

Categories of Physical Processes

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Part I
A non-topological TQFT

Construction Sketch

Phys \longrightarrow ***Mod**

Construction Sketch

- ▶ $*\mathbf{Mod}$ = representations of C^* -algebras + isometric relative homomorphisms

$\mathbf{Phys} \longrightarrow * \mathbf{Mod}$

$$H \xrightarrow{h} H' \quad (*\mathbf{Mod})$$

$$A \xrightarrow{f} B \quad (C^* \mathbf{Alg})$$

$$h(av) = f(a)h(v)$$

Construction Sketch

Phys \longrightarrow $*\mathbf{Mod}$

$$\mathcal{S}(A) = \{\varphi : A \longrightarrow \mathbb{C}\}$$

φ positive

- ▶ $\mathcal{S} : C^* \mathbf{Alg}^{op} \longrightarrow \mathbf{Set}$
- ▶ $\mathbf{Phys} = 1 \downarrow \mathcal{S}$
Pairs $(A, \varphi), \varphi \in \mathcal{S}(A)$
- ▶ \mathcal{S} monoidal \implies \mathbf{Phys} monoidal
 $(A, \varphi) \otimes (B, \psi) = (A \otimes B, \varphi \otimes \psi)$

Construction Sketch

$$\begin{array}{ccc} \mathbf{Phys} & \longrightarrow & * \mathbf{Mod} \\ \mathcal{O} \downarrow & & \\ C^* \mathbf{Alg}^{op} & & \end{array}$$

- ▶ $(A, \varphi) \mapsto A$
- ▶ “Noncommutative spaces”
- ▶ Not Morita invariant

Construction Sketch

$$\mathbf{Phys} \xrightarrow{GNS} *Mod$$

- ▶ What is *GNS*?

The GNS Construction

Definition

A pointed A -module (H, v) **represents** $\varphi : A \rightarrow \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$

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Theorem (The Gelfand-Naimark-Segal Theorem)

- ▶ Positive φ have an *initial representation*
- ▶ A representation is initial iff it is cyclic
(cyclic = generated by the chosen vector)

Notation

- ▶ Initial representation of $\varphi = GNS(\varphi)$
- ▶ Representing vector = Ω
- ▶ Write H for (H, v)

The GNS Functor

H represents $\varphi \implies f^*H$ represents $f^*\varphi$

$$f^*H \longrightarrow H$$

$$B \xrightarrow{f} A \xrightarrow{\varphi} \mathbb{C}$$

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$GNS(f^*\varphi)$

$GNS(\varphi)$

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$$GNS(f^*\varphi) \quad f^*GNS(\varphi) \longrightarrow GNS(\varphi)$$

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$$B \xrightarrow{f} A \xrightarrow{\varphi} \mathbb{C}$$

Theorem

This gives a symmetric monoidal functor

$$\text{GNS} : \mathbf{Phys}^{op} \longrightarrow * \mathbf{Mod}$$

Proof.

Things exist by initiality. Diagrams commute by cyclicity. \square

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It's going the wrong way!

The Covariant GNS Functor

Physically Correct Direction

$$\mathbf{Phys} \xrightarrow{GNS^{op}} *Mod^{op} \xrightarrow{\text{adjoint}} *Mod_{adj}$$

GNS_c

The diagram illustrates the relationship between three categories: \mathbf{Phys} , $*Mod^{op}$, and $*Mod_{adj}$. A horizontal arrow labeled GNS^{op} points from \mathbf{Phys} to $*Mod^{op}$. A second horizontal arrow labeled "adjoint" points from $*Mod^{op}$ to $*Mod_{adj}$. A curved arrow labeled GNS_c points directly from \mathbf{Phys} to $*Mod_{adj}$, representing the composition of the two horizontal arrows.

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Definition

- ▶ $*Mod_{adj}$ is $*$ -modules with adjoint homomorphisms
- ▶ Adjoint homomorphisms: coisometries h such that

$$ah(v) = h(f(a)v)$$

Part II
Physics From a Functor

The Schrödinger Picture – Example Factory

1. H – faithful A -module
2. $U : H \rightarrow H'$ – isometric linear map
3. $f : A \rightarrow B = UAU^*$ – algebra map given by $a \mapsto UaU^*$

Theorem (Lifting Schrödinger)

For any $\psi \in H$ we have $f : U\psi \rightarrow \psi \in \mathbf{Phys}$, and

$$\begin{array}{ccc} GNS(\psi) & \xrightarrow{GNS(f)} & GNS(U\psi) \\ \downarrow & & \downarrow \\ H & \xrightarrow{U} & H' \end{array}$$

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Corollary

If U is unitary, then $g(a) = U^*aU$ gives $g : \psi \rightarrow U\psi \in \mathbf{Phys}$, and

$$\begin{array}{ccc} GNS(\psi) & \xrightarrow{GNS_c(g)} & GNS(U\psi) \\ \downarrow & & \downarrow \\ H & \xrightarrow{U} & H' \end{array}$$

Symmetries and Unitary Representations

Why does a G -equivariant state give a unitary representation of G ?

$$G \longrightarrow \mathbf{Phys} \xrightarrow{GNS_c} * \mathbf{Mod}_{adj}$$

Symmetries and Unitary Representations

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Because of composition!

$$G \begin{array}{c} \xrightarrow{\quad} \mathbf{Phys} \xrightarrow{GNS_c} \ast\mathbf{Mod}_{adj} \\ \searrow \quad \quad \quad \nearrow \end{array}$$

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Bonus items:

- ▶ Groupoids of symmetries
- ▶ Equivariant GNS:

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$$\mathbf{Phys}^G \xrightarrow{GNS_c^G} *Mod_{adj}^G \xrightarrow{U} \mathbf{Rep}(G)$$

- ▶ Compatibility with composite systems:

$$\varphi \otimes \psi \text{ has symmetry } G \times G'$$

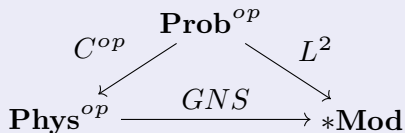
Relation to Probability Theory

Prob – compact probability spaces. From (X, μ) we construct:

- ▶ A state on $C(X)$ – the expectation value $\mathbb{E}_\mu(a) = \int_X a d\mu$
- ▶ $L^2(\mu)$, a $C(X)$ -module

Theorem

The following diagram of symmetric monoidal functors commutes



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The following diagram of symmetric monoidal functors commutes

$$\begin{array}{ccc} & \mathbf{Prob}^{op} & \\ C^{op} \swarrow & & \searrow L^2 \\ \mathbf{Phys}^{op} & \xrightarrow{GNS} & *Mod \end{array}$$

Proof.

1. $L^2(\mu)$ is cyclic
2. $1 \in L^2(\mu)$ represents the expectation value \mathbb{E}_μ



Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_\varphi(a) = (\text{Spec}(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Theorem (Eigenvalue-Eigenvector Link)

The following are equivalent:

1. $a\Omega = \lambda\Omega$
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Proof.

The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R : \varphi \rightarrow P_\varphi(a) \in \mathbf{Phys}$
Previous theorem computes $GNS(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \rightarrow GNS(\varphi)$$

Thus: $a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$ in $L^2 \iff a = \lambda$ a.e. □

Classical Markov Processes

Definition (Markov Processes)

- ▶ $M(X)$ = probability measures on X
- ▶ Markov process $X \rightarrow Y = \text{map } X \rightarrow M(Y)$
- ▶ Category of Markov processes = $Kleisli(M)$

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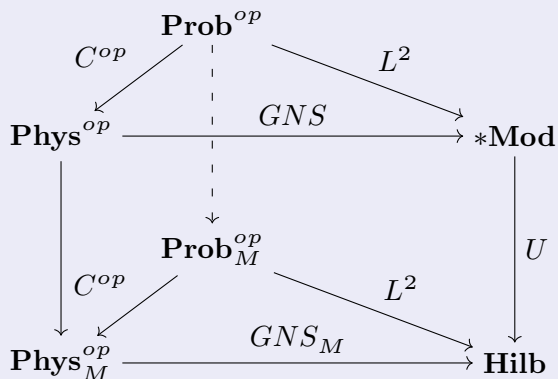
Theorem (Generalized Gelfand Duality; Furber & Jacobs 2015)

*Gelfand duality extends to a contravariant equivalence between Markov processes and **completely positive unital maps** between C^* -algebras*

Quantum Markov Processes

Theorem (Non-Unitary GNS Representation)

There is a commuting prism of symmetric monoidal functors:



Example: State Vector Collapse

- ▶ $P \in A$ – self-adjoint projection (i.e. idempotent)
- ▶ $\Phi : A \rightarrow A$ given by $a \mapsto PaP$

Theorem

- ▶ φ represented by $\Omega \implies \Phi^*\varphi$ represented by $P\Omega$
- ▶ $GNS_M(\Phi)$ is the composite

$$GNS(\Phi^*\varphi) \hookrightarrow GNS(\varphi) \xrightarrow{P} GNS(\varphi)$$

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Corollary

$GNS_{M,c}(\Phi)$ is cyclic ($\Omega \mapsto P\Omega$), and acts as

$$GNS(\varphi) \xrightarrow{P} GNS(\varphi) \xrightarrow{\text{orth. proj.}} GNS(\Phi^*\varphi)$$

Example: Particle Scattering

- ▶ H – Hilbert space
- ▶ $S : \mathcal{F}(H) \rightarrow \mathcal{F}(H)$ – unitary scattering matrix
- ▶ $H_\alpha, H_\beta \subseteq \mathcal{F}(H)$ – particles of type α and β

Proposition

There is a process $S_{\alpha\beta} : \alpha \rightarrow \beta \in \mathbf{Phys}_M$ such that

$$H_\alpha \begin{array}{c} \xrightarrow{\text{inclusion}} \\ \searrow \\ \xrightarrow{\text{projection}} \end{array} \mathcal{F}(H) \xrightarrow{S} \mathcal{F}(H) \begin{array}{c} \xrightarrow{\text{projection}} \\ \nearrow \\ \xrightarrow{\text{inclusion}} \end{array} H_\beta$$

$GNS_{M,c}(S_{\alpha\beta})$

If you believe in QED:

$$\gamma + \gamma \rightarrow e^- + e^+$$

Part III
The Frontier

Internalizing in a Topos

- ▶ *GNS* in a topos $E =$ monoidal morphism in $\mathbf{Stacks}(E)$
- ▶ Which definition? All equivalent in \mathbf{Set} !
- ▶ In models of synthetic differential geometry:
 - ▶ Infinitesimal processes, like symmetries
(Heisenberg \iff Schrödinger)
 - ▶ Deformation quantization = Infinitesimal \hbar -families
- ▶ Few examples – must use C^∞ -*-algebras (work in progress)

Questions for the Future

What is ...

- ▶ ...a gauge theory? **No spacetime!**
- ▶ ...extended locality **in general?**
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Target Theorem (Witten)

The \hbar -family of vacua of super Yang-Mills theory is trivial.
(in four dimensions, $N = 2$)

Thank You!