Custom Hypergraph Categories via Generalized Relations

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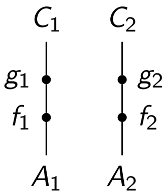
Outline

- Compact closed categories and diagrammatic calculi
- Some ad-hoc procedures for constructing new compact closed categories, using variations on the notion of binary relation
- A parameterized theory of categories of generalized relations

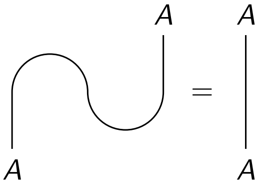
Motivation

Graphical Languages I

Pictures of morphisms in symmetric monoidal categories.



Compact closed categories



Motivation

Standard Settings

Some examples of compact closed categories

- The category of finite dimensional Hilbert spaces and linear maps, **FdHilb** - Pure state QM
- The category of finite dimensional Hilbert spaces and completely positive maps, **CPM(FdHilb)** - Mixed state QM
- The category of sets and binary relations, **Rel** - Non-deterministic computation
- Cospans - Networks
- Corelations - Networks
- Spans - ?

Where do we find more?

CPM

Selinger's CPM construction and relatives of it generate new compact closed categories. They are 1 parameter constructions, but what exactly does CPM do?

Motivations

Options

CPM

Selinger's CPM construction and relatives of it generate new compact closed categories. They are 1 parameter constructions, but what exactly does CPM do?

Decorated Corelations and Cospans

These constructions, due to Brendan Fong, are parameterized by various "technical" parameters such as factorization systems and lax monoidal functors.

- The corelation construction is completely generic
- Constructing particular examples involves clever choices of the technical parameters. Can we make things easier?

Motivation

Convexity

Mathematical models of cognition (Gärdenfors) emphasize convexity, how can we address this in a compact closed setting?

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- The finite distribution monad D

$$X \mapsto \{d : X \to [0,1] \mid d \text{ has finite support and } \sum d(x) = 1\}$$

- Algebras in $\mathbf{EM}(D)$ are sets with a "convex mixing operation" $D(X) \to X$
- $\mathbf{EM}(D)$ is regular so we can form a compact closed category $\mathbf{Rel}(D)$ in which morphisms are binary relations $R : A \to B$ such that

$$R(a_1, b_1) \wedge ... \wedge R(a_n, b_n) \Rightarrow R(\sum_i p_i a_i, \sum_i p_i b_i)$$

Motivation

Metrics I

Cognition also requires metrics, but categories of metric spaces are not regular so we cannot use the previous trick. What to do?

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Definition

A (unital) quantale Q is a complete join semilattice with a monoid structure $\otimes, k$ such that

$$x\otimes\left[\bigvee U\right] = \{x\otimes u \mid u \in U\} \quad \text{and} \quad \left[\bigvee U\right]\otimes x = \{u\otimes x \mid x \in U\}$$

We can define a category of relations $\mathbf{Rel}(Q)$ with relations maps $A \times B \to Q$ and

$$(S \circ R)(a, c) = \bigvee R(a, b) \otimes S(b, c)$$

We refer to these relations as Q-relations. If Q is a commutative quantale, this category is compact closed.

Motivation

Metrics II

As **Rel**(Q) is poset enriched via

$$R \subseteq R' \quad \text{if} \quad \forall a, b. R(a, b) \leq R(a', b')$$

Therefore, we can talk about **Rel**(Q)-internal monads. These are endomorphisms satisfying

$$1 \subseteq R \quad \text{and} \quad R \circ R \subseteq R$$

Paralleling the usual notion of a monad on a category being an endofunctor $T : \mathcal{C} \to \mathcal{C}$ with

$$\eta : 1 \Rightarrow T \quad \text{and} \quad T \circ T \to T$$

Motivation

Metrics III

Example (Internal monads in $\mathbf{Rel}(Q)$)

- For quantale $B = \{0, 1\}$

$$R(a,a) \quad \text{and} \quad R(a,b) \wedge R(b,c) \Rightarrow R(a,c)$$

- For quantale $I = [0, 1]$

$$R(a,a) = 1 \quad \text{and} \quad R(a,b) \wedge R(b,c) \leq R(a,c)$$

- For quantale $C = ([0, \infty], \bigvee = \inf, k = 0, \otimes = +)$

$$R(a,a) = 0 \quad \text{and} \quad R(a,b) + R(b,c) \geq R(a,c)$$

- For quantale $F = ([0, \infty], \bigvee = \inf, k = 0, \otimes = \max)$

$$R(a,a) = 0 \quad \text{and} \quad \max(R(a,b), R(b,c)) \geq R(a,c)$$

Motivation

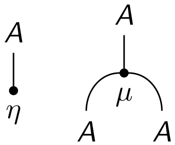
From the ad-hoc to theory

- We used a couple of ad-hoc tricks
  - Relations in regular categories, particularly from algebraic structure
  - Relations with truth values in a commutative quantale
- Questions
  - Can we relate / combine these two schemes?
  - Can relations be varied in other ways to generate yet more examples?
  - How can we relate constructions with different parameters?

Motivation

Hypergraph Categories (Kissinger, Fong)

Commutative monoid:



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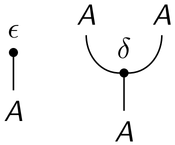

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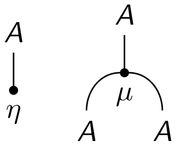

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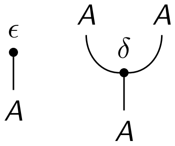
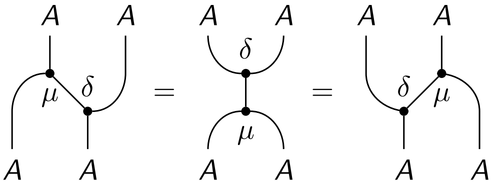
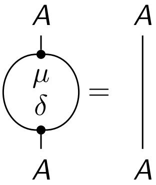


Frobenius axiom:



Special axiom:



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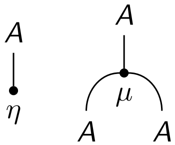

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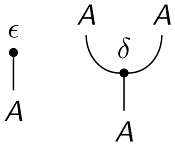

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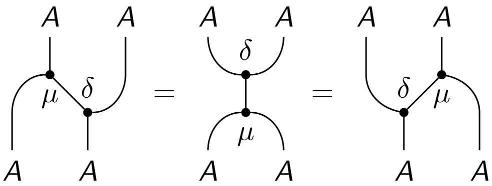

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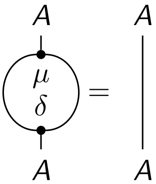


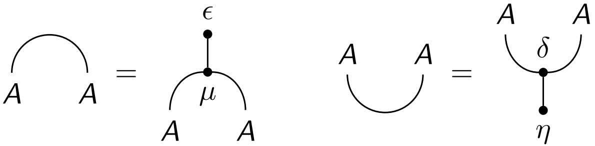
Example

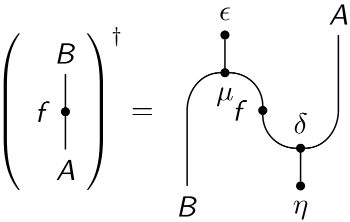
The categories **Rel**, **Rel**(**EM**(D)) and **Rel**(Q) are hypergraph categories.

Motivation

From Hypergraph to Compact Closure

Every hypergraph category is †-compact closed.





Starting with Q-relations, we aim to incorporate algebraic structure.

Algebraic Structure for Q-relations

To incorporate algebraic structure, we fix a signature of operation symbols Σ, and a set of equations over that signature. We need to generalize the condition

$$R(a_1, b_1) \wedge \ldots \wedge R(a_n, b_n) \Rightarrow R(\sigma(a_1, \ldots, a_n), \sigma(b_1, \ldots, b_n))$$

We exploit the operations of our quantale, leading to the following condition for each $\sigma \in \Sigma$

$$R(a_1, b_1) \otimes \ldots \otimes R(a_n, b_n) \leq R(\sigma(a_1, \ldots, a_n), \sigma(b_1, \ldots, b_n))$$

We refer to such a Q-relation as **algebraic**.

Generalized Relations

Identities and Composition

We define our identities as

$$1(a, a') = \begin{cases} k \text{ if } a = a' \\ \bot \text{ otherwise} \end{cases}$$

Composition of relations as before

$$(S \circ R)(a, c) = \bigvee_b R(a, b) \otimes S(b, c)$$

Generalized Relations

Varying the Ambient Topos

We can generalize further by allowing our ambient topos to vary. To do this we consider quantales internal to our chosen topos. A tweak to the identities is required

$$1(a, a') = \bigvee \{ k \mid a = a' \}$$

Theorem

If $\mathcal{E}$ is a topos, Q an internal commutative quantale, and (Σ, E) an algebraic variety in $\mathcal{E}$ then

- *There is a category $\mathbf{Rel}_{(\Sigma,E)}(Q)$ with objects (Σ, E)-algebras, and morphisms algebraic Q-relations.*
- *$\mathbf{Rel}_{(\Sigma,E)}(Q)$ has a symmetric monoidal structure given by products in $\mathcal{E}$.*
- *$\mathbf{Rel}_{(\Sigma,E)}(Q)$ is poset enriched with respect to the ordering*

$$R \subseteq R' \text{ if } \forall a, b. R(a, b) \leq R'(a, b)$$

- *$\mathbf{Rel}_{(\Sigma,E)}(Q)$ is a hypergraph category*

Details of the Structure

- Tensors

$$(R \otimes S)(a, a', b, b') = R(a, b) \otimes S(a', b')$$

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- Graphs

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \to \mathbf{Rel}_{(\Sigma, E)}(Q)$$
$$f_\circ(a, b) = \bigvee \{k \mid f(a) = b\}$$

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- Hypergraph structure from cartesian structure in $\mathcal{E}$

$$\epsilon_A = !_\circ \qquad \delta_A = \Delta_\circ$$

Generalized Relations

Examples

The following examples can be constructed using this procedure.

- **Rel**
- **Rel**(C) giving a category of relations with internal monads the generalized metric spaces
- **Rel**(F) giving a category of relations with internal monads the generalized ultrametric spaces
- The category **Rel**(**EM**(D)) of convex relations arises for a suitable choice of (Σ, E) and Q
- The category of linear relations used in models of linear dynamical systems

We get new examples worthy of further investigation

- C-valued convex relations, blending both convexity and metrics
- Models varying with context using presheaf toposes

Spans

Spans are Constructive Relations I

A span of sets consists of the data

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ A & & B \end{array}$$

Interpretation - Constructive Relations

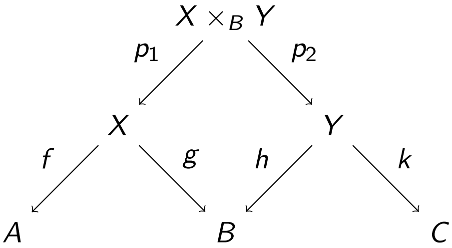
Elements of the apex X are proof witnesses for relatedness, we write

$$\begin{array}{ccc} & x & \\ \swarrow & & \searrow \\ a & & b \end{array} \quad \text{if} \quad f(x) = a \quad \text{and} \quad g(x) = b$$

Spans

Spans are Constructive Relations I

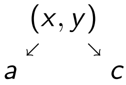
Spans are composed using pullbacks



We recall that

$$X \times_B Y \cong \{(x, y) \mid x \in X, y \in Y, g(x) = h(y)\}$$

So



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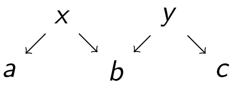

Spans

Incorporating Truth Values

We introduce a monoid $Q = (Q, \otimes, k)$ of truth values, and a characteristic morphism

$$\begin{array}{ccccc} & & X & \xrightarrow{\ \chi\ } & Q \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ A & & & & B \end{array}$$

We call such a span a Q-span, and write

$$\begin{array}{ccc} & x^q & \\ \swarrow & & \searrow \\ a & & b \end{array} \quad \text{if} \quad f(x) = a \quad \text{and} \quad g(x) = b \quad \text{and} \quad \chi(x) = q$$

Spans

Composing Spans with Truth Values

Q-spans compose via pullbacks as before. We must also define the resulting characteristic function

$$\begin{array}{ccc} & (x,y)^{p\otimes q} & \\ \swarrow & & \searrow \\ a & & c \end{array} \quad \text{if} \quad \begin{array}{ccccc} & x^p & & y^q & \\ \swarrow & & \searrow \swarrow & & \searrow \\ a & & b & & c \end{array}$$

Theorem

For a finitely complete category $\mathcal{E}$ and internal monoid Q, Q-spans form a category **Span**(Q). *If Q is commutative,* **Span**(Q) *is a hypergraph category.*

Spans

Algebraic Structure

We fix variety (Σ, E). An **algebraic** Q-span is a Q-span with domain and codomain (Σ, E)-algebras, satisfying the condition that if for every $\sigma \in \Sigma$

$$a_1 \swarrow x_1^{q_1} \searrow b_1 \quad \wedge \ldots \wedge \quad a_n \swarrow x_n^{q_n} \searrow b_n$$

Then there exists x such that

$$\sigma(a_1, \ldots, a_n) \swarrow x^q \searrow \sigma(b_1, \ldots, b_n) \qquad \text{and } q_1 \otimes \ldots \otimes q_n \leq q$$

Note the need for order structure on the truth values.

Theorem

If $\mathcal{E}$ is a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid then

- *Algebraic Q-spans form a category $\mathbf{Span}_{(\Sigma,E)}(Q)$*
- *The category $\mathbf{Span}_{(\Sigma,E)}(Q)$ is a hypergraph category*

Spans

Algebraic Q-spans

Details of the Structure

- Tensors given by products in $\mathbf{Alg}(\Sigma, E)$ and monoid multiplication
- Graphs

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \to \mathbf{Span}_{(\Sigma, E)}(Q)$$

$$f_\circ = \begin{array}{ccccc} & & A & \xrightarrow{\;!\;} 1 & \xrightarrow{\;k\;} Q \\ & {\scriptstyle 1}\swarrow & & \searrow{\scriptstyle f} & \\ A & & & & B \end{array}$$

- Hypergraph structure given by graphs of diagonals and terminal maps in the base.

We say that

$$(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$$

if there is a $\mathcal{E}$-monomorphism $m : X_1 \to X_2$ such that

$$f_1 = f_2 \circ m \text{ and } g_1 = g_2 \circ m \text{ and } \forall a, b. \chi_1(x) \leq \chi_2(m(x))$$

This relation makes $\mathbf{Span}_{(\Sigma,E)}(Q)$ a preordered monoidal category.

If Q is a commutative quantale, we can turn an algebraic Q-span into an algebraic Q-relation via

$$V(X, f, g, \chi)(a, b) = \bigvee \{\chi(x) \mid f(x) = a \wedge g(x) = b\}$$

Theorem

Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$ and Q an internal commutative quantale. There is a strict monoidal, identity and surjective on objects, preorder-functor

$$V : \mathbf{Span}_{(\Sigma,E)}(Q) \to \mathbf{Rel}_{(\Sigma,E)}(Q)$$

This functor "commutes with graphs"

$$\begin{array}{ccc} \mathbf{Span}_{(\Sigma,E)}(Q) & \xrightarrow{V} & \mathbf{Rel}_{(\Sigma,E)}(Q) \\ {\scriptstyle (-)_\circ} \nwarrow & & \nearrow {\scriptstyle (-)_\circ} \\ & \mathbf{Alg}(\Sigma, E) & \end{array}$$

Parameterized Constructions

We have shown a procedure for constructing preordered hypergraph categories. These categories can be customized along 4 axes of variation

1. The ambient mathematical universe
2. The truth values
3. The relevant algebraic structure
4. Proof relevance versus provability

Conclusion

- Conceptually motivated parameterized construction of hypergraph categories (today's talk)
- This construction is functorial in the choice of truth values (done)
- It is also functorial in the algebraic structure in interesting ways (done)
- Structure of generalized categories of relations, zero objects, biproducts etc. (ongoing)
- The construction should also be functorial in the choice of topos (ongoing)
- Our category of spans should really be a symmetric monoidal bicategory (future work)
- We can take truth values in monoidal categories - unify the span and relation constructions (future work)