A TYPE THEORY FOR OPETOPES

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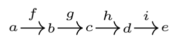
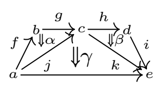
Introduction. Opetopes were originally introduced by Baez and Dolan in order to formulate a definition of weak ω-categories [BD98]. Their name reflects the fact that they encode the possible shapes for higher-dimensional operations: they are *ope*ration poly*topes*. They have been the subject of many investigations in order to provide good working definitions of opetopes allowing to explore their combinatorics [HMP00, Lei04]. One of the most commonly used nowadays is the formulation based on polynomial functors and the corresponding graphical representation using "zoom complexes" [KJBM10].

In order to grasp quickly the nature of opetopes, consider a sequence of four composable arrows as in figure (1). There are various ways we can compose them. For instance, we can compose f with g, as well as h with i, and then $g \circ f$ with $i \circ h$. Or we can compose f, g and h together all at once, and then the result with i. These two schemes for composing can respectively be pictured in (2) and (3). From there, the general idea of getting "higher-dimensional" is that we should take these compositions as "2-operations", which can themselves be composed. Opetopes describe all the ways in which these compositions can be meaningfully specified, in arbitrary dimension. We can expect (and it is indeed the case) that the combinatorics of these objects is not easy to describe.

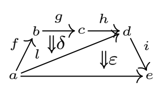
(1)



(2)

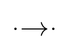


(3)

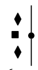


Constructing low dimensional opetopes. Opetopes are defined by induction on their dimension, the base cases being dimensions 0 and 1. The unique 0-opetope is written $\blacklozenge$ and called the *point*. The unique 1-opetope is written $\blacksquare$, called the *arrow*, and is graphically represented in (4). The arrow can be seen as a 1-dimensional "operation", taking a point as input, and outputting another point, whence the tree representation (5). Say that the unique input edge of that tree is $*$. Then the "source pasting scheme" of $\blacksquare$, i.e. the way its inputs are arranged, is completely described by the expression (6). An expression of this form is called a *preopetope*. In the case of $\blacksquare$, those three representations are a bit trivial, but those approaches will become of relevance in the higher dimensional cases.

(4)



(5)

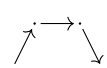


(6) $\left\{ * \leftarrow \blacklozenge \right.$

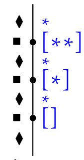
Now, the arrow can be used to create "1-pasting schemes", i.e. meaningful compositions of cells whose shapes are the arrow. An example of such a pasting scheme is given in (7), and the corresponding composition tree is represented in (8). As previously mentioned, the input edge of each $\blacksquare$ node is called $*$, and so we can associate an *address* to every node in that tree (in blue), which is a bracket-enclosed sequence of names giving "walking instructions from the root node". Syntactically, this can be expressed by the preopetope in (9), and since $\blacksquare$ is encoded by the preopetope $\{* \leftarrow \blacklozenge$, this expression can be further expanded as in (10).

Now, the pasting scheme (7) on the left can be "filled" with a 2-cell representing its "compositor", as depicted in (11). This compositor, which we shall denote by **3**, has three input cells, located at address $[]$, $[*]$, and $[**]$, and so we may represent it as a corolla (12), i.e., a tree consisting of a unique node labeled by **3**, whose three input edges are named $[]$, $[*]$, and $[**]$, and labeled by the opetopes at those addresses, in this case, $\blacksquare$ for all three. Further, the output of **3** is the target in (11), i.e., an arrow, and thus the root edge is labeled by $\blacksquare$ as well. This process can be iterated by forming n-pasting schemes, and filling them in order to obtain $(n+1)$-opetopes, that can in turn by assembled into $(n+1)$-pasting schemes. Figure (13) is an example of a 2-pasting scheme, and its tree representation (the 2-opetopes **1** and **2** are defined similarly to **3**) is given in (14). As before, the pasting scheme can be expressed as a preopetope (15), and fully expanded as in (16).

(7)



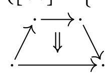
(8)



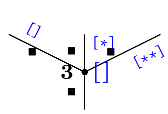
(9) $\begin{cases} [] \leftarrow \blacksquare \\ [*] \leftarrow \blacksquare \\ [**] \leftarrow \blacksquare \end{cases}$

(10) $\begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \\ [**] \leftarrow \{* \leftarrow \blacklozenge \end{cases}$

(11)



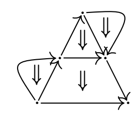
(12)



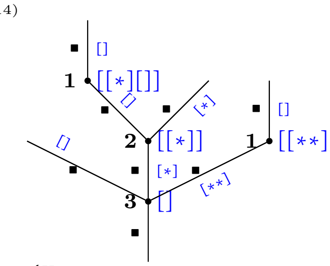
If $n \geq 2$, the set $\mathbb{P}_n$ of n-preopetopes is no longer a singleton as in dimension 0 and 1. Consequently, edge labelings in $(n+1)$-pasting schemes are not trivial, and dictate which opetope can be adjacent to which. Since preopetopes do not keep track of edge labels, some of them describe pasting schemes that are not "well-formed", and thus do not correspond to an actual opetope. First, some preopetopes do not even describe a tree: $\{[*] \leftarrow \{* \leftarrow \blacklozenge$ does not have a root node (that would necessarily be located at address $[]$). But, more importantly, from dimension 3 and higher, some compatibility conditions have to be verified when building pasting diagrams. For example, the corolla associated with (14) has four inputs, one of them being decorated by **2**.. When attaching another corolla to that corolla at this particular input, we must make sure that the target of the opetope decorating the node ot the attached corolla is **2**.

Syntax. The set $\mathbb{A}_n$ of addresses (or more precisely, *n-addresses*) locating n-opetopes in n-pasting schemes, is inductively defined as follows: $\mathbb{A}_0 = \{*\}$, and $\mathbb{A}_{n+1}$ is the set of finite words over the alphabet $\mathbb{A}_n$, written with enclosing brackets. For example, $[[[**][*]][][[*][]]] \in \mathbb{A}_3$, while the empty address $[]$ is in $\mathbb{A}_n$ for all $n \geq 1$. The set $\mathbb{P}_n$ of *n-preopetopes* is also defined inductively: $\mathbb{P}_0 = \{\blacklozenge\}$, and $\mathbb{P}_{n+1}$ is the set of expression of the form $\begin{cases} [p_1] \leftarrow \mathbf{p}_1 \\ \vdots \\ [p_k] \leftarrow \mathbf{p}_k \end{cases}$ and $\{\{\mathbf{q}$, where $[p_1], \ldots, [p_k] \in \mathbb{A}_n$ are distinct n-addresses, $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{P}_n$, and $\mathbf{q} \in \mathbb{P}_{n-1}$ if $n \geq 2$. The expression $\{\{\mathbf{q}$ describes an empty n-pasting scheme, i.e. a tree with no nodes and a unique edge labeled by the $(n-1)$-preopetope $\mathbf{q}$.

(13)



(14)



(15) $\begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][]] \leftarrow \mathbf{1} \\ [[**]] \leftarrow \mathbf{1} \end{cases}$

(16) $\begin{cases} [] \leftarrow \begin{cases} * \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \\ [**] \leftarrow \{* \leftarrow \blacklozenge \end{cases} \\ [[*]] \leftarrow \begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \end{cases} \\ [[*][]] \leftarrow \{[] \leftarrow \{* \leftarrow \blacklozenge \\ [[**]] \leftarrow \{[] \leftarrow \{* \leftarrow \blacklozenge \end{cases}$

As previously mentioned, not all preopetopes correspond to an opetope. We shall present a derivation system named $\mathrm{O{\small PT}}^?$ that characterizes opetopes among preopetopes as those that are derivable. The inference rules of the system essentially follow the construction procedure presented above: rule `point` derives the 0-preopetope $\blacklozenge$ with no prior assumption; rule `degen` takes a n-preopetope $\mathbf{q}$ and constructs the empty pasting scheme $\{\{\mathbf{q}$; rule `shift` takes a n-preopetope $\mathbf{p}$ and considers it as the unique cell of a n-pasting scheme $\{[] \leftarrow \mathbf{p}$; and finally, rule `graft` takes a $(n+1)$-preopetope $\mathbf{r} = \begin{cases} [p_1] \leftarrow \mathbf{p}_1 \\ \vdots \\ [p_k] \leftarrow \mathbf{p}_k \end{cases}$ as above, an n-address $[p_{k+1}]$, and a n-preopetope $\mathbf{p}_{k+1}$, and, extends $\mathbf{r}$ as the pasting scheme $\begin{cases} [p_1] \leftarrow \mathbf{p}_1 \\ \vdots \\ [p_{k+1}] \leftarrow \mathbf{p}_{k+1} \end{cases}$ obtained from $\mathbf{r}$ by adding the cell $\mathbf{p}_{k+1}$ at address $[p_{k+1}]$, embodying the well-formedness property as side conditions.

Further developments. Higher addresses are an extremely convenient tool when dealing with the combinatorics of opetopes. They allow for a succinct and precise formalism of preopetopes, and are a cornerstone to the definition of $\mathbb{O}$, the category of opetopes, based on the intuition carried by the graphical representations of pasting schemes. In the preprint [CHM18], we also present another syntax for opetopes, using variables instead of higher addresses, and a corresponding derivation system $\mathrm{O{\small PT}}^!$, is presented. The latter system is more user-friendly and easy to read, but does not lend itself so nicely to a mathematical treatment, especially when it come to organizing opetopes into a category. The results developed in thhe latter approach have been submitted elsewhere. Variations of systems $\mathrm{O{\small PT}}^!$ and $\mathrm{O{\small PT}}^?$, designed for syntactical representations of finite opetopic sets (finite presheaves over $\mathbb{O}$), are also presented in [CHM18], and a Python implementation of all of them is available in [Ho 18]. Together with an adequate formulation of opetopic higher categories, it is our hope that this work will be used productively for mechanical proofs of coherence in opetopic ω-categories or opetopic ω-groupoids.

References

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