Quantale-valued dissimilarity

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Abstract
Motivated by the theory of apartness relations of Scott [3], a positive theory of dissimilarity valued in an involutive quantale
\[ Q = (Q, \& , k, ^\circ) \]
is established without the aid of negation. The notion of Q-valued dissimilarity dualizes that of Q-valued set (i.e., a set equipped with a Q-valued similarity) in the sense of Höhle–Kubiak [2], whose prototype comes from the theory of Ω-sets of Fourman–Scott [1].

It is demonstrated that sets equipped with a Q-valued dissimilarity are symmetric categories enriched in the quantaloid
\[ B(Q) \]
of back diagonals of Q [4]. Moreover, it is shown that similarities and dissimilarities are interdefinable if Q is a Girard quantale, in which case there is an isomorphism
\[ D(Q) \cong B(Q) \]
of quantaloids, where D(Q) is the quantaloid of diagonals of Q [5]. In the case that Q is a commutative quantale, it is proved that the above isomorphism holds if, and only if, Q is a Girard quantale.

Keywords: dissimilarity, similarity, back diagonal, diagonal, quantale, quantaloid

References