Grading monads, comonads, distributive laws

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building on recent works of
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What is this?

- We organize (co)effectful computations with monads, idioms (lax monoidal endofunctors), comonads, arrows, relative monads etc.
- Often it is useful to track the “degree” of effectfulness, e.g., for ensuring safety (honoring of given resource usage bounds) or optimizations.
- Enter grading of monads, idioms etc.
- This is revisiting the old idea of effect systems and in particular of the marriage of monads and effects (with effect inference and all that).
- This time we are guided by a mathematically driven foundation.
Married monads and effects:
Graded monadic metalanguage

\[
\begin{align*}
\Gamma & \vdash t : A \\
\Gamma & \vdash \text{ret } t : T \text{i } A \\
\Gamma & \vdash t : T \text{e } A \\
\Gamma, x : A & \vdash u : T \text{f } B \\
\Gamma & \vdash \text{case } t \text{ of (ret } x) \rightarrow u : T (e \ast f) B \\
\Gamma & \vdash t : T \text{e } B \\
\Gamma & \vdash e \leq e' \\
\Gamma & \vdash t : T e' B
\end{align*}
\]
Outline

- Graded monads
  - Kleisli and Eilenberg-Moore categories for graded monads
- Graded “monads of monoids” (MonadPlus instances)
- Graded distributive laws
- Graded comonads
Graded monads

- Given a monoid \((E, i, \ast)\). A graded monad on a category \(\mathbb{C}\) is
  - for any \(e : E\), a functor \(T e : \mathbb{C} \to \mathbb{C}\)
  - a nat. transf. \(\eta : \text{Id} \to T i\)
  - for any \(e, f : E\), a nat. transf. \(\mu^{e,f} : T e \cdot T f \to T (e \ast f)\)

such that

\[
\begin{align*}
\text{Id} \cdot T e & \xrightarrow{T e \eta} T i \cdot T e && T e \cdot \text{Id} \xrightarrow{T e \eta} T e \cdot T i \\
T e & \xrightarrow{T (i \ast e)} T (i \ast e) && T e & \xrightarrow{T (e \ast i)} T (e \ast i) \\
T e \cdot (T f \cdot T g) & \xrightarrow{\mu^{e,f} \cdot T g} (T e \cdot T f) \cdot T g && (T e \cdot T f) \cdot T g & \xrightarrow{T e \cdot T (f \ast g)} T (e \ast (f \ast g)) \xrightarrow{T ((e \ast f) \ast g)} T ((e \ast f) \ast g)
\end{align*}
\]

- In short, a graded monad on \(\mathbb{C}\) is a lax monoidal functor from \((E, i, \ast)\) as a discrete monoidal category to \(([\mathbb{C}, \mathbb{C}], \text{Id}, \cdot)\).
Graded monads ctd

- A very useful generalization is to generalize from a monoid to a pomonoid $((E, \leq), i, \ast)$, i.e., a set $E$ with a partial order $\leq$ and a monoid structure $(i, \ast)$ such that $\ast$ is monotone wrt. $\leq$.

- NB! $i$ need not be the least element (nor the greatest).

- Then a graded monad has also
  - for any $e \leq e'$, a natural transformation $T(e \leq e') : T e \to T e'$ such that

$$T(e \leq e) = \text{id}_{T e}$$
$$T(e' \leq e'') \circ T(e \leq e') = T(e \leq e' \leq e'')$$

$$T e \cdot T f \xrightarrow{\mu_{e,f}} T(e \ast f)$$
$$T(e \leq e') \cdot T(f \leq f') \downarrow \quad \downarrow T(e \ast f \leq e' \ast f')$$

$$T e' \cdot T f' \xrightarrow{\mu_{e',f'}} T(e' \ast f')$$

- Again, a graded monad is a lax monoidal functor, this time from a thin monoidal category.

- One can also grade with a general monoidal category.
Example: Graded maybe

\[ E = \{\text{pure}, \text{fail}, \text{mf}\} \]

\[ i = \text{pure} \]

\[
\begin{array}{c|ccc}
  \ast & \text{pure} & \text{fail} & \text{mf} \\
  \hline
  \text{pure} & \text{pure} & \text{fail} & \text{mf} \\
  \text{fail} & \text{fail} & \text{fail} & \text{fail} \\
  \text{mf} & \text{mf} & \text{fail} & \text{mf} \\
\end{array}
\]

\[
T \text{ pure } X = X \\
T \text{ fail } X = 1 \\
T \text{ mf } X = X + 1 \\
T (\text{pure} \leq \text{mf}) X = X \xrightarrow{\text{inl}} X + 1 \\
T (\text{fail} \leq \text{mf}) X = 1 \xrightarrow{\text{inr}} X + 1
\]
Example: Graded state

Given a set $S$ of states.

$$E = \{\text{pure, ro, wo}^+, \text{wo, rw}\}$$

$$T_{\text{pure}} \quad X = \quad X$$
$$T_{\text{ro}} \quad X = \quad S \Rightarrow X$$
$$T_{\text{wo}^+} \quad X = \quad S \times X$$
$$T_{\text{wo}} \quad X = \quad (S + 1) \times X \cong (S \times X) + X$$
$$T_{\text{rw}} \quad X = \quad S \Rightarrow S \times X$$
Example: Graded writer

Given an alphabet $\Sigma$.

Option 1:

$$(E, \leq, i, \ast) = (\mathbb{N}, \leq, 0, +)$$

$$T n X = \Sigma^{\leq n} \times X$$

Option 2:

$$(E, \leq, i, \ast) = (\mathcal{P}\Sigma^*, \subseteq, [], ++)$$

(or we could use any class of languages closed under $[]$ and $++$)

where

$[] = \{[]\}$

$L ++ L' = \{w ++ w' | w \in L, w' \in L'\}$

$$T L X = L \times X$$
Kleisli category of a graded monad

- Given a pomonoid \((E, i, \ast)\) and a graded monad \(T\) on \(\mathbb{C}\).
- An object of the *Kleisli category* of \(T\) is given by an element \(e\) of \(E\) and object \(X\) of \(\mathbb{C}\).
- A map between \((e, X), (e', Y)\) is given by an element \(f\) of \(E\) such that \(e \ast f \leq e'\) and a map \(k : X \to T f Y\), modulo the equivalence relation \(\sim\) given by the rule

\[
(f, e \ast f \leq e \ast f' \leq e', k) \sim (f', e \ast f' \leq e', T (f \leq f') \circ k)
\]
Eilenberg-Moore category of a graded monad

- An object of the *E-M category* of $T$ (an algebra) is given by
  - for any $e : E$, an object $A e$
  - for any $e \leq e'$, a map $A(e \leq e') : A e \to A e'$,
  - for any $e, f : E$, a map $a^{e,f} : T e (A f) \to A(e * f)$

such that

\[
\begin{array}{ccc}
T e (A f) & \xrightarrow{a^{e,f}} & A(e * f) \\
\downarrow T (e \leq e') & & \downarrow A(e \leq e' * f') \\
T e' (A f') & \xrightarrow{a^{e',f'}} & A(e' * f')
\end{array}
\begin{array}{ccc}
A e & \xrightarrow{\eta_{A e}} & T i(A e) \\
\downarrow T (e \leq e') & & \downarrow a^{i,e} \\
A e & \xrightarrow{\mu_{A g}} & T (e * f) (A g)
\end{array}
\begin{array}{ccc}
T e (T f (A g)) & \xrightarrow{\mu_{A g}} & T (e * f) (A g) \\
\downarrow T (e \leq e') & & \downarrow a^{e,f,g} \\
T e (A (f * g)) & \xrightarrow{a^{e,f,g}} & A(e * f * g)
\end{array}
\]

- A morphism (algebra map) between $(A, a), (B, b)$ is given by
  - for any $e : E$, a map $h^e : A e \to B e$

such that

\[
\begin{array}{ccc}
A e & \xrightarrow{h^e} & B e \\
\downarrow A(e \leq e') & & \downarrow B(e \leq e') \\
A e' & \xrightarrow{h^{e'}} & B e'
\end{array}
\begin{array}{ccc}
T e (A f) & \xrightarrow{a^{e,f}} & A(e * f) \\
\downarrow T e h^f & & \downarrow h^{e*f} \\
T e (B f) & \xrightarrow{b^{e,f}} & B(e * f)
\end{array}
\]
Resolutions of graded monads

- A resolution of $T$ is given by
  - a category $\mathcal{D}$,
  - a strict monoidal functor $A : (\mathcal{E}, i, \ast) \rightarrow ([\mathcal{C}, \mathcal{C}], \text{Id}, \cdot)$,
  - adjoint functors $L, R$ between $\mathcal{C}$ and $\mathcal{D}$

such that

- $T e = R \cdot A e \cdot L$
- (appropriate conditions on $\eta, \mu$)

- The Kleisli category is the initial resolution, the E-M category is the final resolution.
Graded monads of monoids (MonadPlus instances)

- Given a *right near-semiring* \((E, i, *, o, +)\), i.e., a set \(E\) with two monoid structures \((i, *)\), \((o, +)\), with \(*\) distributing over \(o\) and \(+\) from the right. (Left distributivity and commutativity of \(+\) are not required.)
- A *graded monad of monoids* on a Cartesian category \((C, 1, \times)\) is a \((E, i, *)\)-graded monad on \(C\) with
  - a nat. transf. \(mze : 1 \to T o\)
  - a nat. transf. \(mpl^{e,f} : T e \times T f \to T (e + f)\)
such that

\[
\begin{align*}
1 \times T e & \xrightarrow{mze \times T e} T o \times T e \\
T e & \xrightarrow{T e \times mze} T e \times T o
\end{align*}
\]

\[
\begin{align*}
T e \times (T f \times T g) & \xleftarrow{\alpha_{T e, T f, T g}} (T e \times T f) \times T g \\
T e \times T (f + g) & \xrightarrow{mpl^{e,f+g}} T (e + (f + g))
\end{align*}
\]
In short, a graded monad of monoids is a lax right near-semiring
functor from \((E, i, *, o, +)\) as a discrete semiring category to
\([(\mathcal{C}, \mathcal{C}), \text{Id}, \cdot, 1, \times)\).

The cartesian structure \((1, \times)\) on \(\mathcal{C}\) (which is lifted to \([\mathcal{C}, \mathcal{C}]\)) can be
replaced with a general monoidal structure \((I, \otimes)\).

Similarly to the monad case, it makes sense to generalize to grading
with an ordered right near-semiring or with a general right
near-semiring category.
Example: Graded nondeterminism

$$(\mathbb{E}, i, *, o, +) = ((\mathbb{N}, \leq), 1, *, 0, +)$$

$$T n X = X^{\leq n}$$

$$\quad$$

$$(\mathbb{E}, i, *, o, +) = ((\mathbb{N}, \geq), 1, *, 0, +)$$

$$T n X = X^{\geq n}$$
Matching pairs of actions

- Given two monoids \((E_0, i_0, *)\) and \((E_1, i_1, *)\).
- A matching pair is a pair of functions \(\downarrow: E_1 \times E_0 \to E_0\), \(\uparrow: E_1 \times E_0 \to E_1\) such that

\[
\begin{align*}
e_1 \downarrow i_0 &= i_0 \\
e_1 \downarrow (e_0 * e'_0) &= (e_1 \downarrow e_0) * (e_1 \downarrow e_0) \\
i_1 \downarrow e_0 &= e_0 \\
(e_1 \ast_1 e'_1) \downarrow e_0 &= e_1 \downarrow (e'_1 \downarrow e_0)
\end{align*}
\]

\[
\begin{align*}
e_1 \uparrow i_0 &= e_1 \\
e_1 \uparrow (e_0 * e'_0) &= (e_1 \uparrow e_0) \uparrow e'_0 \\
i_1 \uparrow e_0 &= i_1 \\
(e_1 \ast_1 e'_1) \uparrow e_0 &= (e_1 \uparrow (e'_1 \downarrow e_0)) \ast_1 (e'_1 \downarrow e_0)
\end{align*}
\]

- A matching pair equips \(E_0 \times E_1\) with a monoid structure by
  \(i = (i_0, i_1)\) and \((e_0, e_1) \ast (e'_0, e'_1) = (e_0 \ast e'_0) (e_1 \downarrow e'_0), (e_1 \downarrow e'_0) \ast_1 (e'_1, e'_1)\), a Zappa-Szép product structure on \(E_0 \times E_1\).
- Matching pairs and Zappa-Szép product structures are in a bijection.
Graded distributive laws

- Given two monoids \((E_0, i_0, *_0)\), \((E_1, i_1, *_1)\) with a matched pair and graded monads \((T_0, \eta_0, \mu_0)\) and \((T_1, \eta_1, \mu_1)\).

- A **graded distributive law** consists of, for any \(e_1 : E_1, e_0 : E_0\), a nat. transf. \(\theta^{e_1, e_0} : T_1 e_1 \cdot T_0 e_0 \to T_0 (e_1 \downarrow e_0) \cdot T_1 (e_1 \uparrow e_0)\) such that

\[
\begin{array}{ccc}
T_1 e_1 \\
\downarrow T_1 e_1 \cdot \eta_0 \\
T_1 e_1 \cdot T_0 i_0 \quad \theta^{e_1, i_0} \\
\downarrow \quad \downarrow \\
T_0 (e_1 \downarrow i_0) \cdot T_1 (e_1 \uparrow i_0) \\
\end{array}
\]

and three more equations hold.

- Let \(T (e_0, e_1) = T_0 e_0 \cdot T_1 e_1\). A graded distributive law equips \(T\) with a graded monad structure for the Zappa-Szép product by

\[
\eta = \eta_0 \cdot \eta_1
\]

\[
\mu^{(e_0, e_1),(e'_0, e'_1)} = \mu_0^{e_0, e_1 \downarrow e'_0} \cdot \mu_1^{e_1 \uparrow e'_0, e'_1} \circ T_0 e_0 \cdot \theta^{e_1, e'_0} \cdot T_1 e_1
\]

a **compatible graded monad** structure.

- Distributive laws and compatible graded monad structures are in a bijection.
Example: Distributing graded maybe

Let $(E_1, i_1, \ast_1)$ and $(T_1, \eta_1, \mu_1)$ be the pomonoid and graded monad from the graded maybe example.

For any pomonoid $(E_0, i_0, \ast_0)$ that has joins and graded monad $(T_0, \eta_0, \mu_0)$, the following is a matching pair for which we have a graded distributive law:

$$\begin{align*}
\text{pure } \downarrow e_0 &= e_0 & e_1 \downarrow e_0 &= e_1 \\
\text{fail } \downarrow e_0 &= i_0 \\
\text{mf } \downarrow e_0 &= e_0 \lor i_0
\end{align*}$$

$$\begin{align*}
\theta^{e_1, e_0} &: T_1 e_1 \cdot T_0 e_0 \to T_0 (e_1 \downarrow e_0) \cdot T_1 (e_1 \downarrow e_0) \\
\theta^{\text{pure}, e_0} &: T_0 e_0 X \quad T_0 e_0 X \\
\theta^{\text{fail}, e_0} &: 1 \xrightarrow{\eta_0} T_0 i_0 1 \\
\theta^{\text{mf}, e_0} &: T_0 e_0 X + 1 \xrightarrow{T_0 e_0 X + \eta_0} T_0 e_0 X + T_0 i_0 1 \to T_0 (e_0 \lor i_0) (X + 1)
\end{align*}$$
Grading the stack writer monad

\[ (\mathbb{E}_0, i_0, *_0) = ((\mathbb{N}, \geq), 0, +) \]
\[ (\mathbb{E}_1, i_1, *_1) = ((\mathbb{N}, \leq), 0, +) \]
\[ n_1 \downarrow n_0 = n_0 - n_1 \]
\[ n_1 \uparrow e_0 = n_1 - n_0 \]
\[ i = (0, 0) \]
\[ (n_0, n_1) * (n'_0, n'_1) = (n_0 + (n'_0 - n_1), (n_1 - n'_0) + n_1) \]

\[ T_0 n_0 X = \mathbb{N} \geq n_0 \times X \]
\[ T_1 n_1 X = \Sigma \leq n_1 \times X \]

\[ \theta^{n_0, n_1}_X : \Sigma \leq n_1 \times (\mathbb{N} \geq n_0 \times X) \rightarrow \mathbb{N} \geq n_0 - n_1 \times (\Sigma \leq n_1 - n_0 \times X) \]
\[ (w, (k, x)) \mapsto (k - |w|, (\text{drop } k \ w, \ xs)) \]
**Graded comonads**

- Given a *monoid* \((E, i, \ast)\). A *graded comonad* on a category \(C\) is
  - for any \(e : E\), a functor \(D e : C \to C\)
  - a nat. transf. \(\varepsilon : D i \to \text{Id}\)
  - for any \(e, f : E\), a nat. transf. \(\delta^{e,f} : D (e \ast f) \to D e \cdot D f\)

such that

\[
\begin{align*}
D (i \ast e) & \xrightarrow{\delta^{i,e}} D i \cdot D e \\
D e & \xrightarrow{\varepsilon \cdot D e} D i \cdot D e \\
D (e \ast i) & \xrightarrow{\delta^{e,i}} D e \cdot D i \\
D e & \xrightarrow{D e \cdot \varepsilon} D e \cdot \text{Id}
\end{align*}
\]

\[
\begin{align*}
D (e \ast (f \ast g)) & \xrightarrow{\delta^{e,f,g}} D ((e \ast f) \ast g) \\
D (e \ast (f \ast g)) & \xrightarrow{\delta^{e,f,g}} D (e \ast f) \cdot D g \\
D e \cdot (f \ast g) & \xrightarrow{D e \cdot \delta^{f,g}} D e \cdot (D f \cdot D g) \\
D e \cdot (f \ast g) & \xrightarrow{D e \cdot \delta^{f,g}} (D e \cdot D f) \cdot D g
\end{align*}
\]

- In short, a graded comonad on \(C\) is an oplax monoidal functor from \((E, i, \ast)\) as a discrete monoidal category to \(([C, C], \text{Id}, \cdot)\).
Graded comonads ctd

- Again it makes sense to generalize from a monoid \((E, i, \ast)\) to a pomonoid \((E, i, \ast)\).
- But now a graded comonad is reasonably defined as an oplax monoidal functor from \((E, i, \ast)^\text{op}\) to \([\mathcal{C}, \mathcal{C}], \text{Id}, \cdot\).

(Note the additional \text{op}.)
Example: Graded dataflow

\((\mathcal{E}, i, \ast) = ((\mathbb{N}, \leq), 0, +)\)

\(D n X = X \times X^{\leq n}\)

\(D (n \leq n') X : X \times X^{\leq n'} \times X \times X^{\leq n}\)

\(\varepsilon X : X \rightarrow X\)

\(\delta^{n,m} X : X \times X^{\leq n+m} \rightarrow (X \times X^{\leq m}) \times (X \times X^{m})^{\leq n}\)
Takeaway

- Graded monads etc are natural concepts both theoretically and in terms of programming examples.
- Marrying monads and effects works!
- But as ever we see that it pays off to look at the categorical generalities to get everything right.