Indefinite causal structures using diagrammatic methods

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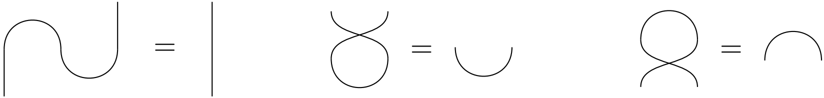
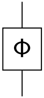
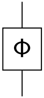


- *Processes* are morphisms 

- *Processes* are morphisms 

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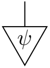
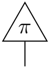
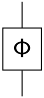
states := 

effects := 

- *Processes* are morphisms 
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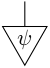
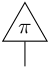
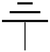
states :=    *effects* := 

- Want a special effect *discard*

d_A := 

compatible with monoidal structure

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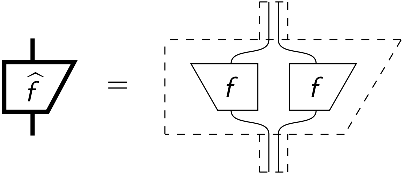
- 

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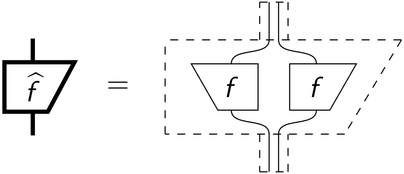
- 

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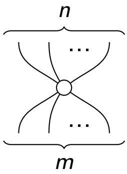
- corresponds to pure map via $\Phi_f(\rho) = f\rho f^\dagger$

spiders

- 

$$= \sum_j \underbrace{|j \ldots j\rangle}_{n} \underbrace{\langle j \ldots j|}_{m}$$

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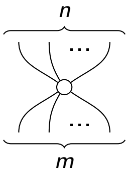
- 

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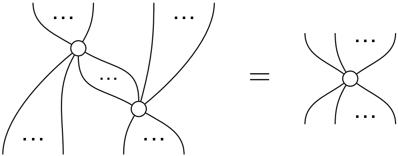
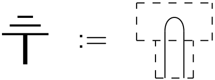
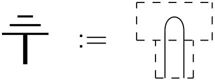
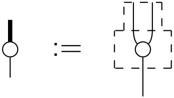
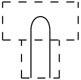
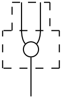
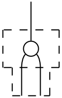
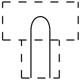
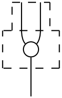
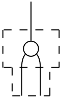
- Spiders fuse 

- Discard: 

- Discard: 
- Encode: 

- Discard:  $:=$ 
- Encode:  $:=$ 
- Measure:  $:=$ 

- Discard:  := 
- Encode:  := 
- Measure:  := 
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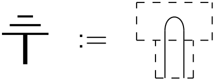
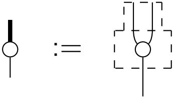
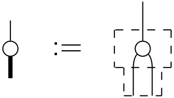
- Discard: 

- Encode: 

- Measure: 

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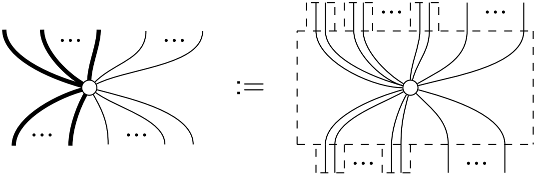
- "*doubling* = *quantum*"

- Spiders with both classical and quantum wires are *bastards*



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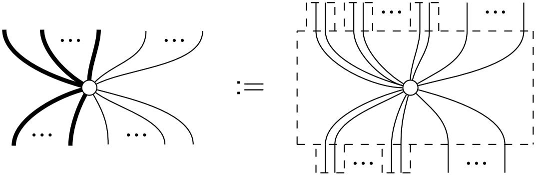
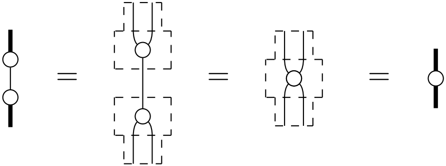


- Decoherence: first measure, then encode



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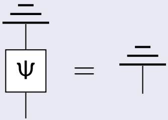

Signaling

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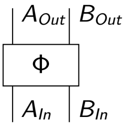

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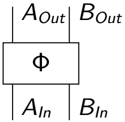
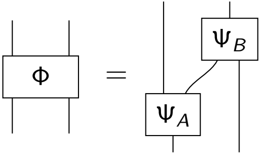


- Say $A \preceq B$ if



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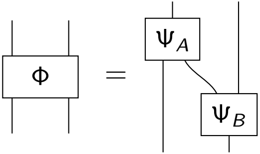

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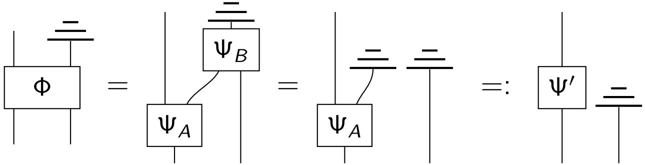
- 

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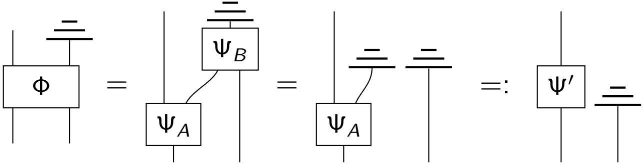
- 

- *No signaling* if Φ admits both factorisations $A \preceq B$ and $B \preceq A$

Example

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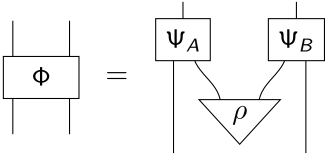
- 

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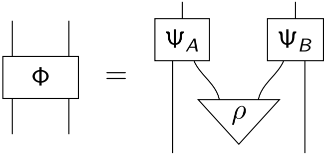
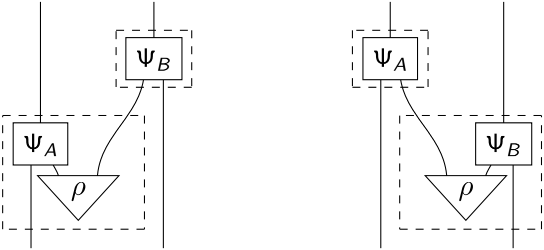
- 

- 

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$$\text{Processes} \qquad \eta^{A}_{a|x} : B(H_{A_I}) \to B(H_{A_O})$$

$$\text{and} \qquad \xi^{B}_{b|y} : B(H_{B_I}) \to (H_{B_O})$$

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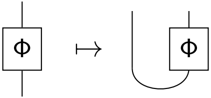

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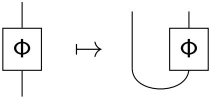

- Process-State duality

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for all cptp $\hat{\eta}_{cptp}, \hat{\xi}_{cptp}$

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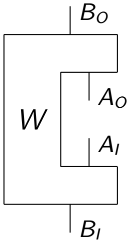

- Quantum combs

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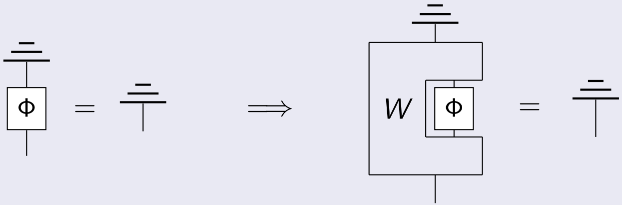

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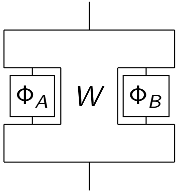

is causal for all causal Φ_A and Φ_B

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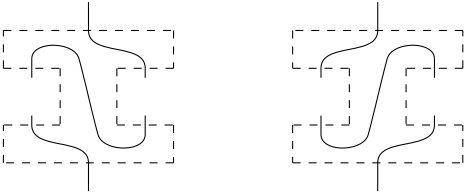
- 

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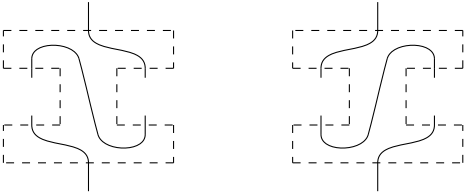
- 
- Fixed causal order

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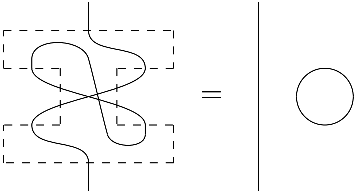
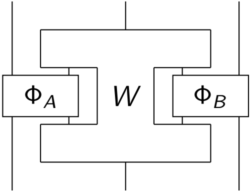
- Example: swap



- What if A and B have ancilla systems



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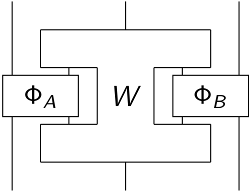
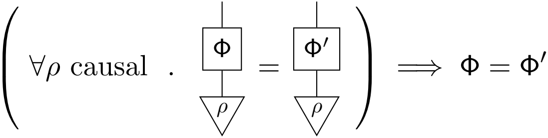


Theorem

$$SOC_2 \Rightarrow Completely\, SOC_2$$

proof

- Enough causal states if



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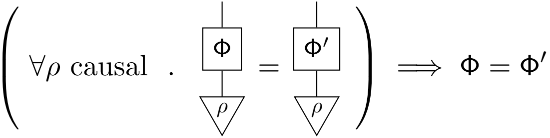

- This is the case for operator algebras with cp maps

- For causal states ρ_A, ρ_B

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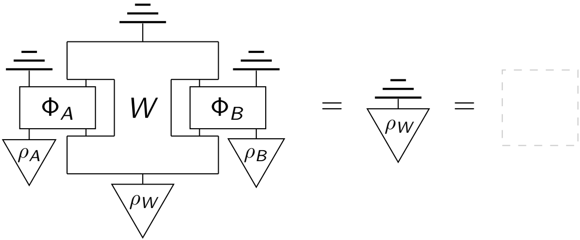
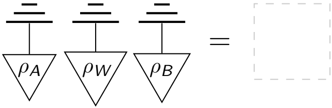
- So



- but also



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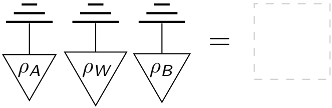
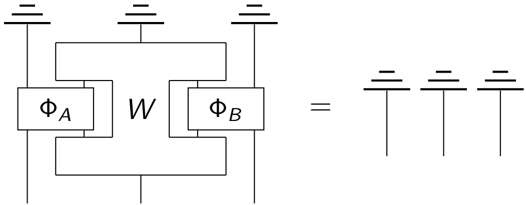


- Hence



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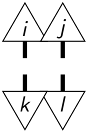

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$$W = {}_{B_O}W + {}_{A_O}W - {}_{A_O B_O}W$$

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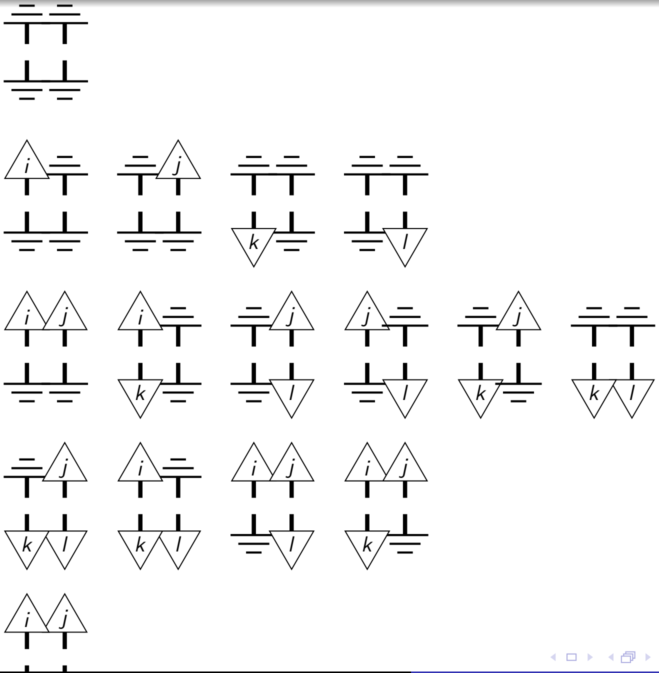
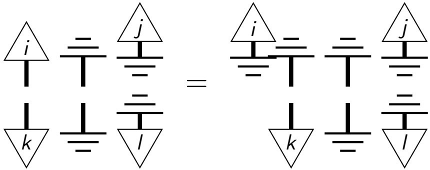
- Consider all possible terms in W


i
j
k
l

- ${}_{B_I B_O}W = {}_{A_O B_I B_O}W$
- 

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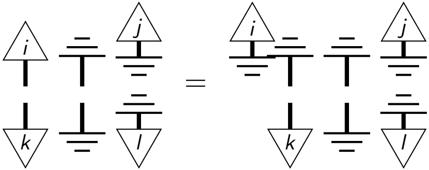
- 

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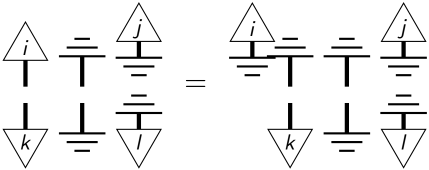
- 

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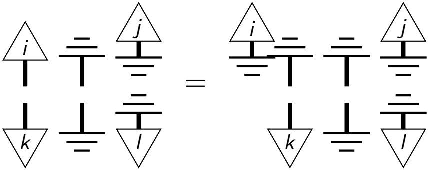
- 

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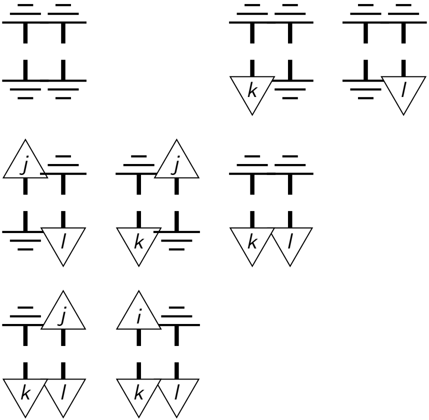
- At least one of i,j equals 0 (trace)


k
l
j
l
j
k
k
l
j
k
l
i
k
l

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- $W \geq 0$ restricts to positive cone
- Example: $W = \frac{1}{4}\left[1111 + \frac{1}{\sqrt{2}}(1ZZ1 + Z1XZ)\right]$
- Can show this W breaks causal inequality bound for Guess Your Neighbours Input game
  We obtain probabilities incompatible with a fixed causal order

- Consider again $W = \frac{1}{4}\left[1111 + \frac{1}{\sqrt{2}}(1ZZ1 + Z1XZ)\right]$

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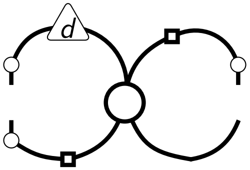
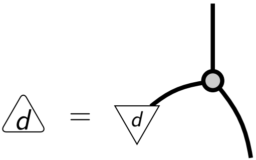
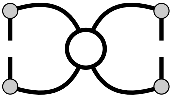
- $\frac{1}{4}W =$ 
- Where 

,

  d the state corresponding to $\cos(\frac{\pi}{8})\,|0\rangle + \sin(\frac{\pi}{8})\,|1\rangle$ and gray spiders correspond to X-basis: $|+\rangle\,, |-\rangle$

- Consider very simpel W



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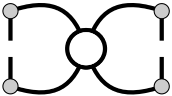

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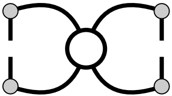

- Not valid due to constrains on Pauli matrices (one term fails)
- d kills precisely this Pauli term

Open questions

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