Categories of Physical Processes

Stanisław Szawiel

University of Warsaw

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Part I
A non-topological TQFT
Construction Sketch

\[
\begin{align*}
\text{Phys} & \longrightarrow \ast\text{Mod} \\
H & \xrightarrow{h} H' \quad (\ast\text{Mod}) \\
A & \xrightarrow{f} B \quad (C^*\text{Alg}) \\
\end{align*}
\]

\[h(\alpha v) = f(a)h(v)\]

\ast\text{Mod} = \text{representations of } C^*-\text{algebras + isometric relative homomorphisms}
Construction Sketch

\[
\begin{align*}
\text{Phys} & \longrightarrow \ast\text{Mod} \\
S(A) & = \{\varphi : A \rightarrow \mathbb{C}\} \\
\varphi & \text{ positive}
\end{align*}
\]

\begin{itemize}
\item \(S : C^*\text{Alg}^{op} \rightarrow \text{Set}\)
\item \(\text{Phys} = 1 \downarrow S\)
\item Pairs \((A, \varphi), \varphi \in S(A)\)
\item \(S\) monoidal \(\implies\) \(\text{Phys}\) monoidal
\item \((A, \varphi) \otimes (B, \psi) = (A \otimes B, \varphi \otimes \psi)\)
\end{itemize}
Construction Sketch

Phys $\rightarrow$ *Mod

$C^*\text{Alg}^{op}$

- $(A, \varphi) \mapsto A$
- “Noncommutative spaces”
- Not Morita invariant

$\mathcal{G}N\mathcal{S}$
Construction Sketch

\[
\begin{align*}
\text{Phys} & \xrightarrow{GNS} \star \text{Mod} \\
\end{align*}
\]

What is \(GNS\)?
The GNS Construction

Definition

A pointed $A$-module $(H, v)$ represents $\varphi : A \rightarrow \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$
The GNS Construction

Definition
A pointed $A$-module $(H, v)$ represents $\varphi : A \to \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$

Theorem (The Gelfand-Naimark-Segal Theorem)
- Positive $\varphi$ have an initial representation
- A representation is initial iff it is cyclic
  (cyclic = generated by the chosen vector)

Notation
- Initial representation of $\varphi = GNS(\varphi)$
- Representing vector = $\Omega$
- Write $H$ for $(H, v)$
The GNS Functor

\( H \) represents \( \varphi \implies f^*H \) represents \( f^*\varphi \)

\[
\begin{align*}
  f^*H &\longrightarrow H \\
  B &\xrightarrow{f} A & \varphi &\xrightarrow{} C
\end{align*}
\]
The GNS Functor

\[ H \text{ represents } \varphi \implies f^*H \text{ represents } f^*\varphi \]

\[ \text{GNS}(f^*\varphi) \quad \text{GNS}(\varphi) \]

\[ B \xrightarrow{f} A \xrightarrow{\varphi} C \]
The GNS Functor

$H$ represents $\varphi \implies f^*H$ represents $f^*\varphi$

$$GNS(f^*\varphi) \quad f^*GNS(\varphi) \to GNS(\varphi)$$

$$\exists! GNS(f)$$

This gives a symmetric monoidal functor $GNS : \text{Phys}^{\text{op}} \to \ast \text{Mod}$
The GNS Functor

\[ H \text{ represents } \varphi \implies f^* H \text{ represents } f^* \varphi \]

\[
\begin{align*}
\exists! \quad GNS(f^* \varphi) &\rightarrow f^* GNS(\varphi) & GNS(\varphi) \\
B &\rightarrow f & A &\varphi & C
\end{align*}
\]
The GNS Functor

\[ H \text{ represents } \varphi \implies f^*H \text{ represents } f^*\varphi \]

\[
\begin{align*}
GNS(f^*\varphi) & \xrightarrow{\exists!} f^*GNS(\varphi) \\
& \xrightarrow{GNS(f)} GNS(\varphi)
\end{align*}
\]

\[
\begin{array}{c}
B \xrightarrow{f} A \xrightarrow{\varphi} C
\end{array}
\]
The GNS Functor

\( H \) represents \( \varphi \implies f^*H \) represents \( f^*\varphi \)

\[
\begin{array}{c}
GNS(f^*\varphi) \\
\exists! \\
GNS(f)
\end{array} \xrightarrow{\exists!} f^*GNS(\varphi) \xrightarrow{\exists!} GNS(\varphi)
\]

\[
\begin{array}{c}
B \\
f \\
A \\
\varphi \\
C
\end{array} \xrightarrow{f} \xrightarrow{\varphi} \xrightarrow{\text{cyclicity}}
\]

**Theorem**

*This gives a symmetric monoidal functor*

\[
GNS : \text{Phys}^{\text{op}} \longrightarrow \star\text{Mod}
\]

**Proof.**

Things exist by initiality. Diagrams commute by cyclicity.
The GNS Functor

\( H \) represents \( \varphi \) \( \implies \) \( f^*H \) represents \( f^*\varphi \)

\[
GNS(f^*\varphi) \xrightarrow{\exists!} f^*GNS(\varphi) \xrightarrow{} GNS(\varphi)
\]

\[
GNS(f)
\]

\[
B \xrightarrow{f} A \xrightarrow{\varphi} C
\]

Theorem

This gives a symmetric monoidal functor

\[
GNS : \text{Phys}^{\text{op}} \longrightarrow \ast\text{Mod}
\]

It’s going the wrong way!
The Covariant GNS Functor
Physically Correct Direction

\[ \text{Phys} \xrightarrow{GNS_{c}^{op}} \star \text{Mod}_{op} \xrightarrow{\text{adjoint}} \star \text{Mod}_{adj} \]

Definition

$\star \text{Mod}_{adj}$ is $\star$-modules with adjoint homomorphisms

Adjoint homomorphisms: coisometries $h$ such that $a h(v) = h(f(a)v)$
Definition

- $\ast\text{Mod}_{\text{adj}}$ is $\ast$-modules with adjoint homomorphisms.
- Adjoint homomorphisms: coisometries $h$ such that $ah(v) = h(f(a)v)$. 

The Covariant GNS Functor
Physically Correct Direction
Part II
Physics From a Functor
1. $H$ – faithful $A$-module
2. $U : H \rightarrow H'$ – isometric linear map
3. $f : A \rightarrow B = UAU^*$ – algebra map given by $a \mapsto UaU^*$

**Theorem (Lifting Schrödinger)**

*For any $\psi \in H$ we have $f : U\psi \rightarrow \psi \in \text{Phys},$ and*

$$
\begin{array}{c}
\text{GNS}(\psi) \\
\downarrow \\
H
\end{array} \xrightarrow{\text{GNS}(f)} \xrightarrow{U} \begin{array}{c}
\text{GNS}(U\psi) \\
\downarrow \\
H'
\end{array}
$$
1. \( H \) – faithful \( A \)-module
2. \( U : H \rightarrow H' \) – isometric linear map
3. \( f : A \rightarrow B = UAU^* \) – algebra map given by \( a \mapsto UaU^* \)

**Corollary**

If \( U \) is unitary, then \( g(a) = U^*aU \) gives \( g : \psi \rightarrow U\psi \in \text{Phys} \), and

\[
\begin{array}{c}
GNS(\psi) \xrightarrow{GNS_c(g)} GNS(U\psi) \\
\downarrow \quad \downarrow \\
H \xrightarrow{U} H'
\end{array}
\]
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$?

\[ G \rightarrow \text{Phys} \xrightarrow{GNS_c} \ast \text{Mod}_{adj} \]
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$? Because of composition!

$G \rightarrow \text{Phys} \xrightarrow{GNS_c} \star \text{Mod}_{adj} \rightarrow \text{Unitary representation of } G$!
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$?
Because of composition!

![Diagram](attachment:image.png)

Unitary representation of $G$!

Bonus items:
- Groupoids of symmetries
- Equivariant GNS:

$$
\begin{align*}
\text{Phys} & \xrightarrow{GNS_c} \star\text{Mod}_{adj} \\
\end{align*}
$$
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$? Because of composition!

$G \rightarrow \text{Phys} \xrightarrow{GNS_c} \star\text{Mod}_{adj}$

Unitary representation of $G$!

Bonus items:
- Groupoids of symmetries
- Equivariant GNS:

$\text{Phys}^G \xrightarrow{GNS_c^G} \star\text{Mod}_{adj}^G$
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$? Because of composition!

$G \xrightarrow{	ext{Phys}} *\text{Mod}_{adj} \xrightarrow{\text{GNS}_c} \text{Rep}(G)$

Bonus items:
- Groupoids of symmetries
- Equivariant GNS:

$\text{Phys}_G^G \xrightarrow{\text{GNS}_c^G} *\text{Mod}_{adj}^G \xrightarrow{U} \text{Rep}(G)$
Symmetries and Unitary Representations

Why does a $G$-equivariant state give a unitary representation of $G$? Because of composition!

$$G \rightarrow \text{Phys} \xrightarrow{GNS_c} \ast \text{Mod}_{ad,j}$$

Unitary representation of $G$!

Bonus items:

▶ Groupoids of symmetries
▶ Equivariant GNS:

$$\text{Phys}^G \xrightarrow{GNS_c^G} \ast \text{Mod}_{ad,j}^G \xrightarrow{U} \text{Rep}(G)$$

▶ Compatibility with composite systems:

$\varphi \otimes \psi$ has symmetry $G \times G'$
Relation to Probability Theory

**Prob** – compact probability spaces. From \((X, \mu)\) we construct:

- A state on \(C(X)\) – the expectation value \(\mathbb{E}_\mu(a) = \int_X a \, d\mu\)
- \(L^2(\mu)\), a \(C(X)\)-module

**Theorem**

The following diagram of symmetric monoidal functors commutes

\[
\begin{array}{ccc}
C^\text{op} & \xrightarrow{\text{Prob}^\text{op}} & L^2 \\
\downarrow{\text{Phys}^\text{op}} & & \downarrow{\text{GNS}} \\
\text{GNS} & \xrightarrow{\ast\text{Mod}} & \ast\text{Mod}
\end{array}
\]
**Relation to Probability Theory**

**Prob** – compact probability spaces. From \((X, \mu)\) we construct:
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---

**Theorem**

*The following diagram of symmetric monoidal functors commutes*

\[
\begin{array}{ccc}
C^{op} & \overset{\text{Prob}^{op}}{\longrightarrow} & L^2 \\
\downarrow & & \downarrow \\
\text{Phys}^{op} & \overset{GNS}{\longrightarrow} & \ast\text{Mod}
\end{array}
\]

**Proof.**

1. \(L^2(\mu)\) is cyclic
2. \(1 \in L^2(\mu)\) represents the expectation value \(\mathbb{E}_\mu\)
Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_\varphi(a) = (\text{Spec}(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Theorem (Eigenvalue-Eigenvector Link)

The following are equivalent:

1. $a\Omega = \lambda\Omega$
2. $a = \lambda$ a.e. in $P_\varphi(a)$

Proof. The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R: \varphi \rightarrow P_\varphi(\langle a \rangle)$. The previous theorem computes $\mathcal{G}_\mathcal{N}\mathcal{S}(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \rightarrow \mathcal{G}_\mathcal{N}\mathcal{S}(\varphi)$$

Thus:

$$a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$$

in $L^2 \iff a = \lambda$ a.e.
Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_\varphi(a) = (\text{Spec}(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Theorem (Eigenvalue-Eigenvector Link)

The following are equivalent:

1. $a\Omega = \lambda\Omega$
2. $a = \lambda$ a.e. in $P_\varphi(a)$

Proof.

The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R : \varphi \rightarrow P_\varphi(a) \in \text{Phys}$

Previous theorem computes $GNS(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \rightarrow GNS(\varphi)$$

Thus: $a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$ in $L^2 \iff a = \lambda$ a.e.
Classical Markov Processes

**Definition (Markov Processes)**
- $M(X) = \text{probability measures on } X$
- Markov process $X \rightarrow Y = \text{map } X \rightarrow M(Y)$
- Category of Markov processes $= \text{Kleisli}(M)$

**Theorem (Generalized Gelfand Duality; Furber & Jacobs 2015)**
Gelfand duality extends to a contravariant equivalence between Markov processes and completely positive unital maps between $C^*$-algebras.
Classical Markov Processes

Definition (Markov Processes)

- $M(X) =$ probability measures on $X$
- Markov process $X \rightarrow Y =$ map $X \rightarrow M(Y)$
- Category of Markov processes $= Kleisli(M)$

Theorem (Generalized Gelfand Duality; Furber & Jacobs 2015)

*Gelfand duality extends to a contravariant equivalence between Markov processes and completely positive unital maps between C*-algebras*
Theorem (Non-Unitary GNS Representation)

There is a commuting prism of symmetric monoidal functors:

\[
\begin{array}{cccccc}
\text{Prob}^{op} & \xrightarrow{L^2} & \text{GNS} & \xrightarrow{\ast \text{Mod}} & \\
\text{Phys}^{op} & \xrightarrow{C^{op}} & \\
\text{Phys}^{op}_M & \xrightarrow{C^{op}_M} & \text{Prob}^{op}_M & \xrightarrow{L^2} & \text{GNS}_M & \xrightarrow{U} & \text{Hilb}
\end{array}
\]
Example: State Vector Collapse

- $P \in A$ – self-adjoint projection (i.e. idempotent)
- $\Phi : A \rightarrow A$ given by $a \mapsto PaP$

**Theorem**

- $\varphi$ represented by $\Omega \mapsto \Phi^* \varphi$ represented by $P\Omega$
- $GNS_M(\Phi)$ is the composite

\[
GNS(\Phi^* \varphi) \xleftarrow{} GNS(\varphi) \xrightarrow{P} GNS(\varphi)
\]
Example: State Vector Collapse

- \( P \in A \) – self-adjoint projection (i.e. idempotent)
- \( \Phi : A \rightarrow A \) given by \( a \mapsto PaP \)

**Theorem**

- \( \varphi \) represented by \( \Omega \mapsto \Phi^* \varphi \) represented by \( P\Omega \)
- \( GNS_M(\Phi) \) is the composite

\[
\begin{align*}
GNS(\Phi^* \varphi) & \xleftarrow{P} GNS(\varphi) \\
GNS(\varphi) & \xrightarrow{P} GNS(\varphi)
\end{align*}
\]

**Corollary**

\( GNS_{M,c}(\Phi) \) is cyclic \( (\Omega \mapsto P\Omega) \), and acts as

\[
\begin{align*}
GNS(\varphi) & \xrightarrow{P} GNS(\varphi) \\
& \xrightarrow{\text{orth. proj.}} GNS(\Phi^* \varphi)
\end{align*}
\]
Example: Particle Scattering

- $H$ – Hilbert space
- $S : \mathcal{F}(H) \rightarrow \mathcal{F}(H)$ – unitary scattering matrix
- $H_\alpha, H_\beta \subseteq \mathcal{F}(H)$ – particles of type $\alpha$ and $\beta$

Proposition

There is a process $S_{\alpha\beta} : \alpha \rightarrow \beta \in \text{Phys}_M$ such that

$$H_\alpha \xrightarrow{\text{inclusion}} \mathcal{F}(H) \xrightarrow{S} \mathcal{F}(H) \xrightarrow{\text{projection}} H_\beta$$

$GNS_{M,c}(S_{\alpha\beta})$

If you believe in QED:

$$\gamma + \gamma \rightarrow e^- + e^+$$
Part III
The Frontier
Internalizing in a Topos

- \( GNS \) in a topos \( E = \) monoidal morphism in \( \text{Stacks}(E) \)
- Which definition? All equivalent in \( \text{Set} \)!
- In models of synthetic differential geometry:
  - Infinitesimal processes, like symmetries
    (Heisenberg \( \Leftrightarrow \) Schrödinger)
  - Deformation quantization = Infinitesimal \( \hbar \)-families
- Few examples – must use \( C^{\infty} \)-\(*\)-algebras (work in progress)
Questions for the Future

What is ...

- ...a gauge theory? *No spacetime!*
- ...extended locality *in general*?
- ...a family of vacuum states?
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What is ...

- ...a gauge theory? No spacetime!
- ...extended locality in general?
- ...a family of vacuum states?

Target Theorem (Witten)

The $\hbar$-family of vacua of super Yang-Mills theory is trivial.
(in four dimensions, $N = 2$)
Thank You!