Inverse semigroups and étale groupoids

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1. Background

- J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer, 1980.

- J. Kellendonk, The local structure of tilings and their integer group of coinvariants, *Comm. Math. Phys* **187** (1997), 115–157.

The goal is to understand the connection between inverse semigroups and étale groupoids and the rôle this connection plays in C^*-algebras and beyond.

2. Inverse semigroups

"Symmetry denotes that sort of concordance of several parts by which they integrate into a whole." − Hermann Weyl

Symmetry is more than groups.

As groups are algebraic tools for studying symmetry, so *inverse semigroups* are tools for studying partial symmetry.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Example: the symmetric inverse monoid

Let X be a set equipped with the discrete topology. Denote by $I(X)$ the set of all partial bijections of X. This is an example of an inverse semigroup called the *symmetric inverse monoid*.

Theorem [Vagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

The natural partial order

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup.*

It is called the *natural partial order*.

Example In symmetric inverse monoids the natural partial order is nothing other than the restriction ordering on partial bijections.

Let S be an inverse semigroup. Elements of the form $a^{-1}a$ and aa^{-1} are idempotents. Denote by $E(S)$ the set of idempotents of S.

Remarks

1. $E(S)$ is a commutative subsemigroup or *semilattice*.

2. $E(S)$ is an order ideal of S.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A non-empty subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

Example

The idempotents in $I(X)$ are the identity functions defined on the subsets of X. Denote them by 1_A, where $A \subseteq X$, called *partial identities*. Then

$$1_A \leq 1_B \iff A \subseteq B$$

and

$$1_A 1_B = 1_{A \cap B}.$$

Thus the semilattice of idempotents on $I(X)$ is isomorphic to $\mathcal{P}(X)$.

Partial bijections f and g are compatible if and only if $f \cup g$ is a partial bijection.

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each non-empty finite (resp. arbitrary) compatible subset has a join.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse semigroup is said to be *Boolean* if it is distributive and its semilattice of idempotents is a (generalized) Boolean algebra.

Un petit peu d'histoire

Pseudogroups of transformations are pseudogroups of partial homeomorphisms between the open subsets of a topological space.

They play an important rôle in geometry. See page 110 of *Three-dimensional geometry and topology* by William P. Thurston.

They are the origin of inverse semigroup theory.

Important in the work of Charles Ehresmann, who was the first to apply category theory to differential geometry.

3. Groupoids

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'.

Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms an inverse monoid.*

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

Etale groupoids therefore have a strong algebraic character.

4. The main adjunction theorem

There are two basic constructions.

- Let G be an étale groupoid. Denote by B(G) the set of all open local bisections of G. Then B(G) is a pseudogroup.

- Let S be a pseudogroup. Denote by G(S) the set of all *completely prime filters* of S. Then G(S) is an étale groupoid. [This is the 'hard' direction].

Denote by **Inv** a suitable category of pseudogroups and by **Etale** a suitable category of étale groupoids.

Theorem 1 *The functor* G$\colon \mathbf{Inv}^{op} \to \mathbf{Etale}$ *is right adjoint to the functor* B$\colon \mathbf{Etale} \to \mathbf{Inv}^{op}$.

5. Boolean inverse semigroups

A *locally compact Boolean space* is a locally compact Hausdorff space with a basis of clopen subsets.

A *Boolean groupoid* is an étale topological groupoid whose space of identities is a locally compact Boolean space.

If G is a Boolean groupoid denote by KB(G) the set of all compact-open local bisections.

If S is a Boolean inverse monoid denote by G(S) the set of ultrafilters of S.

The following may be deduced from Theorem 1.

Theorem 2

1. KB(G) *is a Boolean inverse semigroup.*

2. G(S) *is a Boolean groupoid.*

3. *Boolean inverse semigroups are in duality with Boolean groupoids.*

Connections . . .

- Universal algebra. Boolean inverse semi-groups form a congruence-permutable variety (Wehrung).

- Logic. MV-algebras.

- Etale groupoids.

- Groups of dynamical origin.

- Dynamical systems.

- Aperiodic tilings.

- Coarse structures.

- C^*-algebras of real rank zero.