Categories of Physical Processes

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Part I
A non-topological TQFT

Construction Sketch

$$\mathbf{Phys} \longrightarrow {*}\mathbf{Mod}$$

Construction Sketch

- $*\mathbf{Mod}$ = representations of C^*-algebras + isometric relative homomorphisms

$$\mathbf{Phys} \longrightarrow {\color{red}*\mathbf{Mod}}$$

$$H \xrightarrow{\quad h \quad} H' \quad (*\mathbf{Mod})$$

$$A \xrightarrow{\quad f \quad} B \quad (C^*\mathbf{Alg})$$

$$h(av) = f(a)h(v)$$

Construction Sketch

$$\textbf{Phys} \longrightarrow *\textbf{Mod}$$

$$\mathcal{S}(A) = \{\varphi : A \longrightarrow \mathbb{C}\}$$

φ positive

- $\mathcal{S} : C^*\textbf{Alg}^{op} \longrightarrow \textbf{Set}$
- $\textbf{Phys} = 1 \downarrow \mathcal{S}$
  Pairs $(A, \varphi), \varphi \in \mathcal{S}(A)$
- $\mathcal{S}$ monoidal $\Longrightarrow$ $\textbf{Phys}$ monoidal
  $(A, \varphi) \otimes (B, \psi) = (A \otimes B, \varphi \otimes \psi)$

Construction Sketch

$$\begin{tikzcd} \mathbf{Phys} \arrow[r] \arrow[d, "\mathcal{O}"'] & *\mathbf{Mod} \\ C^*\mathbf{Alg}^{op} & \end{tikzcd}$$

- $(A, \varphi) \longmapsto A$
- "Noncommutative spaces"
- Not Morita invariant

Construction Sketch

$$\mathbf{Phys} \xrightarrow{GNS} {*}\mathbf{Mod}$$

- What is GNS?

The GNS Construction

Definition

A pointed A-module (H, v) represents $\varphi : A \longrightarrow \mathbb{C}$ if

$$\varphi(a) = \langle av, v \rangle_H$$

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Theorem (The Gelfand-Naimark-Segal Theorem)

- *Positive φ have an initial representation*
- *A representation is initial iff it is cyclic (cyclic $=$ generated by the chosen vector)*

Notation

- Initial representation of $\varphi = GNS(\varphi)$
- Representing vector $= \Omega$
- Write H for (H, v)

The GNS Functor

H represents $\varphi \Longrightarrow f^*H$ represents $f^*\varphi$

$$f^*H \longrightarrow H$$

$$B \xrightarrow{f} A \xrightarrow{\varphi} \mathbb{C}$$

The GNS Functor

H represents $\varphi \Longrightarrow f^*H$ represents $f^*\varphi$

$$GNS(f^*\varphi) \qquad\qquad GNS(\varphi)$$

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The GNS Functor

H represents $\varphi \Longrightarrow f^*H$ represents $f^*\varphi$

$$GNS(f^*\varphi) \xrightarrow{\exists!} f^*GNS(\varphi) \longrightarrow GNS(\varphi)$$

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The GNS Functor

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$$\begin{tikzcd} GNS(f^*\varphi) \arrow[r, "\exists!"] \arrow[rr, bend right, "GNS(f)"'] & f^*GNS(\varphi) \arrow[r] & GNS(\varphi) \end{tikzcd}$$

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Theorem

This gives a symmetric monoidal functor

$$GNS : \mathbf{Phys}^{op} \longrightarrow *\mathbf{Mod}$$

Proof.

Things exist by initiality. Diagrams commute by cyclicity. $\square$

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It's going the wrong way!

The Covariant GNS Functor

Physically Correct Direction

$$
\begin{tikzcd}
\mathbf{Phys} \arrow[r, "GNS^{op}"] \arrow[rr, bend right, "GNS_c"'] & *\mathbf{Mod}^{op} \arrow[r, "\text{adjoint}"] & *\mathbf{Mod}_{adj}
\end{tikzcd}
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Definition

- $*\mathbf{Mod}_{adj}$ is $*$-modules with adjoint homomorphisms
- Adjoint homomorphisms: coisometries h such that

$$ah(v) = h(f(a)v)$$

Part II
Physics From a Functor

The Schrödinger Picture – Example Factory

1. H – faithful A-module
2. $U : H \longrightarrow H'$ – isometric linear map
3. $f : A \longrightarrow B = UAU^*$ – algebra map given by $a \longmapsto UaU^*$

Theorem (Lifting Schrödinger)

For any $\psi \in H$ we have $f : U\psi \longrightarrow \psi \in \mathbf{Phys}$, and

$$\begin{tikzcd} GNS(\psi) \arrow[r, "GNS(f)"] \arrow[d, hook] & GNS(U\psi) \arrow[d, hook] \\ H \arrow[r, "U"] & H' \end{tikzcd}$$

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Corollary

If U is unitary, then $g(a) = U^*aU$ gives $g : \psi \longrightarrow U\psi \in \mathbf{Phys}$, and

$$\begin{tikzcd}
GNS(\psi) \arrow[r, "GNS_c(g)"] \arrow[d, hook] & GNS(U\psi) \arrow[d, hook] \\
H \arrow[r, "U"] & H'
\end{tikzcd}$$

Symmetries and Unitary Representations

Why does a G-equivariant state give a unitary representation of G?

$$G \longrightarrow \mathbf{Phys} \xrightarrow{GNS_c} *\mathbf{Mod}_{adj}$$

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Why does a G-equivariant state give a unitary representation of G?

Because of composition!

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{Unitary representation of } G!"'] & \mathbf{Phys} \arrow[r, "GNS_c"] & *\mathbf{Mod}_{adj} \end{tikzcd}$$

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Bonus items:

- Groupoids of symmetries
- Equivariant GNS:

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$$\mathbf{Phys}^G \xrightarrow{GNS_c^G} *\mathbf{Mod}_{adj}^G \xrightarrow{U} \mathbf{Rep}(G)$$

- Compatibility with composite systems:

$$\varphi \otimes \psi \text{ has symmetry } G \times G'$$

Relation to Probability Theory

$\mathbf{Prob}$ – compact probability spaces. From (X, μ) we construct:

- A state on $C(X)$ – the expectation value $\mathbb{E}_\mu(a) = \int_X a \, d\mu$
- $L^2(\mu)$, a $C(X)$-module

Theorem

The following diagram of symmetric monoidal functors commutes

$$\begin{tikzcd} & \mathbf{Prob}^{op} \arrow[dl, "C^{op}"'] \arrow[dr, "L^2"] & \\ \mathbf{Phys}^{op} \arrow[rr, "GNS"] & & *\mathbf{Mod} \end{tikzcd}$$

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Proof.

1. $L^2(\mu)$ is cyclic
2. $1 \in L^2(\mu)$ represents the expectation value $\mathbb{E}_\mu$ $\square$

Application: Eigenvalue-Eigenvector Link

Any normal $a \in \mathcal{O}(\varphi)$ determines a probability space

$$P_{\varphi}(a) = (Spec(\langle a \rangle), \varphi|_{\langle a \rangle})$$

Theorem (Eigenvalue-Eigenvector Link)

The following are equivalent:

1. $a\Omega = \lambda\Omega$
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Proof.

The inclusion $\langle a \rangle \subseteq \mathcal{O}(\varphi)$ gives a map $R : \varphi \longrightarrow P_\varphi(a) \in \mathbf{Phys}$
Previous theorem computes $GNS(R)$:

$$L^2(\varphi|_{\langle a \rangle}) \longrightarrow GNS(\varphi)$$

Thus: $a\Omega = \lambda\Omega \iff a \cdot 1 = \lambda \cdot 1$ in $L^2 \iff a = \lambda$ a.e. $\square$

Classical Markov Processes

Definition (Markov Processes)

- $M(X) =$ probability measures on X
- Markov process $X \longrightarrow Y$ = map $X \longrightarrow M(Y)$
- Category of Markov processes $= Kleisli(M)$

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Theorem (Generalized Gelfand Duality; Furber & Jacobs 2015)

Gelfand duality extends to a contravariant equivalence between Markov processes and completely positive unital maps between C^*-algebras

Quantum Markov Processes

Theorem (Non-Unitary GNS Representation)

There is a commuting prism of symmetric monoidal functors:

$$
\begin{tikzcd}
 & \mathbf{Prob}^{op} \arrow[dl, "C^{op}"'] \arrow[drr, "L^2"] \arrow[dd, dashed] & & \\
\mathbf{Phys}^{op} \arrow[rrr, "GNS"] \arrow[dd] & & & *\mathbf{Mod} \arrow[dd, "U"] \\
 & \mathbf{Prob}_M^{op} \arrow[dl, "C^{op}"'] \arrow[drr, "L^2"] & & \\
\mathbf{Phys}_M^{op} \arrow[rrr, "GNS_M"] & & & \mathbf{Hilb}
\end{tikzcd}
$$

Example: State Vector Collapse

- $P \in A$ – self-adjoint projection (i.e. idempotent)
- $\Phi : A \longrightarrow A$ given by $a \longmapsto PaP$

Theorem

- *φ represented by $\Omega \Longrightarrow \Phi^*\varphi$ represented by $P\Omega$*
- *$GNS_M(\Phi)$ is the composite*

$$GNS(\Phi^*\varphi) \hookrightarrow GNS(\varphi) \xrightarrow{P} GNS(\varphi)$$

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Corollary

$GNS_{M,c}(\Phi)$ is cyclic ($\Omega \longmapsto P\Omega$), and acts as

$$GNS(\varphi) \xrightarrow{P} GNS(\varphi) \xrightarrow{\text{orth. proj.}} GNS(\Phi^*\varphi)$$

Example: Particle Scattering

- H – Hilbert space
- $S : \mathcal{F}(H) \longrightarrow \mathcal{F}(H)$ – unitary scattering matrix
- $H_\alpha, H_\beta \subseteq \mathcal{F}(H)$ – particles of type α and β

Proposition

There is a process $S_{\alpha\beta} : \alpha \longrightarrow \beta \in \mathbf{Phys}_M$ such that

$$\begin{tikzcd}
H_\alpha \arrow[r, hook, "\text{inclusion}"] \arrow[rrr, bend right, "GNS_{M,c}(S_{\alpha\beta})"'] & \mathcal{F}(H) \arrow[r, "S"] & \mathcal{F}(H) \arrow[r, two heads, "\text{projection}"] & H_\beta
\end{tikzcd}$$

If you believe in QED:

$$\gamma + \gamma \longrightarrow e^- + e^+$$

Part III
The Frontier

Internalizing in a Topos

- GNS in a topos E = monoidal morphism in $\mathbf{Stacks}(E)$
- Which definition? All equivalent in $\mathbf{Set}$!
- In models of synthetic differential geometry:
  - Infinitesimal processes, like symmetries (Heisenberg $\Longleftrightarrow$ Schrödinger)
  - Deformation quantization = Infinitesimal $\hbar$-families
- Few examples – must use C^∞-$*$-algebras (work in progress)

Questions for the Future

What is ...

- ...a gauge theory? No spacetime!
- ...extended locality in general?
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Target Theorem (Witten)

The $\hbar$-family of vacua of super Yang-Mills theory is trivial.
(in four dimensions, $N = 2$)

Thank You!