c} \Psi \\ \downarrow \\ B \end{array} = A \begin{array}{c} \Phi \\ C \downarrow B \downarrow \end{array} = \downarrow = 1$$

Applying (C4):

$$A \begin{array}{c} \rho \\ A \downarrow B \downarrow \rho \end{array} \quad \Rightarrow \quad A \begin{array}{c} A \\ \downarrow \end{array} = A \begin{array}{c} \rho \\ \downarrow \end{array}$$

for some $\rho$ causal. So $\rho \circ \downarrow = 1_A$ and $\downarrow \circ \rho = 1_I$ is causality.
Causal states

A process is causal, a.k.a. *first order causal*, if and only if it preserves the set of causal states:

\[
\begin{array}{c}
\downarrow \rho \\
\text{causal}
\end{array} \\implies \\
\begin{array}{c}
\downarrow \rho \\
\text{causal}
\end{array}
\]

That is, it preserves:

\[
c = \left\{ \rho : A \mid \downarrow \rho = 1 \right\} \subseteq C(I, A)
\]

We define \text{Caus}[C] by equipping each object with a *generalisation of* the set \( c \), and requiring processes to preserve it.
Duals and closure

Note any set of states $c \subseteq C(I, A)$ admits a dual, which is a set of effects:

$$c^* := \left\{ \pi : A^* \mid \forall \rho \in c . \begin{prooftree} \pi \end{prooftree} \Downarrow \rho = 1 \right\}$$

The double-dual $c^{**}$ is a set of states again.

Definition
A set of states $c \subseteq C(I, A)$ is closed if $c = c^{**}$.
Flatness

If $c$ is the set of causal states, discarding $\in c^*$, and up to some rescaling, discarding-transpose:

$$\frac{1}{D} \quad \bot$$

i.e. the maximally mixed state $\in c$.

We make this symmetric $c \leftrightarrow c^*$, and call this property flatness:

**Definition**

A set of states $c \subseteq C(I, A)$ is *flat* if there exist invertible numbers $\lambda, \mu$ such that:

$$\lambda \quad \bot \quad \in c \quad \mu \quad \top \quad \in c^*$$
The main definition

Definition
For a precausal category $\mathcal{C}$, the category $\text{Caus}[\mathcal{C}]$ has as objects pairs:

$$\mathbf{A} := (A, c_A \subseteq \mathcal{C}(I, A))$$

where $c_A$ is closed and flat. A morphism $f : \mathbf{A} \to \mathbf{B}$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\rho \in c_A \implies f \circ \rho \in c_B$$
The main theorem

Theorem
Caus[C] is a *-autonomous category, where:

\[ A \otimes B := (A \otimes B, (c_A \otimes c_B)^{**}) \quad I := (I, \{1_I\}) \]

\[ A^* := (A^*, c_A^*) \]
Connectives

One connective $\otimes$ becomes 3 interrelated ones:

$$A \otimes B$$

$$A \gfrak B := (A^* \otimes B^*)^*$$

$$A \rightarrow B := A^* \gfrak B \cong (A \otimes B^*)^*$$

- $\otimes$ is the smallest joint state space that contains all product states
- $\gfrak$ is the biggest joint state space normalised on all product effects:

$$c_{A \gfrak B} = \left\{ \rho : A \otimes B \mid \forall \pi \in c_A^*, \xi \in c_B^* \ . \ \begin{array}{c} \pi \\ \rho \\ \xi \end{array} = 1 \right\}$$

- $\rightarrow$ is the space of causal-state-preserving maps
Example: first-order systems

First order := systems of the form $A = (A, \{ \rightarrow \}^*)$

$c_{A \otimes B} := (c_A \otimes c_B)^{**} = (\rightarrow \rightarrow)^* =$ all causal states

$c_{A \lozenge B} := \left\{ \rho : A \otimes B \mid \forall \pi \in c_A^*, \xi \in c_B^* \cdot \frac{\pi}{\rho} \frac{\xi}{\rho} = 1 \right\} =$ all causal states

**Theorem**

For first order systems, $A \otimes B \simeq A \lozenge B$. 
When $\otimes \neq \otimes$

For f.o. $A, A', B, B'$:

$$(A \rightarrow A') \otimes (B \rightarrow B')$$

\[
\begin{align*}
& A^* \otimes A' \otimes B^* \otimes B' \\
& A^* \otimes B^* \otimes A' \otimes B' \\
& (A \otimes B)^* \otimes A' \otimes B' \\
& (A \otimes B)^* \otimes (A' \otimes B') \\
& A \otimes B \rightarrow A' \otimes B'
\end{align*}
\]

$$(A \rightarrow A') \otimes (B \rightarrow B') = \text{all causal processes}$$
Theorem

\[(A \rightarrow A') \otimes (B \rightarrow B') = \text{causal, non-signalling processes}\]

Proof. (idea) The causal states for \((A \rightarrow A') \otimes (B \rightarrow B')\) are:

\[
\{\Phi_1, \Phi_2\}^{**}
\]

We show:

\[
\begin{array}{c}
A' \\
\downarrow \quad w \\
A \\
\end{array}
\quad \begin{array}{c}
B' \\
\downarrow \\
B \\
\end{array}
\end{array}
\in

\[
\{\Phi_1, \Phi_2\}^*
\]

is also normalised for all non-signalling processes:

\[
\begin{array}{c}
\text{NS} \\
\end{array}
\quad w
\]

This follows from a graphical proof using all 4 precausal axioms.
Refining causal structure

Since $I \cong I^* = (I, \{1\})$, a standard theorem of $*$-autonomous gives a canonical embedding:

$$(A \rightarrow A') \otimes (B \rightarrow B') \hookrightarrow (A \rightarrow A') \otimes (B \rightarrow B')$$

What about in between?

$$(A \rightarrow A') \otimes (B \rightarrow B') \hookrightarrow \cdots \hookrightarrow (A \rightarrow A') \otimes (B \rightarrow B')$$
Theorem

One-way signalling processes are processes of the form:

\[
\begin{array}{c}
\begin{array}{c}
A' \\
\phi \\
A
\end{array}
\end{array}
\quad : \quad A \rightarrow (A' \rightarrow B) \rightarrow B'
\]
One-way signalling

Proof. Exploiting the relationship between one-way signalling and second-order causal:

\[
\Phi \Psi = \Psi \Phi' = \Psi' \Phi = \Phi'
\]

we have:

\[
\begin{array}{c}
A' \\
\Phi \\
A
\end{array} \quad \begin{array}{c}
B' \\
\Phi' \\
B
\end{array} : \quad (A' \rightarrow B) \rightarrow (A \rightarrow B')
\]

Then ∗-autonomous structure gives a canonical iso:

\[
(A' \rightarrow B) \rightarrow (A \rightarrow B') \cong A \rightarrow (A' \rightarrow B) \rightarrow B'
\]
Further examples

- $n$-party non-signalling:
  \[
  \Phi : (A_1 \rightarrow A'_1) \otimes \cdots \otimes (A_n \rightarrow A'_n)
  \]

- Quantum $n$-combs:
  \[
  w : A_1 \rightarrow (A'_1 \rightarrow (\cdots \rightarrow A_n) \rightarrow A'_n)
  \]
Further examples

- Compositions of those things:
Further examples

- Indefinite causal structures (e.g. quantum switch, OCB $W$-process, Baumeler-Wolf):

\[
\left[ (A_1 \rightarrow A'_1) \otimes \ldots \otimes (A_n \rightarrow A'_n) \right]^* 
\]
Automation

The internal logic of $\ast$-autonomous categories is multiplicative linear logic (MLL):

\[ \vdash \Gamma, A \quad \vdash \Delta, B \]
\[ \vdash \Gamma, \Delta, A \otimes B \]

\[ \vdash \Gamma, A, B \]
\[ \vdash \Gamma, A \otimes B \]

\[ \vdash 1 \]
\[ \vdash \Gamma, \bot \]

$\Rightarrow$ use off-the-shelf theorem provers to prove causality theorems.
For example, we can show using \texttt{1lprover} that:

\[
(A \rightarrow A') \otimes (B \rightarrow B') \\
\Downarrow \\
A \rightarrow (A' \rightarrow B) \rightarrow B' \\
\Downarrow \\
(A \rightarrow A') \ltimes (B \rightarrow B')
\]
Thanks

...and some refs:

- **A categorical semantics for causal structure.** arXiv:1701.04732
- **Causal structures and the classification of higher order quantum computation.** Paulo Perinotti. arXiv:1612.05099