What is a quantum symmetry?

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Road map

- Classical maths: symmetries = actions of groups

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- Quantum maths: actions of Hopf algebras (or even Hopf algebroids, Hopf monads...)
- Main message today: Classical objects can have interesting quantum symmetries
- More specifically: Many (maybe all) singular plane curves are quantum homogeneous spaces

Part I: Quantum groups (Hopf algebras)

Algebras (aka monoids)

Definition

An **algebra** in a monoidal category $\mathcal{C}$ is an object G with morphisms $\mu : G \otimes G \to G$, and $\eta : \mathbb{I} \to G$ such that

$$\begin{array}{ccc} G \otimes G \otimes G & \xrightarrow{\mu \otimes \mathrm{id}} & G \otimes G \\ {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \otimes G & \xrightarrow{\mu} & G \end{array}$$

$$\begin{array}{ccccc} \mathbb{I} \otimes G & \xleftarrow{\cong} & G & \xrightarrow{\cong} & G \otimes \mathbb{I} \\ & {\scriptstyle \eta\otimes\mathrm{id}}\searrow & \uparrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{id}\otimes\eta} & \\ & & G \otimes G & & \end{array}$$

commute.

Examples

- Logic: $\mathcal{C} = \textbf{Set}$, $\otimes = \times$, $\mathbb{I} = \{\emptyset\}$, algebras = monoids, i.e. semigroups G with a unit element e_G
- Physics: $\mathcal{C} = \textbf{Vect}_k$ (k some field), $\otimes = \otimes_k$, $\mathbb{I} = k$, algebras = unital associatve k-algebras
- Categories: $\mathcal{C} = \textbf{End}_{\mathcal{D}}$ ($\mathcal{D}$ some category), $\otimes = \circ$, $\mathbb{I} = \mathrm{id}$, algebras = monads on $\mathcal{D}$
- The last example is sort of universal if we identify an algebra G in $\mathcal{C}$ with the monad $G \otimes -$ on $\mathcal{C}$

Modules (aka algebras) over a monad

Definition

A **module** over a monad T on $\mathcal{D}$ is an object M in $\mathcal{D}$ with an action $\alpha\colon \mathrm{T}(M) \to M$ of T,

$$\begin{array}{ccc} \mathrm{T}(\mathrm{T}(M)) & \xrightarrow{\mathrm{T}(\alpha)} & \mathrm{T}(M) \\ {\scriptstyle \mu(\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathrm{T}(M) & \xrightarrow[\alpha]{} & M \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{\eta(\mathrm{id})} & \mathrm{T}(M) \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & M \end{array}$$

- $\mathcal{D} = \mathbf{Set}$, $\mathrm{T} = G \times -$, G monoid: T-modules are G-sets X, $\alpha\colon G \times X \to X$, $(g, x) \mapsto gx$ satisfies

$$g(hx) = (gh)x, \quad e_G x = x, \quad \forall g, h \in G, x \in X.$$

Hopf algebras

- Co(al)gebras in $\mathcal{C}$ = algebras in $\mathcal{C}^\circ$,

$$\Delta : G \to G \otimes G, \quad \varepsilon : G \to \mathbb{I}.$$

- In $\mathcal{C} = \mathbf{Set}$, $\Delta : G \to G \times G$ must be $g \mapsto (g, g)$, in $\mathcal{C} = \mathbf{Vect}_k$ things are more flexible (linear duality!)

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- Braidings $\chi : G \otimes G \to G \otimes G$ (e.g. the flip for **Set** and $\mathbf{Vect}_k$) turn $\mathbf{Alg}_\mathcal{C}$ into a monoidal category.

Definition

A **Hopf algebra** is a coalgebra in $\mathbf{Alg}_\mathcal{C}$ for which

$$\Gamma : G \otimes G \xrightarrow{\Delta \otimes \mathrm{id}} G \otimes G \otimes G \xrightarrow{\mathrm{id} \otimes \mu} G \otimes G$$

is an isomorphism (this is called the **Galois map**).

$(\mathbf{Set}, \times, \{\emptyset\})$ in detail

- An algebra in **Set** is a semigroup with unit element,

$$\mu\colon G \times G \to G, \quad (g,h) \mapsto gh, \quad f(gh) = (fg)h,$$

$$\eta\colon \{\emptyset\} \to G, \quad e_G g = g e_G = g, \quad e_G := \eta(\emptyset).$$

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- Every set G is a coalgebra in a unique way,

$$\Delta\colon G \to G \times G, \quad g \mapsto (g,g), \quad \varepsilon\colon G \to \{\emptyset\}.$$

- This turns an algebra into a coalgebra in $\mathbf{Alg}_{\mathbf{Set}}$ with respect to the braiding $\chi(g,h) = (h,g)$,

$$\Delta(gh) = (gh, gh) = (g,g)(h,h) = \Delta(g)\Delta(h).$$

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- The Galois map is $\Gamma(g,h) = (g, gh)$, so a Hopf algebra is a group, with $\Gamma^{-1}(g,h) = (g, g^{-1}h)$.

Part II: Quantum homogeneous spaces

Homogeneous spaces

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Fixing $x \in X$ defines a G-equivariant surjective map

$$\pi \colon G \to X, \quad g \mapsto gx$$

and identifies X with the G-set

$$G/H := \{gH \mid g \in G\} \subset \mathcal{P}(G)$$

of all cosets $gH = \{gh \mid h \in H\}$ of the **isotropy group**

$$H := \{g \in G \mid gx = x\}.$$

Affine schemes over k

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- $\mathbf{Sch}_k := \mathbf{CommAlg}_k^{\circ}$: an **affine scheme** over k is a commutative k-algebra, morphisms $A \to B$ of affine schemes are k-algebra homomorphisms $B \to A$.
- The set underlying a scheme is $X = \mathrm{Hom}_{\mathbf{Sch}_k}(k, B)$.

Hilbert's Nullstellensatz

- Historical motivation: If $k = \mathbb{C}$ and A is finitely generated and reduced (no nilpotents), then

$$A \cong \mathcal{O}(X)$$

for some **algebraic set**

$$X = \{x \in k^n \mid f_1(x) = \dots f_d(x) = 0\}$$

given by $f_1, \dots, f_d \in k[t_1, \dots, t_n]$, where $\mathcal{O}(X)$ is the **coordinate ring** of X, i.e. the set of all polynomial functions $X \to k$.

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given by $f_1, \dots, f_d \in k[t_1, \dots, t_n]$, where $\mathcal{O}(X)$ is the **coordinate ring** of X, i.e. the set of all polynomial functions $X \to k$. Also,

$$X \cong \mathrm{Hom}_{\mathbf{Sch}_k}(k, A) \cong \mathrm{MaxSpec}(A).$$

Affine group schemes

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Affine group schemes

- $\mathbf{Sch}_k$ has a good notion of "$\times$" to make sense of groups and actions, namely $A \otimes_k B$.
- $\rightsquigarrow$ comm. Hopf algebras A = affine group schemes
- $\rightsquigarrow$ A-comodule algebras B = affine schemes with an action of an affine group scheme.
- $G = \mathrm{Hom}_{\mathbf{Sch}_k}(k, A)$ is a group that acts on $X = \mathrm{Hom}_{\mathbf{Sch}_k}(k, B)$ via convolution

$$\varphi * \psi := \mu_k \circ (\varphi \otimes \psi) \circ \rho,$$

where $\mu_k \colon k \otimes_k k \to k$ is the multiplication map and

$$\rho \colon B \to A \otimes_k B$$

is the coaction.

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- This means: a chain complex

$$L \to M \to N$$

of B-modules is exact iff so is the induced complex

$$A \otimes_B L \to A \otimes_B M \to A \otimes_B N$$

of A-modules.

Motivation

- In particular: if $I \subset B$ is a proper ideal, then $AI \cong A \otimes_B I$ is an ideal in $A \cong A \otimes_B B$, and since

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is not exact (as $B/I \neq 0$), faithful flatness implies

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is not exact, so AI is a proper ideal.

- $\rightsquigarrow$ a faithfully flat ring map $B \to A$ induces a surjection $\mathrm{MaxSpec}(A) \to \mathrm{MaxSpec}(B)$.

Homogeneous spaces

- Summing up, this leads to:

Definition

A **homogeneous space** of an affine group scheme A is a left coideal subalgebra $B \subset A$, $\Delta(B) \subset A \otimes_k B$, for which A is faithfully flat as a B-module.

- **Quantum groups** and **quantum homogeneous spaces**: allow A and B to be noncommutative.
- Main examples obtained by deformation quantisation of Poisson homogeneous spaces.

Final slide despite the number

Question

Which affine schemes are quantum homogeneous spaces?

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Observation

The cusp $X \subset k^2$ *given by* $y^2 = x^3$ *and the nodal cubic* $Y \subset k^2$ *given by* $y^2 = x^2 + x^3$ *both are!*

Interesting, as "homogenous" means all points look alike, but these curves are **singular** so they are definitely not homogeneous; an algebraic set over $k = \mathbb{R}$ which is a homogeneous space must be smooth (a manifold).

The cusp

- Fix a field k and $q \in k$ with $q^3 = 1$. Then the algebra A with generators x, y, a, a^{-1} and relations

$$ay = -ya, \;\; ax = qxa, \;\; a^6 = 1, \;\; xy = yx, \;\; y^2 = x^3$$

is a Hopf algebra with

$$\Delta(y) = 1 \otimes y + y \otimes a^3, \quad \Delta(x) = 1 \otimes x + x \otimes a^2,$$

$$\Delta(a) = a \otimes a, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(a) = 1,$$

$$S(y) = -ya^{-3}, \quad S(x) = -xa^{-2}, \quad S(a) = a^{-1}.$$

- The subalgebra generated by x, y is the algebra of polynomial functions on the cusp and is a quantum homogeneous space.

The nodal cubic I

- Similarly, fix $(p, q) \in k^2$ such that $p^2 = q^2 + q^3$ and define the algebra A with generators $a, b, x, y, a^{-1}, b^{-1}$ and relations

$$aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = 1, \qquad y^2 = x^2 + x^3,$$
$$ba = ab, \qquad ya = ay, \qquad bx = xb, \qquad yx = xy,$$
$$a^2x = -xa^2 - axa - a^2 + (1 + 3q)\, a^3,$$
$$ax^2 = -ax - xa - x^2a - xax + (2 + 3q)qa^3,$$
$$a^3 = b^2, \qquad by + yb = 2pb^2.$$

The nodal cubic II

- This is a Hopf algebra with

$$\Delta(x) = 1 \otimes (x - qa) + x \otimes a,$$
$$\Delta(y) = 1 \otimes (y - pb) + y \otimes b,$$
$$\Delta(a) = a \otimes a, \qquad \Delta(b) = b \otimes b,$$
$$\varepsilon(x) = q, \quad \varepsilon(y) = p, \quad \varepsilon(a) = \varepsilon(b) = 1,$$
$$S(x) = q - (x - q)\,a^{-1}, \quad S(y) = p - (y - p)\,b^{-1},$$
$$S(a) = a^{-1}, \quad S(b) = b^{-1}$$

and the subalgebra generated by x, y embeds the coordinate ring of the nodal cubic as a quantum homogeneous space.

Where do the relations come from?

- Wild guessing and hard labour in particular by Angela, but observe that

$$X = x - q, \qquad Y = y - p$$

are twisted primitive,

$$\Delta(X) = 1 \otimes X + X \otimes a, \quad \Delta(Y) = 1 \otimes Y + Y \otimes b$$

and satisfy the defining relation of the curve again,

$$Y^2 = X^2 + X^3.$$

References and outlook

- A. Masuoka, D. Wigner 1994 - background theory (with references to M. Takeuchi 1979)
- U.K, A. Tabiri 2016 - the plane curve story