What is a quantum symmetry?

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Road map

- Classical maths: symmetries = actions of groups

Main message today: Classical objects can have interesting quantum symmetries

More specifically: Many (maybe all) singular plane curves are quantum homogeneous spaces
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- Quantum maths: actions of Hopf algebras (or even Hopf algebroids, Hopf monads...)

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Part I: Quantum groups (Hopf algebras)
Algebras (aka monoids)

Definition

An **algebra** in a monoidal category $\mathcal{C}$ is an object $G$ with morphisms $\mu : G \otimes G \to G$, and $\eta : \mathbb{I} \to G$ such that

\[
\begin{array}{c}
G \otimes G \otimes G \\
\downarrow \text{id} \otimes \mu \\
G \otimes G \\
\downarrow \mu \\
G
\end{array}
\]

and

\[
\begin{array}{c}
\mathbb{I} \otimes G \\
\sim \\
G \\
\sim \\
G \otimes \mathbb{I}
\end{array}
\]

commute.
Examples

- **Logic:** $\mathcal{C} = \textbf{Set}$, $\otimes = \times$, $\mathbb{I} = \{\emptyset\}$, algebras = monoids, i.e. semigroups $G$ with a unit element $e_G$

- **Physics:** $\mathcal{C} = \textbf{Vect}_k$ ($k$ some field), $\otimes = \otimes_k$, $\mathbb{I} = k$, algebras = unital associative $k$-algebras

- **Categories:** $\mathcal{C} = \textbf{End}_\mathcal{D}$ ($\mathcal{D}$ some category), $\otimes = \circ$, $\mathbb{I} = \text{id}$, algebras = monads on $\mathcal{D}$

- The last example is sort of universal if we identify an algebra $G$ in $\mathcal{C}$ with the monad $G \otimes -$ on $\mathcal{C}$
**Definition**

A _module_ over a monad $T$ on $\mathcal{D}$ is an object $M$ in $\mathcal{D}$ with an action $\alpha : T(M) \to M$ of $T$,

\[
\begin{array}{ccc}
T(T(M)) & \xrightarrow{T(\alpha)} & T(M) \\
\mu(id) & \downarrow & \downarrow \alpha \\
T(M) & \xrightarrow{\alpha} & M
\end{array}
\quad
\begin{array}{ccc}
M & \xrightarrow{\eta(id)} & T(M) \\
\text{id} & \downarrow \downarrow \alpha \\
M & \xrightarrow{\alpha} & M
\end{array}
\]

- $\mathcal{D} = \textbf{Set}$, $T = G \times -$, $G$ monoid: $T$-modules are $G$-sets $X$, $\alpha : G \times X \to X$, $(g, x) \mapsto gx$ satisfies
  \[
g(hx) = (gh)x, \quad e_Gx = x, \quad \forall g, h \in G, x \in X.
\]
Hopf algebras

- Co(al)gebras in $\mathcal{C} = \text{algebras in } \mathcal{C}^\circ$, 
  \[
  \Delta : G \rightarrow G \otimes G, \quad \varepsilon : G \rightarrow \mathbb{I}.
  \]

- In $\mathcal{C} = \text{Set}$, $\Delta : G \rightarrow G \times G$ must be $g \mapsto (g, g)$, in $\mathcal{C} = \text{Vect}_k$ things are more flexible (linear duality!)

- Definition
  A Hopf algebra is a coalgebra in $\text{Alg } \mathcal{C}$ for which 
  \[
  \Gamma : G \otimes G \Delta \otimes \text{id} \rightarrow G \otimes G \otimes G \text{id} \otimes \mu \rightarrow G \otimes G
  \]
  is an isomorphism (this is called the Galois map).

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Hopf algebras

- Co(al)gebras in $\mathcal{C} = \text{algebras in } \mathcal{C}^\circ$:
  $$\Delta : G \rightarrow G \otimes G, \quad \varepsilon : G \rightarrow \mathbb{1}.$$  

- In $\mathcal{C} = \text{Set}$, $\Delta : G \rightarrow G \times G$ must be $g \mapsto (g, g)$, in $\mathcal{C} = \text{Vect}_k$ things are more flexible (linear duality!)

- Braidings $\chi : G \otimes G \rightarrow G \otimes G$ (e.g. the flip for $\text{Set}$ and $\text{Vect}_k$) turn $\text{Alg}_\mathcal{C}$ into a monoidal category.

**Definition**

A **Hopf algebra** is a coalgebra in $\text{Alg}_\mathcal{C}$ for which

$$\Gamma : G \otimes G \xrightarrow{\Delta \otimes \text{id}} G \otimes G \otimes G \xrightarrow{\text{id} \otimes \mu} G \otimes G$$

is an isomorphism (this is called the **Galois map**).
An algebra in $\textbf{Set}$ is a semigroup with unit element,

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh, \quad f(gh) = (fg)h,$$

$$\eta : \{\emptyset\} \to G, \quad e_G g = ge_G = g, \quad e_G := \eta(\emptyset).$$
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Every set $G$ is a coalgebra in a unique way,

$$\Delta : G \to G \times G, \quad g \mapsto (g, g), \quad \varepsilon : G \to \{\emptyset\}.$$

This turns an algebra into a coalgebra in $\textbf{Alg}_{\textbf{Set}}$ with respect to the braiding $\chi(g, h) = (h, g),

$$\Delta(gh) = (gh, gh) = (g, g)(h, h) = \Delta(g)\Delta(h).$$
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\]

The Galois map is \( \Gamma(g, h) = (g, gh) \), so a Hopf 
algebra is a group, with \( \Gamma^{-1}(g, h) = (g, g^{-1}h) \).
Part II: Quantum homogeneous spaces
Homogeneous spaces

**Definition**

A **homogeneous space** of a group $G$ is a set $X$ with a transitive action $G \times X \to X$, $(g, x) \mapsto gx$. Fixing $x \in X$ defines a $G$-equivariant surjective map $\pi: G \to X$, $g \mapsto gx$ and identifies $X$ with the $G$-set $G/H := \{gH | g \in G\} \subset \mathcal{P}(G)$ of all cosets $gH = \{gh | h \in H\}$ of the isotropy group $H := \{g \in G | gx = x\}$. 
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of all cosets $gH = \{gh \mid h \in H\}$ of the **isotropy group**

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Affine schemes over $k$

We can give $X$ more structure and lift $\textbf{Set}$ by any category in which groups, actions, and surjective morphisms $G \to X$ make sense, e.g.
We can give $X$ more structure and lift $\text{Set}$ by any category in which groups, actions, and surjective morphisms $G \to X$ make sense, e.g.

- $\text{Sch}_k := \text{CommAlg}_k^\circ$: an affine scheme over $k$ is a commutative $k$-algebra, morphisms $A \to B$ of affine schemes are $k$-algebra homomorphisms $B \to A$.
- The set underlying a scheme is $X = \text{Hom}_{\text{Sch}_k}(k, B)$. 

**Uli (U Glasgow)**

What is a quantum symmetry?

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Hilbert’s Nullstellensatz

- Historical motivation: If \( k = \mathbb{C} \) and \( A \) is finitely generated and reduced (no nilpotents), then

\[
A \cong \mathcal{O}(X)
\]

for some algebraic set

\[
X = \{ x \in k^n \mid f_1(x) = \ldots f_d(x) = 0 \}
\]

given by \( f_1, \ldots, f_d \in k[t_1, \ldots, t_n] \), where \( \mathcal{O}(X) \) is the \textbf{coordinate ring} of \( X \), i.e. the set of all polynomial functions \( X \to k \).
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given by $f_1, \ldots, f_d \in k[t_1, \ldots, t_n]$, where $\mathcal{O}(X)$ is the coordinate ring of $X$, i.e. the set of all polynomial functions $X \to k$. Also,

$$X \cong \text{Hom}_{\text{Sch}_k}(k, A) \cong \text{MaxSpec}(A).$$
Affine group schemes

- $\textbf{Sch}_k$ has a good notion of “$\times$” to make sense of groups and actions, namely $A \otimes_k B$. 

- Commutative Hopf algebras $A$ are affine group schemes $\Rightarrow A$-comodule algebras $B$ associated with affine schemes that carry an action of an affine group scheme.

- $G = \text{Hom}_{\text{Sch}_k}(k, A)$ is a group that acts on $X = \text{Hom}_{\text{Sch}_k}(k, B)$ via convolution $\phi^* \psi := \mu_k \circ (\phi \otimes \psi) \circ \rho$, where $\mu_k : k \otimes k \rightarrow k$ is the multiplication map and $\rho : B \rightarrow A \otimes_k B$ is the coaction.

- What is a quantum symmetry?
Affine group schemes

- $\text{Sch}_k$ has a good notion of “$\times$” to make sense of groups and actions, namely $A \otimes_k B$.
- $\Rightarrow$ comm. Hopf algebras $A = \text{affine group schemes}$
- $\Rightarrow A$-comodule algebras $B = \text{affine schemes with an action of an affine group scheme}$. 
**Affine group schemes**

- \( \text{Sch}_k \) has a good notion of “\( \times \)” to make sense of groups and actions, namely \( A \otimes_k B \).
- \( \hookrightarrow \) comm. Hopf algebras \( A = \) affine group schemes
- \( \hookrightarrow A \)-comodule algebras \( B = \) affine schemes with an action of an affine group scheme.
- \( G = \text{Hom}_{\text{Sch}_k}(k, A) \) is a group that acts on \( X = \text{Hom}_{\text{Sch}_k}(k, B) \) via convolution

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\varphi \ast \psi := \mu_k \circ (\varphi \otimes \psi) \circ \rho,
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where \( \mu_k : k \otimes_k k \to k \) is the multiplication map and \( \rho : B \to A \otimes_k B \) is the coaction.
Faithful flatness

- $\text{Sch}_k$ also has a good notion of “surjective map”, namely a **faithfully flat** algebra embedding $B \to A$. 

This means: a chain complex $L \to M \to N$ of $B$-modules is exact iff so is the induced complex $A \otimes B L \to A \otimes B M \to A \otimes B N$ of $A$-modules.

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of \( A \)-modules.
Motivation

In particular: if \( I \subset B \) is a proper ideal, then \( AI \cong A \otimes_B I \) is an ideal in \( A \cong A \otimes_B B \), and since

\[
I \to B \to 0
\]

is not exact (as \( B/I \neq 0 \)), faithful flatness implies

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AI \to A \to 0
\]

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is not exact, so \( AI \) is a proper ideal.

- \( \rightsquigarrow \) a faithfully flat ring map \( B \rightarrow A \) induces a surjection \( \text{MaxSpec}(A) \rightarrow \text{MaxSpec}(B) \).
Homogeneous spaces

- Summing up, this leads to:

**Definition**

A **homogeneous space** of an affine group scheme $A$ is a left coideal subalgebra $B \subset A$, $\Delta(B) \subset A \otimes_k B$, for which $A$ is faithfully flat as a $B$-module.

- **Quantum groups** and **quantum homogeneous spaces**: allow $A$ and $B$ to be noncommutative.
- Main examples obtained by deformation quantisation of Poisson homogeneous spaces.
Question

*Which affine schemes are quantum homogeneous spaces?*

Observation

The cusp $X \subset \mathbb{k}^2$ given by $y^2 = x^3$ and the nodal cubic $Y \subset \mathbb{k}^2$ given by $y^2 = x^2 + x^3$ both are!

Interesting, as "homogenous" means all points look alike, but these curves are singular so they are definitely not homogeneous; an algebraic set over $\mathbb{k} = \mathbb{R}$ which is a homogeneous space must be smooth (a manifold).
Question

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The cusp

- Fix a field $k$ and $q \in k$ with $q^3 = 1$. Then the algebra $A$ with generators $x, y, a, a^{-1}$ and relations

$$ay = -ya, \ ax = qx a, \ a^6 = 1, \ xy = yx, \ y^2 = x^3$$

is a Hopf algebra with

$$\Delta(y) = 1 \otimes y + y \otimes a^3, \quad \Delta(x) = 1 \otimes x + x \otimes a^2,$$

$$\Delta(a) = a \otimes a, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(a) = 1,$$

$$S(y) = -ya^{-3}, \quad S(x) = -xa^{-2}, \quad S(a) = a^{-1}.$$

- The subalgebra generated by $x, y$ is the algebra of polynomial functions on the cusp and is a quantum homogeneous space.
Similarly, fix \((p, q) \in k^2\) such that \(p^2 = q^2 + q^3\) and define the algebra \(A\) with generators \(a, b, x, y, a^{-1}, b^{-1}\) and relations

\[
\begin{align*}
    aa^{-1} &= a^{-1}a = bb^{-1} = b^{-1}b = 1, \\
    y^2 &= x^2 + x^3, \\
    ba &= ab, \\
    ya &= ay, \\
    bx &= xb, \\
    yx &= xy, \\
    a^2x &= -xa^2 - axa - a^2 + (1 + 3q)a^3, \\
    ax^2 &= -ax - xa - x^2a - xax + (2 + 3q)qa^3, \\
    a^3 &= b^2, \\
    by + yb &= 2pb^2.
\end{align*}
\]
This is a Hopf algebra with

\[ \Delta(x) = 1 \otimes (x - qa) + x \otimes a, \]
\[ \Delta(y) = 1 \otimes (y - pb) + y \otimes b, \]
\[ \Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b, \]
\[ \varepsilon(x) = q, \quad \varepsilon(y) = p, \quad \varepsilon(a) = \varepsilon(b) = 1, \]
\[ S(x) = q - (x - q) a^{-1}, \quad S(y) = p - (y - p) b^{-1}, \]
\[ S(a) = a^{-1}, \quad S(b) = b^{-1} \]

and the subalgebra generated by \( x, y \) embeds the coordinate ring of the nodal cubic as a quantum homogeneous space.
Wild guessing and hard labour in particular by Angela, but observe that

\[ X = x - q, \quad Y = y - p \]

are twisted primitive,

\[ \Delta(X) = 1 \otimes X + X \otimes a, \quad \Delta(Y) = 1 \otimes Y + Y \otimes b \]

and satisfy the defining relation of the curve again,

\[ Y^2 = X^2 + X^3. \]
References and outlook

- A. Masuoka, D. Wigner 1994 - background theory (with references to M. Takeuchi 1979)
- U.K, A. Tabiri 2016 - the plane curve story