

Indefinite causal structures using diagrammatic methods

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- 1 Diagrams and processes
- 2 Causal processes
- 3 Probabilities
- 4 Second order causality
- 5 W processmatrix
- 6 Diagrammatic W

Setting

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- For every object A there is a cup $\eta_A : I \rightarrow A \otimes A$
- and a cap $\eta_A : A \otimes A \rightarrow I$

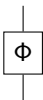
Setting


- We work in a self-dual compact closed category
- For every object A there is a cup $\eta_A : I \rightarrow A \otimes A$
- and a cap $\eta_A : A \otimes A \rightarrow I$ which satisfy

Three equations illustrating the simplification of string diagrams:

- Equation 1: A diagram with a vertical line on the left, a cup shape on the right, and a vertical line on the far right, connected by a curved line, is equal to a single vertical line.
- Equation 2: A diagram with a vertical line on the left, a cap shape on the right, and a vertical line on the far right, connected by a curved line, is equal to a single vertical line.
- Equation 3: A diagram with a vertical line on the left, a loop shape on the right, and a vertical line on the far right, connected by a curved line, is equal to a single vertical line.


- *Processes* are morphisms



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- We have special morphisms from and to the trivial system I

$$\text{states} := \begin{array}{c} \downarrow \\ \psi \\ \triangle \end{array}$$

$$\text{effects} := \begin{array}{c} \triangle \\ \pi \\ \uparrow \end{array}$$

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- Want a special effect *discard*

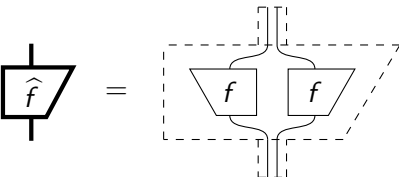
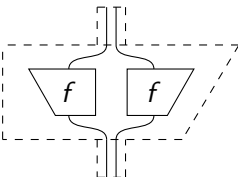
$$d_A := \begin{array}{c} \text{---} \\ \text{---} \\ \uparrow \end{array}$$

compatible with monoidal structure

- Work in **FHilb**

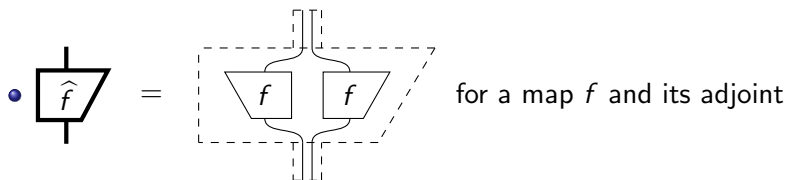
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•  =  for a map f and its adjoint

The diagram on the left shows a square box with a trapezoidal shape inside, containing the symbol \hat{f} . It has a vertical line extending upwards from the top and a vertical line extending downwards from the bottom. The diagram on the right shows a dashed trapezoidal box containing two trapezoidal boxes, each labeled f . The top and bottom of these two boxes are connected by vertical lines, and the left and right sides are connected by curved lines, forming a larger trapezoidal shape within the dashed box.

- Work in **FHilb**
- $H \otimes H \cong B(H)$



- corresponds to pure map via $\Phi_f(\rho) = f\rho f^\dagger$

spiders

$$\begin{array}{c}
 \overbrace{\quad\quad\quad}^n \\
 \begin{array}{c} \diagup \quad \dots \quad \diagdown \\ \circ \\ \diagdown \quad \dots \quad \diagup \end{array} \\
 \underbrace{\quad\quad\quad}_m
 \end{array}
 = \sum_j \underbrace{\langle j \dots j |}_n \underbrace{|j \dots j \rangle}_m$$


spiders





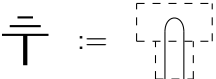
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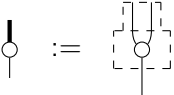
- Spiders fuse

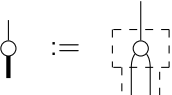
$$\begin{array}{c}
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

- Discard: $\overline{\top}$ $:=$ 


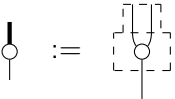

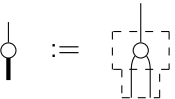

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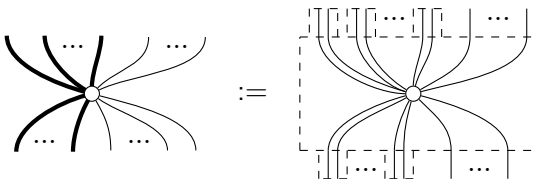
• Encode: 

• Measure: 

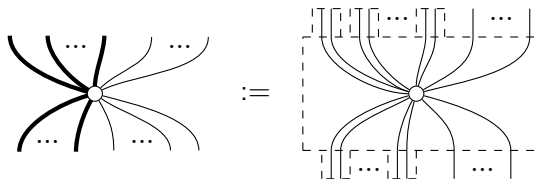
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- This gives a subcategory of **FHilb**
- “*doubling = quantum*”

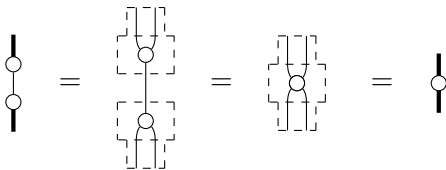
- Spiders with both classical and quantum wires are *bastards*



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- Decoherence: first measure, then encode



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causality

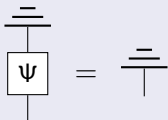
Definition

A process $\Psi : A \rightarrow B$ is called causal if $d_B \circ \Psi = d_A$. That is:

causality

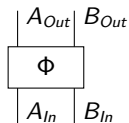
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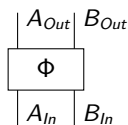
Signaling

- Imagine two observers Alice and Bob acting on a system

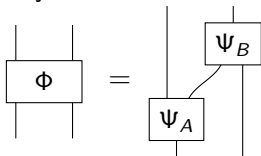


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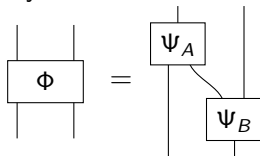
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- Say $A \preceq B$ if

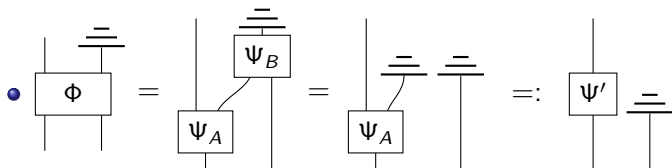


- Say $B \preceq A$ if

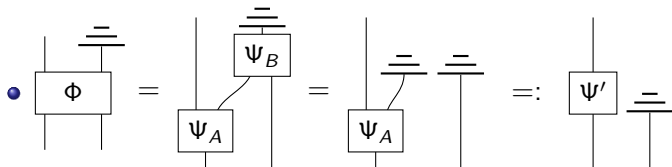


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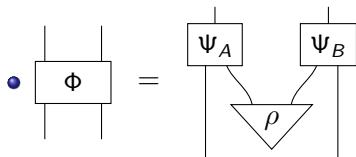
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- *No signaling* if Φ admits both factorisations $A \preceq B$ and $B \preceq A$

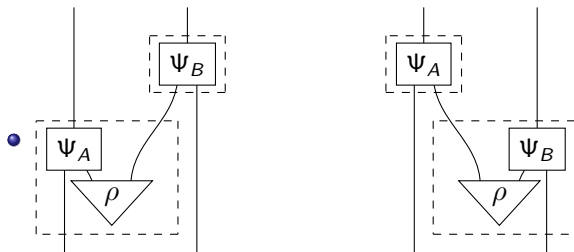
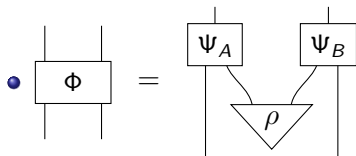
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- Alice and Bob get input x, y

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- Similar: $r^{B \preceq A}$
- Question: is p of the form $p = \lambda q^{A \preceq B} + (1 - \lambda)r^{B \preceq A}$?

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- Alice and Bob have a lab where they act on a system which enters once

$$\begin{array}{ll} \text{Processes} & \eta_{a|x}^A : B(H_{A_I}) \rightarrow B(H_{A_O}) \\ \text{and} & \xi_{b|y}^B : B(H_{B_I}) \rightarrow B(H_{B_O}) \end{array}$$

Choi-Jamiołkowski isomorphism

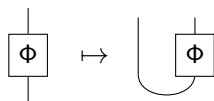
- $\eta \rightarrow \hat{\eta} \in B(H_{A_I} \otimes H_{A_O}), \quad \xi \rightarrow \hat{\xi} \in B(H_{B_I} \otimes H_{B_O})$

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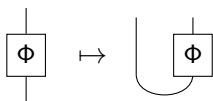
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- Process-State duality

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for all cptp $\hat{\eta}_{cptp}, \hat{\xi}_{cptp}$

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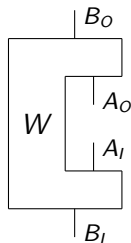
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- Diagrammatically:



- Quantum combs

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Second order causality

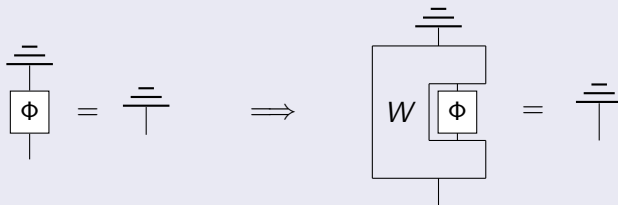
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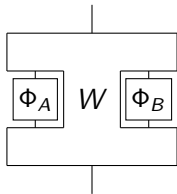
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- Consider $W : (A_I \otimes A_O) \otimes (B_I \otimes B_O) \rightarrow C_I \otimes C_O$

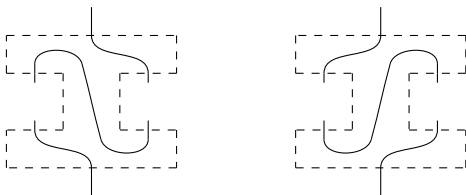
- Consider $W : (A_I \otimes A_O) \otimes (B_I \otimes B_O) \rightarrow C_I \otimes C_O$
- W is *bipartite second order causal* (SOC_2) if



is causal for all causal Φ_A and Φ_B

- Example: connect Alice to Bob or vice versa

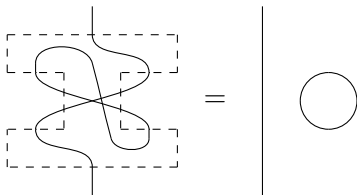
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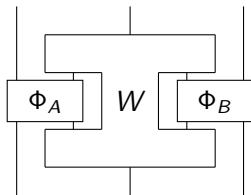
- Fixed causal order

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- Example: swap



- What if A and B have ancilla systems



proof

- Enough causal states if

$$\left(\forall \rho \text{ causal} \cdot \begin{array}{c} \square \Phi \\ \downarrow \\ \triangle \rho \end{array} = \begin{array}{c} \square \Phi' \\ \downarrow \\ \triangle \rho \end{array} \right) \implies \Phi = \Phi'$$

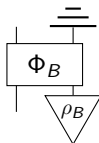
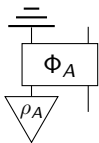
proof

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$$\left(\forall \rho \text{ causal} \cdot \begin{array}{c} \text{---} \\ | \\ \boxed{\Phi} \\ | \\ \nabla \rho \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{\Phi'} \\ | \\ \nabla \rho \end{array} \right) \implies \Phi = \Phi'$$

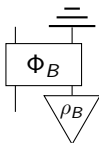
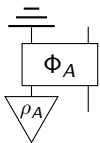
- This is the case for operator algebras with cp maps

- For causal states ρ_A, ρ_B



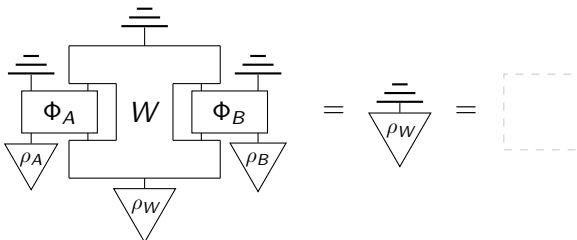
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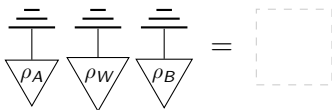


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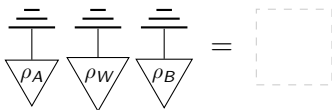
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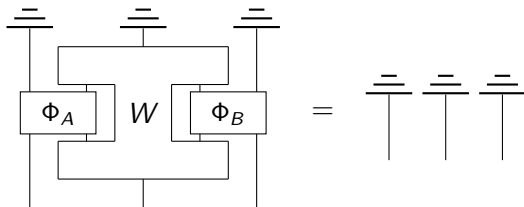
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- Write W in Pauli basis

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$$W = \sum_{i,j,k,l} \lambda_{i,j,k,l} \begin{array}{c} \triangle i \quad \triangle j \\ | \quad | \\ \triangle k \quad \triangle l \end{array}$$

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where ${}_X W = \frac{\mathbf{1}^X}{d_X} \otimes tr_X W$

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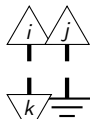
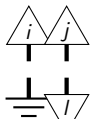
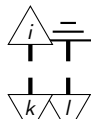
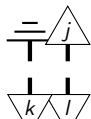
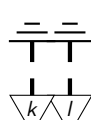
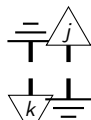
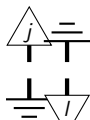
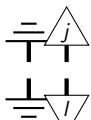
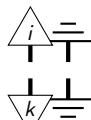
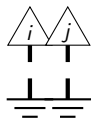
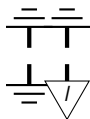
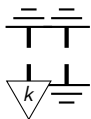
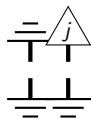
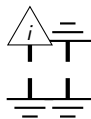
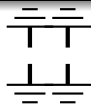
$$Tr(W) = d_{A_0} d_{B_0}$$

$$B_I B_O W = A_O B_I B_O W \quad A_I A_O W = B_O A_I A_O W$$

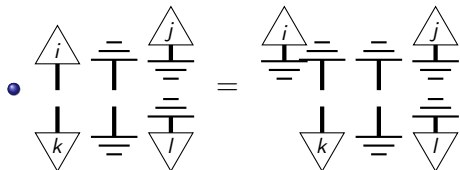
$$W = B_O W + A_O W - A_O B_O W$$

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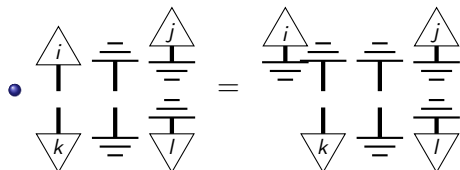
- Consider all possible terms in W



- $B_1 B_0 W = A_0 B_1 B_0 W$

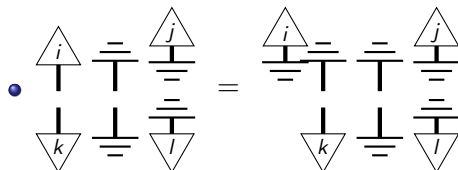


- $B_1 B_0 W = A_0 B_1 B_0 W$



- $j = l = 0 \Rightarrow i = 0$

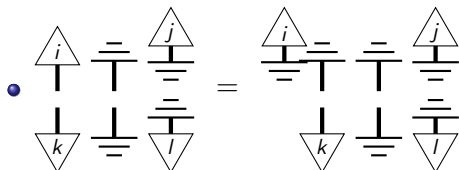
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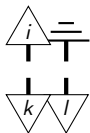
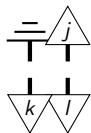
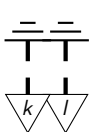
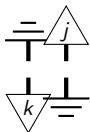
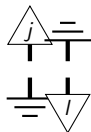
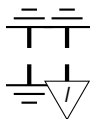
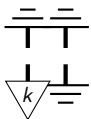
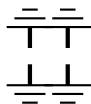


- $j = l = 0 \Rightarrow i = 0$

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- $W = B_0 W + A_0 W - A_0 B_0 W$

- At least one of i, j equals 0 (trace)



- $Tr(W) = d_{A_0} d_{B_0} = 4 \Rightarrow \lambda_{0,0,0,0} = \frac{1}{4}$

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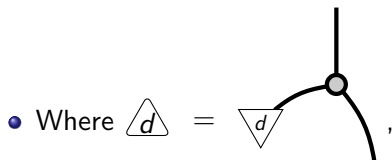
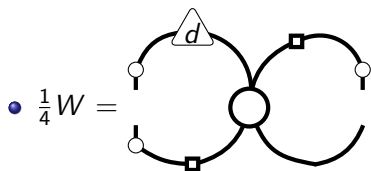
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 - Example: $W = \frac{1}{4} [1111 + \frac{1}{\sqrt{2}}(1ZZ1 + Z1XZ)]$
 - Can show this W breaks causal inequality bound for Guess Your Neighbours Input game
- We obtain probabilities incompatible with a fixed causal order

- 1 Diagrams and processes
- 2 Causal processes
- 3 Probabilities
- 4 Second order causality
- 5 W processmatrix
- 6 Diagrammatic W**

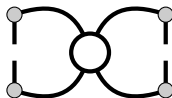
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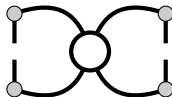


d the state corresponding to $\cos(\frac{\pi}{8}) |0\rangle + \sin(\frac{\pi}{8}) |1\rangle$
and gray spiders correspond to X -basis: $|+\rangle, |-\rangle$

- Consider very simple W

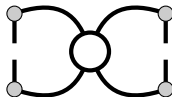


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Open questions

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