# Operational **P**hysics, **L**ogic **a**nd **C**ategories

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**C**ategories, **L**ogic **a**nd **P**hysics
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# The Plan

Operational Physics

Operational Logic

Operational Categories

- ... in partial form
- ... in total form
- Effectus Theory

Outlook

# I. Operational Physics

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- Operational reconstructions of quantum theory due to Hardy, Chiribella-D'Ariano-Perinotti, ...
- Categorical quantum mechanics: models quantum theory using certain symmetric monoidal categories.
- CDP's operational-probabilistic theories: categorical + probabilistic.

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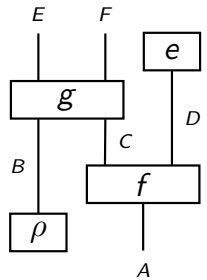

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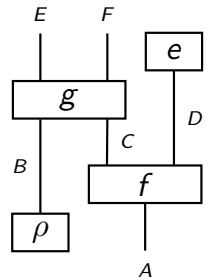


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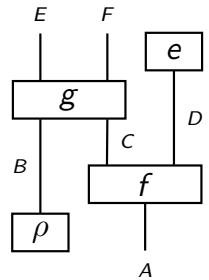


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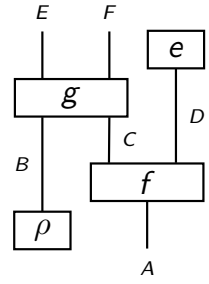


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- Coarse-graining:
  $\{f_x : A \to B\}_{x \in X} \cup \{g_y\}_{y \in Y} \implies \{\bigcirc\!\!\!\!\vee_{x \in X} f_x\} \cup \{g_y\}_{y \in Y}$
  Partial 'addition' $f \bigcirc\!\!\!\!\vee g : A \to B$ on events.

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## Examples

Many! Classical: deterministic or probabilistic.
Quantum: Hilbert spaces/C*-algebras and CP, sub-unital maps.

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Categorically, direct sums are finite coproducts:

$$B_i \xrightarrow{\kappa_i} B_1 + \ldots + B_n = \bigoplus_{k=1}^{n} B_k \xrightarrow{\triangleright_j} B_j\ , \quad \triangleright_j \circ \kappa_i = \begin{cases} \mathrm{id} & i = j \\ 0 & i \neq j \end{cases}$$

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- $\otimes, +$ distribute: $A \otimes (B + C) \simeq A \otimes B + A \otimes C$
- $\maltese_{A+B} = [\maltese_A, \maltese_B]$, $\maltese_I = \mathrm{id}$, $\maltese_{A \otimes B} = \lambda \circ (\maltese_A \otimes \maltese_B)$

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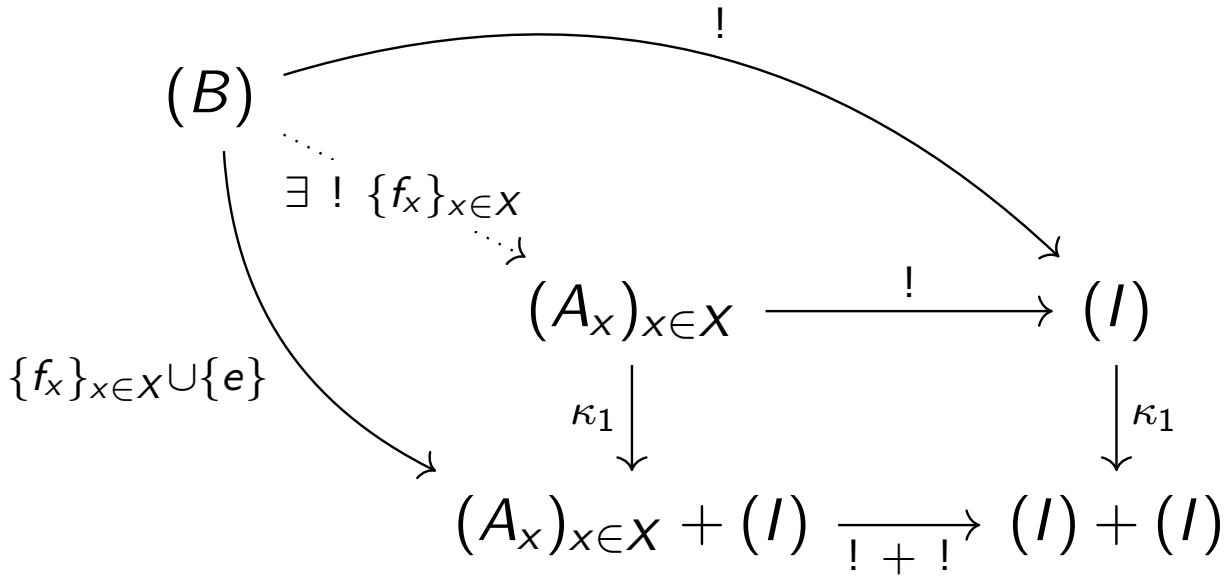

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## Examples

Classical: **Set**. Probabilistic: $\mathsf{Kl}(\mathcal{D})$ for distribution monad $\mathcal{D}$.
Quantum: $\mathbf{CStar}^{\mathrm{op}}_{\mathrm{cpu}}$

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We've seen operational categories in partial form $(\mathbf{C}, \fatslash) = \mathbf{ParTest}(\Theta)$ and total form $\mathbf{B} = \mathbf{Test}(\Theta)$.
$\mathbf{B} = \mathbf{C}_{\text{total}}$, subcategory of $f$ with $\fatslash \circ f = \fatslash$, in fact determines all of $\mathbf{C}$:

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## Examples

| | $\mathbf{B}$ | $\mathrm{Par}(\mathbf{B})$ |
|---|---|---|
| Classical | **Set** | **PFun** |
| Probabilistic | $\mathrm{Kl}(\mathcal{D})$ | $\mathrm{Kl}(\mathcal{D}_{\text{sub}})$ |
| Quantum | $\mathbf{CStar}^{\mathrm{op}}_{\text{cpu}}$ | $\mathbf{CStar}^{\mathrm{op}}_{\text{cpsu}}$ |

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The following structures are equivalent:

- an operational theory with control $\Theta$ with direct sums;
- an operational category in partial form $(\mathbf{C}, \dot{+})$;
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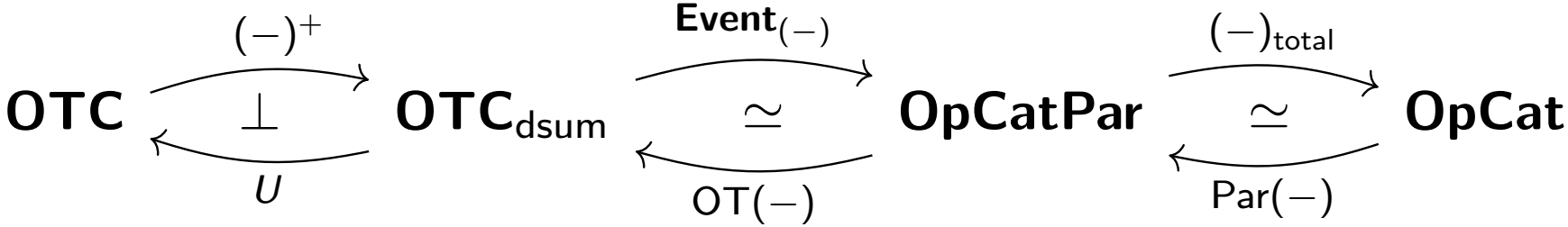

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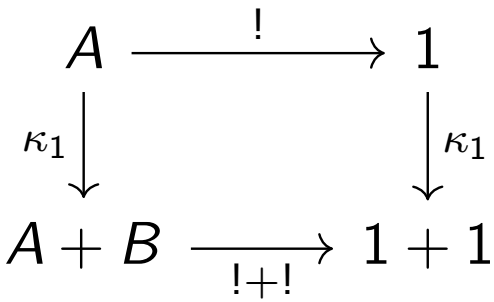


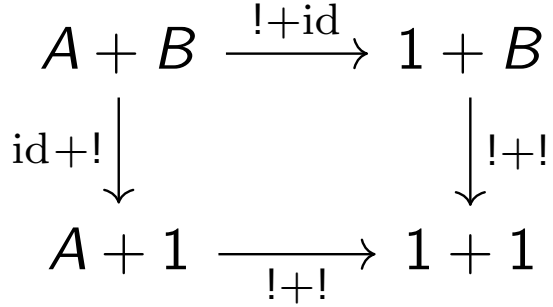

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$\rightsquigarrow$ Each space of effects $e\colon A \to I$ forms an effect algebra.

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Thanks!