The Existential Completion

Davide Trotta

March 7, 2019

Abstract

We determine the existential completion of a primary doctrine, and we prove that the 2-monad obtained from it is lax-idempotent, and that the 2-category of existential doctrines is isomorphic to the 2-category of algebras for this 2-monad. We also show that the existential completion of an elementary doctrine is again elementary. Finally we extend the notion of exact completion of an elementary existential doctrine to an arbitrary elementary doctrine.

1 Introduction

In recent years, many relevant logical completions have been extensively studied in category theory. The main instance is the exact completion, see [3, 4, 5], which is the universal extension of a category with finite limits to an exact category. In [16, 17, 18], Maietti and Rosolini introduce a categorical version of quotient for an equivalence relation, and they study that in a doctrine equipped with a sufficient logical structure to describe the notion of an equivalence relation. In [18] they show that both the exact completion of a regular category and the exact completion of a category with binary products, a weak terminal object and weak pullbacks can be seen as instances of a more general completion with respect to an elementary existential doctrine.

In this paper we present the existential completion of a primary doctrine, and we give an explicit description of the 2-monad $T_e : \mathbf{PD} \to \mathbf{PD}$ constructed from the 2-adjunction, where $\mathbf{PD}$ is the 2-category of primary doctrines.

It is well known that pseudo-monads can express uniformly and elegantly many algebraic structure; we refer the reader to [25, 24, 9] for a detailed description of these topics. We show that every existential doctrine $P : C^{op} \to \mathbf{InfSL}$ admits an action $a : T_e P \to P$ such that $(P, a)$ is a $T_e$-algebra, and that if $(R, b)$ is $T_e$-algebra then the doctrine is existential, and this gives an equivalence between the 2-category $T_e$-$\mathbf{Alg}$ and the 2-category $\mathbf{ED}$ whose objects are existential doctrines.

Here the action encodes the existential structure for a doctrine, and we prove that this structure is uniquely determined to within appropriate isomorphism and that the 2-monad $T_e$ is property-like and lax-idempotent in the sense of Kelly and Lack [9].
We conclude proving that the existential completion preserves elementary doctrines, and then we generalize the bi-adjunction \( \text{EED} \rightarrow \text{Xct} \) presented in [18, 15] to a bi-adjunction from the 2-category \( \text{PED} \) of elementary doctrine to the 2-category of exact categories \( \text{Xct} \).

In the first two sections we recall some definitions and results about the theory of pseudo-monads, primary and existential doctrine which are needed for the rest of the paper.

In section 3 we present the existential completion. We introduce a functor \( \text{E}: \text{PD} \rightarrow \text{PED} \) from the 2-category of primary doctrines to the 2-category of existential doctrines, and we prove that it is a left 2-adjoint to the forgetful functor \( \text{U}: \text{PED} \rightarrow \text{PD} \).

In sections 4 we prove that the 2-monad \( T_e \) constructed from the 2-adjunction is lax-idempotent and, in section 5, that the category \( T_e-\text{Alg} \) is 2-equivalent to the 2-category of existential doctrine.

In section 6 we show that the existential completion of an elementary doctrine is elementary, and we use this fact to extend the notion of exact completion to elementary doctrines.

2 A brief recap of two-dimensional monad theory

This section is devoted to the formal definition of 2-monad on a 2-category and a characterization of the definitions. We use 2-categorical pasting notation freely, following the usual convention of the topic as used extensively in [1, 24, 25].

You can find all the details of the main results of this section in the works of Kelly and Lack [9]. For a more general and complete description of these topics, and a generalization for the case of pseudo-monad, you can see the Ph.D thesis of Tanaka [23], the articles of Marmolejo [20, 19] and the work of Kelly [10]. Moreover we refer to [2, 14] for all the standard results and notions about 2-category theory.

A 2-monad \((T, \mu, \eta)\) on a 2-category \( \mathcal{A} \) is a 2-functor \( T: \mathcal{A} \rightarrow \mathcal{A} \) together 2-natural transformations \( \mu: T^2 \rightarrow T \) and \( \eta: 1_{\mathcal{A}} \rightarrow T \) such that the following diagrams

\[
\begin{array}{ccc}
T^3 & \xrightarrow{T} & T^2 \\
\downarrow{\mu T} & & \downarrow{\mu} \\
T^2 & \xrightarrow{\mu} & T \\
\end{array}
\]

\[
\begin{array}{ccc}
T & \xrightarrow{\eta T} & T^2 \\
\downarrow{id} & & \downarrow{id} \\
T & & T \\
\end{array}
\]

\[
\begin{array}{ccc}
T & \xrightarrow{T \eta} & T \\
\downarrow{id} & & \downarrow{id} \\
T & & T \\
\end{array}
\]
commute. Let \((T, \mu, \eta)\) be a 2-monad on a 2-category \(A\). A \(T\)-algebra is a pair \((A, a)\) where, \(A\) is an object of \(A\) and \(a : TA \rightarrow A\) is a 1-cell such that the following diagrams commute:

\[
\begin{array}{ccc}
T^2A & \xrightarrow{Ta} & TA \\
\downarrow\mu_A & & \downarrow a \\
TA & \xrightarrow{a} & A
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
\downarrow 1_A & & \downarrow a \\
A & \xrightarrow{a} & A
\end{array}
\]

A lax \(T\)-morphism from a \(T\)-algebra \((A, a)\) to a \(T\)-algebra \((B, b)\) is a pair \((f, \tilde{f})\) where \(f\) is a 1-cell \(f : A \rightarrow B\) and \(\tilde{f}\) is a 2-cell

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{f} & B
\end{array}
\]

which satisfies the following coherence conditions:

\[
\begin{array}{ccc}
T^2A & \xrightarrow{T^2f} & TB \\
\downarrow\mu_A & & \downarrow\mu_B \\
TA & \xrightarrow{Tf} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{f} & B
\end{array}
\]

\[
\begin{array}{ccc}
T^2A & \xrightarrow{T^2f} & TB \\
\downarrow\mu_A & & \downarrow\mu_B \\
TA & \xrightarrow{Tf} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{f} & B
\end{array}
\]

and

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow\eta_A & & \downarrow\eta_B \\
TA & \xrightarrow{Tf} & TB \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{f} & B
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow 1_A & & \downarrow 1_B \\
A & \xrightarrow{f} & B
\end{array}
\]
The regions in which no 2-cell is written always commute by the naturality of $\eta$ and $\mu$, and are deemed to contain the identity 2-cell.

If $(f, \bar{f})$ is a lax morphism and $\bar{f}$ is invertible, then it is said T-

morphism and if $\bar{f}$ is the identity it is said strict-T-
morphism.

The category of $T$-algebras and lax $T$-morphisms becomes a 2-category $T$-$\text{Alg}$, when we introduce as 2-cells the $T$-

transformations, where a $T$-transformation from a 1-cell $(f, \bar{f}): (A, a) \rightarrow (B, b)$ to $(g, \bar{g}): (A, a) \rightarrow (B, b)$ is a 2-cell $\alpha: f \Rightarrow g$ in $\mathcal{A}$ satisfies the following coherence condition:

expressing compatibility of $\alpha$ with $\bar{f}$ and $\bar{g}$.

It is observed in [9] that using this notion of T-morphism, we can express more precisely what it might mean to say that an action of a monad $T$ on an object $A$ is unique to within a unique isomorphism. We shall mean that, given two action $a, a': TA \rightarrow A$ there is a unique invertible 2-cell $\alpha: a \Rightarrow a'$ such that $(1_A, \alpha): (A, a) \rightarrow (A, a')$ is a morphism of $T$-algebras (in particular it is an isomorphism of $T$-algebras). In this case we will say that the $T$-algebra structure is essentially unique. More precisely a 2-monad $(T, \mu, \eta)$ is said property-like, if it satisfies the following conditions:

- for every $T$-algebras $(A, a)$ and $(B, b)$, and for every invertible 1-cell $f: A \rightarrow B$ there exists a unique invertible 2-cell $\bar{f}$

such that $(f, \bar{f}): (A, a) \rightarrow (B, b)$ is a morphism of $T$-algebras;

- for every $T$-algebras $(A, a)$ and $(B, b)$, and for every 1-cell $f: A \rightarrow B$ if there exists
such that \((f, \overline{f}) : (A, a) \rightarrow (B, b)\) is a lax morphism of \(T\)-algebras, then it is the unique 2-cell with such property.

We conclude this section recalling a stronger property on a 2-monads \((T, \mu, \eta)\) on \(A\) which implies that \(T\) is property-like: a 2-monad \((T, \mu, \eta)\) is said lax-idempotent, if for every \(T\)-algebras \((A, a)\) and \((B, b)\), and for every 1-cell \(f : A \rightarrow B\) there exists a unique 2-cell \(\overline{f}\)

such that \((f, \overline{f}) : (A, a) \rightarrow (B, b)\) is a lax morphism of \(T\)-algebras. In particular every lax-idempotent monad is property like. See [9, Proposition 6.1].

### 3 Primary and existential doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in series of seminal papers, together with the more general notion of existential elementary doctrine. These were studied in [11, 12, 13]. We recall from loc. cit. some definitions which will be useful in the following. The reader can find all the details about the theory of elementary and existential doctrine in [16, 17, 18].

**Definition 3.1.** Let \(C\) be a category with finite products. A **primary doctrine** if a functor \(P : C^{op} \rightarrow \text{InfSL}\) from the opposite of the category \(C\) to the category of inf-semilattices.

**Definition 3.2.** A primary doctrine \(P : C^{op} \rightarrow \text{InfSL}\) is **elementary** if for every \(A\) in \(C\) there exists an object \(\delta_A\) in \(P(A \times A)\) such that

1. the assignment
   \[
   \mathfrak{E}_{(\text{id}_A, \text{id}_A)}(\alpha) := P_{\text{pr}_1}(\alpha) \land \delta_A
   \]
   for \(\alpha\) in \(PA\) determines a left adjoint to \(P_{(\text{id}_A, \text{id}_A)} : P(A \times A) \rightarrow PA\);
2. for every morphism \( e \) of the form \( \langle \text{pr}_1, \text{pr}_2, \text{pr}_3 \rangle : X \times A \to X \times A \times A \) in \( C \), the assignment

\[
\exists_e(\alpha) := P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\alpha) \land P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\delta_A)
\]

for \( \alpha \) in \( P(X \times A) \) determines a left adjoint to \( P_e : P(X \times A \times A) \to P(X \times A) \).

**Definition 3.3.** A primary doctrine \( P : C^{\text{op}} \to \text{InfSL} \) is **existential** if, for every \( A_1 \) and \( A_2 \) in \( C \), for any projection \( \text{pr}_i : A_1 \times A_2 \to A_i, i = 1, 2 \), the functor

\[
P_{\text{pr}_i} : P(A_i) \to P(A_1 \times A_2)
\]

has a left adjoint \( \exists_{\text{pr}_i} \), and these satisfy:

1. **Beck-Chevalley condition:** for any pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{pr'} & A' \\
\downarrow f' & & \downarrow f \\
X & \xrightarrow{pr} & A
\end{array}
\]

with \( pr \) and \( pr' \) projections, for any \( \beta \) in \( P(X) \) the canonical arrow

\[
\exists_{pr'} P_{f'}(\beta) \leq P_f \exists_{pr}(\beta)
\]

is an isomorphism;

2. **Frobenius reciprocity:** for any projection \( pr : X \to A \), \( \alpha \) in \( P(A) \) and \( \beta \) in \( P(X) \), the canonical arrow

\[
\exists_{pr}(P_{pr}(\alpha) \land \beta) \leq \alpha \land \exists_{pr}(\beta)
\]

in \( P(A) \) is an isomorphism.

**Remark 3.4.** In an existential elementary doctrine, for every map \( f : A \to B \) in \( C \) the functor \( P_f \) has a left adjoint \( \exists_f \) that can be computed as

\[
\exists_{pr_2}(P_{f \times \text{id}_B}(\delta_B) \land P_{pr_1}(\alpha))
\]

for \( \alpha \) in \( P(A) \), where \( pr_1 \) and \( pr_2 \) are the projections from \( A \times B \).

**Examples 3.5.** The following examples are discussed in [11].

1. Let \( C \) be a category with finite limits. The functor

\[
\text{Sub}_C : C^{\text{op}} \to \text{InfSL}
\]
assigns to an object \( A \) in \( C \) the poset \( \text{Sub}_C(A) \) of subobjects of \( A \) in \( C \) and, for an arrow \( B \to A \) the morphism \( \text{Sub}_C(f): \text{Sub}_C(A) \to \text{Sub}_C(B) \) is given by pulling a subobject back along \( f \). The fiber equalities are the diagonal arrows. This is an existential elementary doctrine if and only if the category \( C \) has a stable, proper factorization system \( \langle E, M \rangle \). See [7].

2. Consider a category \( D \) with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

\[
\Psi_D: D^{op} \to \text{InfSL}
\]

where \( \Psi_D(A) \) is the poset reflection of the slice category \( D/A \), and for an arrow \( B \to A \), the homomorphism \( \Psi_D(f): \Psi_D(A) \to \Psi_D(B) \) is given by a weak pullback of an arrow \( X \to A \) with \( f \). This doctrine is existential, and the existential left adjoint are given by the post-composition.

3. Let \( T \) be a theory in a first order language \( \text{Sg} \). We define a primary doctrine

\[
LT: C_T^{op} \to \text{InfSL}
\]

where \( C_T \) is the category of lists of variables and term substitutions:

- **objects** of \( C_T \) are finite lists of variables \( \vec{x} := (x_1, \ldots, x_n) \), and we include the empty list \( () \);
- **morphisms** from \((x_1, \ldots, x_n)\) into \((y_1, \ldots, y_m)\) is a substitution \([t_1/y_1, \ldots, t_m/y_m]\) where the terms \( t_i \) are built in \( \text{Sg} \) on the variable \( x_1, \ldots, x_n \);
- the **composition** of two morphisms \([\vec{t}/\vec{y}]: \vec{x} \to \vec{y} \) and \([\vec{s}/\vec{z}]: \vec{y} \to \vec{z} \) is given by the substitution

\[
[s_1[\vec{t}/\vec{y}]/z_k, \ldots, s_k[\vec{t}/\vec{y}]/z_k]: \vec{x} \to \vec{z}.
\]

The functor \( LT: C_T^{op} \to \text{InfSL} \) sends a list \((x_1, \ldots, x_n)\) in the class \( LT(x_1, \ldots, x_n) \) of all well formed formulas in the context \((x_1, \ldots, x_n)\). We say that \( \psi \leq \phi \) where \( \phi, \psi \in LT(x_1, \ldots, x_n) \) if \( \psi \vdash_T \phi \), and then we quotient in the usual way to obtain a partial order on \( LT(x_1, \ldots, x_n) \). Given a morphism of \( C_T \)

\[
[t_1/y_1, \ldots, t_m/y_m]: (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)
\]

then the functor \( LT|_{[\vec{t}/\vec{y}]} \) acts as the substitution \( LT|_{[\vec{t}/\vec{y}]}(\psi(y_1, \ldots, y_m)) = \psi[\vec{t}/\vec{y}] \).

The doctrine \( LT: C_T^{op} \to \text{InfSL} \) is elementary exactly when \( T \) has an equality predicate and it is existential. For all the detail we refer to [17], and for the case of a many sorted first order theory we refer to [21].


4 Existential completion

In this section we construct an existential doctrine $P^e: C^{op} \rightarrow \text{InfSL}$, starting from a primary doctrine $P: C^{op} \rightarrow \text{InfSL}$. Let $P: C^{op} \rightarrow \text{InfSL}$ be a fixed primary doctrine for the rest of the section, and let $a \subseteq C_1$ be a subset of morphisms closed under pullbacks, compositions and such that it contains the identity morphisms.

For every object $A$ of $C$ consider the following preorder:

- the objects are pairs $(B \xrightarrow{g \in a} A, \alpha \in PB)$;
- $(B \xrightarrow{h \in a} A, \alpha \in PB) \leq (D \xrightarrow{f \in a} A, \gamma \in PD)$ if there exists $w: B \rightarrow D$ such that $B \xrightarrow{w} D \xrightarrow{f} A$ commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder. Let $P^e(A)$ be the partial order obtained by identifying two objects when $(B \xrightarrow{h \in a} A, \alpha \in PB) \cong (D \xrightarrow{f \in a} A, \gamma \in PD)$ in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.

Given a morphism $f: A \rightarrow B$ in $C$, let $P_f^e(C \xrightarrow{g \in a} B, \beta \in PC)$ be the object $(D \xrightarrow{g^f} A, P_{f^*g}(\beta) \in PD)$ where

\[
\begin{array}{ccc}
D & \xrightarrow{g^f} & A \\
\downarrow f^*g & & \downarrow f \\
C & \xrightarrow{g} & B
\end{array}
\]

is a pullback because $g \in a$. Note that $P_f^e$ is well defined, because isomorphisms are stable under pullback.

Proposition 4.1. Let $P: C^{op} \rightarrow \text{InfSL}$ be a primary doctrine. Then $P^e: C^{op} \rightarrow \text{InfSL}$ is a primary doctrine, in particular:

(i) for every object $A$ in $C$, $P^e(A)$ is a inf-semilattice;

(ii) for every morphism $f: A \rightarrow B$ in $C$, $P_f^e$ is an homomorphism of inf-semilattices.
Proof. (i) For every $A$ we have the top element $(A \xrightarrow{id_A} A, \top_A)$. Consider $(A_1 \xrightarrow{h_1} A, \alpha_1 \in PA_1)$ and $(A_2 \xrightarrow{h_2} A, \alpha_2 \in PA_2)$. In order to define the greatest lower bound of the two objects consider a pullback

$$
\begin{tikzcd}
A_1 \times A_2 \ar{rr}{h_2 h_1} \ar{dr}{h_2} & & A_2 \\
A_1 \ar{rr}{h_2} \ar{u}{h_1} & & A
\end{tikzcd}
$$

which exists because $h_1 \in a$ (and $h_2 \in a$). We claim that

$$(A_1 \times A_2 \xrightarrow{h_1 h_2} A, P_{h_2 h_1} (\alpha_1) \wedge P_{h_2 h_2} (\alpha_2))$$

is such an infimum. It is easy to check that

$$(A_1 \times A_2 \xrightarrow{h_1 h_2} A, P_{h_2 h_1} (\alpha_1) \wedge P_{h_2 h_2} (\alpha_2)) \leq (A_i \xrightarrow{h_i} A, \alpha_i \in PA_i)$$

for $i = 1, 2$. Next consider $(B \xrightarrow{g} A, \beta \in PB) \leq (A_i \xrightarrow{h_i} A, \alpha_i \in PA_i) \text{ for } i = 1, 2 \text{ and } g = h_i w_i$. Then there is a morphism $w : C \rightarrow A_1 \times A_2$ such that

$$
\begin{tikzcd}
B \ar{ddr}{w_1} \ar{dr}{w_2} & & \\
& A_1 \times A_2 \ar{rr}{h_2 h_1} \ar{dr}{h_2} & & A_2 \\
A_1 \ar{rr}{h_2} \ar{u}{h_1} & & A
\end{tikzcd}
$$

commutes and $P_{w}(P_{h_2 h_1} (\alpha_1) \wedge P_{h_2 h_2} (\alpha_2)) = P_{w_1} (\alpha_1) \wedge P_{w_2} (\alpha_2) \geq \beta$.

(ii) We first prove that for every morphism $f : A \rightarrow B$ the $P_f$ preserves the order. Consider

$$
(C_1 \xrightarrow{g_1 \in a} B, \alpha_1 \in PC_1) \leq (C_2 \xrightarrow{g_2 \in a} B, \alpha_2 \in PC_2) \text{ with } g_2 w = g_1 \text{ and } P_{w} (\alpha_2) \geq \alpha_1. \text{ We want to prove that}
$$

$$(D_1 \xrightarrow{g_1^* f} A, P_{f^* g_1} (\alpha_1) \in PD_1) \leq (D_2 \xrightarrow{g_2^* f} A, P_{f^* g_2} (\alpha_2) \in PD_1)$$

We can observe that $g_2 w^* g_1 = g_1 f^* g_1 = f g_1^* f$. Then there exists a unique $w : D_1 \rightarrow D_2$
such that the following diagram commutes

Moreover $P_w(P_{f^*g_2}(\alpha_2)) = P_{f^*g_1}(P_w(\alpha_2)) \geq P_{f^*g_1}(\alpha_1)$, and it is easy to see that $P^e_f$ preserves top elements. Finally it is straightforward to prove that $P^e_f(\alpha \land \beta) = P^e_f(\alpha) \land P^e_f(\beta)$. Moreover it is straightforward to prove that $P^e_f(\alpha \land \beta) = P^e_f(\alpha) \land P^e_f(\beta)$.

**Proposition 4.2.** Given a morphism $f : A \rightarrow B$ of a, let

$$\exists_f( C \xrightarrow{h} A, \alpha \in PC) := ( C \xrightarrow{fh} B, \alpha \in PC)$$

when $( C \xrightarrow{h} A, \alpha \in PC)$ is in $P^e(A)$. Then $\exists_f$ is left adjoint to $P^e_f$.

**Proof.** Let $\alpha := ( C_1 \xrightarrow{g_1} B, \alpha_1 \in PC_1)$ and $\beta := ( D_2 \xrightarrow{f_2} A, \beta_2 \in PD_2)$. Now we assume that $\beta \leq P^e_f(\alpha)$. This means that

and $P_w(P_{f^*g_1}(\alpha_1)) \geq \beta_2$. Then we have
and $P_{w f^* g_1}(\alpha_1) \geq \beta$. Then $\mathcal{H}_f(\beta) \leq \alpha$.

Now assume $\mathcal{H}_f(\beta) \leq \alpha$

\[
\begin{array}{ccc}
D_2 & \xrightarrow{f} & A \\
\downarrow \pi & & \downarrow f \\
C_1 & \xrightarrow{g_1} & B
\end{array}
\]

with $P_\pi(\alpha_1) \geq \beta_2$. Then there exists $w: D_2 \rightarrow D_1$ such that the following diagram commutes

\[
\begin{array}{ccc}
D_2 & \xrightarrow{w} & D_1 \xrightarrow{g_1 f} & A \\
\downarrow f_2 & & \downarrow f \\
C_1 \xrightarrow{g_1} & & B
\end{array}
\]

an $P_w(P_{f^* g_1}(\alpha_1) = P_\pi(\alpha_1) \geq \beta_1$. Then we can conclude that $\beta \leq P^e(\alpha)$.

\[ \Box \]

**Theorem 4.3.** For every primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}$, $P^e: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}$ satisfies:

(i) **Beck-Chevalley Condition:** for any pullback

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & A' \\
\downarrow f' & & \downarrow f \\
X & \xrightarrow{g} & A
\end{array}
\]

with $g \in a$ (hence also $g' \in a$), for any $\beta \in P^e(X)$ the following equality holds

\[ \mathcal{H}_{g'} P^e_f(\beta) = P^e_f \mathcal{H}_g(\beta) \]

(ii) **Frobenius Reciprocity:** for every morphism $f: X \rightarrow A$ of $a$, for every $\alpha \in P^e(A)$ and $\beta \in P^e(X)$, the following equality holds

\[ \mathcal{H}_f(P^e_f(\alpha) \wedge \beta) = \alpha \wedge \mathcal{H}_f(\beta) \]
Proof. (i) Consider the following pullback square

\[
\begin{array}{c}
X' \xrightarrow{g'} A' \\
\downarrow f' \quad \downarrow f \\
X \xrightarrow{g} A
\end{array}
\]

where \(g, g' \in a\), and let \(\beta := (C_1 \xrightarrow{h_1} X, \beta_1 \in PC_1) \in P^e(X)\). Consider the following diagram

\[
\begin{array}{c}
D_1 \xrightarrow{h_1 f'} X' \xrightarrow{g'} A' \\
\downarrow f'^{h_1} \quad \downarrow f \\
C_1 \xrightarrow{h_1} X \xrightarrow{g} A
\end{array}
\]

Since the two square are pullbacks, then the big square is a pullback, and then

\[
(D_1 \xrightarrow{g' h_1^* f'} A, P f'^{h_1} (\beta_1)) = (D_1 \xrightarrow{g h_1} A, P f g h_1 (\beta_1))
\]

and these are by definition

\[
\exists^p f P^e f (\beta) = P^e f \exists^p (\beta).
\]

Therefore the Beck-Chevalley Condition is satisfied.

(ii) Now consider a morphism \(f: X \rightarrow A\) of \(a\), an element \(\alpha := (C_1 \xrightarrow{h_1} A, \alpha_1 \in PC_1)\) in \(P^e(A)\), and an element \(\beta = (D_2 \xrightarrow{h_2} X, \beta_2 \in PD_2)\) in \(P^e(X)\). Observe that the following diagram is a pullback

\[
\begin{array}{c}
D_2 \wedge D_1 \xrightarrow{(h_1^* f)_* h_2} D_1 \xrightarrow{f'^{h_1}} C_1 \\
\downarrow (h_1^* f)^* h_2 \quad \downarrow h_1 \quad \downarrow h_1 \\
D_2 \xrightarrow{h_2} X \xrightarrow{f} A
\end{array}
\]

and this means that

\[
\exists^p f (P^e f (\alpha) \wedge \beta) = \alpha \wedge \exists^p f (\beta).
\]

Therefore the Frobenius Reciprocity is satisfied. 

Corollary 4.4. Let \(P: C^{op} \rightarrow \text{InfSL}\) be a primary doctrine, then the doctrine \(P^e: C^{op} \rightarrow \text{InfSL}\) is existential.

12
Remark 4.5. In the case that $a$ is the class of the projections, then from primary doctrine $P : C^{op} \longrightarrow \text{InfSL}$ it can be constructed an existential doctrine $P^e : C^{op} \longrightarrow \text{InfSL}$ in the sense of Definition 3.3. Therefore the notion of existential doctrine can be generalized in the sense that an existential doctrine can be defined as a pair

$$( P : C^{op} \longrightarrow \text{InfSL} , a )$$

where $P : C^{op} \longrightarrow \text{InfSL}$ is a primary doctrine and $a$ is a class of morphisms of $C$ closed by pullbacks, composition and identities, which satisfies the conditions of Theorem 4.3.

Remark 4.6. Let $P : C \longrightarrow \text{PosTop}$ be a functor where $\text{PosTop}$ is the category of posets with top element. We can apply the existential completion since we have not used the hypothesis that $PA$ has infimum during the proofs; we have proved that if it has an infimum it is preserved by the completion. Moreover we can express the Frobenius condition without using infima, and also this condition is preserved by completion.

The requirement of top element for the posets of the category $\text{PosTop}$ is due to the fact that we want an injection from the doctrine $P : C \longrightarrow \text{PosTop}$ into $P^e : C \longrightarrow \text{PosTop}$, since from a logical point of view, we are extending a theory where formulas have no occurrences of the symbol $E$, to an existential one, and we require that the theorems of the previous theory are preserved.

For the rest of this section we assume that the morphisms of $a$ are projections. We define a 2-functor $E : \text{PD} \longrightarrow \text{PED}$ sending a primary doctrine $P : C^{op} \longrightarrow \text{InfSL}$ into the existential doctrine $P^e : C^{op} \longrightarrow \text{InfSL}$. For all the standard notions about 2-category theory we refer to [2] and [14].

**Proposition 4.7.** Consider the category $\text{PD}(P, R)$. We define $E_{P,R} : \text{PD}(P, R) \longrightarrow \text{PED}(P^e, R^e)$ as follow:

- for every 1-cell $(F, b) : (F, b^e) : (F, b) : (F, b^e)$, where $b^e : P^e A \longrightarrow R^e FA$ sends an object $( C \xrightarrow{g} A , \alpha )$ in the object $( FC \xrightarrow{Fg} FA , b_C(\alpha) )$;
- for every 2-cell $\theta : (F, b) \Rightarrow (G, c) : (F, b) \Rightarrow (G, c)$, $E_{P,R}\theta$ is essentially the same.

With the previous assignment $E$ is a 2-functor.

**Proof.** We prove that $(F, b^e) : P^e \longrightarrow R^e$ is a 1-cell of $\text{PED}(P^e, R^e)$. We first prove that for every $A \in C$, $b^e_A$ preserves the order.

If $( C_1 \xrightarrow{g_1} A , \alpha_1 ) \leq ( C_2 \xrightarrow{g_2} A , \alpha_2 )$, we have a morphism $w : C_1 \longrightarrow C_2$ such that the
following diagram commutes

\[ \begin{array}{ccc}
  C_1 & \xrightarrow{w} & C_2 \\
  \downarrow{g_1} & & \downarrow{g_2} \\
  A & \xrightarrow{\alpha} & A
\end{array} \]

and \( \alpha_1 \leq P_w(\alpha_2) \). Since \( b \) is a natural transformation, we have that \( b_{C_1}P_w = R_{Fw}b_{C_2} \).

Then we can conclude that \( ( FC_1 \xrightarrow{Fg_1} FA , \ b_{C_1}(\alpha_1)) \) \( \leq ( FC_2 \xrightarrow{Fg_2} FA , \ b_{C_2}(\alpha_2)) \) because \( Fg_2Fw = Fg_1 \) and \( R_{Fw}(b_{C_2}\alpha_2) = b_{C_1}P_w(\alpha_2) \geq b_{C_1}(\alpha_1) \). Moreover, since \( F \) preserves products, we can conclude that \( b' \) preserves inf.

One can prove that \( b^e : P^e \longrightarrow R^e F^{op} \) is a natural transformation using the facts that \( F \) preserves products. Moreover we can easily see that \( b^e \) preserves the left adjoints along projections. Then \( (F,b^e) \) is a 1-cell of PED.

Now consider a 2-cell \( \theta : (F,b) \Longrightarrow (G,c) \), and let \( \alpha := ( C_1 \xrightarrow{g_1} A , \ \alpha_1) \) be an object of \( P^e(A) \). Then

\[ b'_{\alpha}(\alpha) = ( FC_1 \xrightarrow{Fg_1} FA , \ b_{C_1}(\alpha_1)) \]

and

\[ R_{\theta_A}c_A^e(\alpha) = ( D_1 \xrightarrow{Gg_1^\theta_A} FA , \ R_{Gg_1^\theta_A}c_{C_1}(\alpha_1)) \]

where

Now observe that since \( \theta : F \longrightarrow G \) is a natural transformation, there exists a unique \( w : FC_1 \longrightarrow D_1 \) such that the diagram

\[ \begin{array}{ccc}
  FC_1 & \xrightarrow{w} & D_1 \\
  \downarrow{\theta_{C_1}} & & \downarrow{\theta_A} \\
  GC_1 & \xrightarrow{Gg_1} GA
\end{array} \]

commutes and then \( R_wR_{\theta_A}c_{C_1}(\alpha_1) = R_{\theta_{C_1}}c_{C_1}(\alpha_1) \geq b_{C_1}(\alpha_1) \). Therefore we can conclude that \( b'_{\alpha}(\alpha) \leq R_{\theta_A}c_A^e(\alpha) \), and then \( \theta : F \longrightarrow G \) can is a 2-cell \( \theta : (F,b^e) \Longrightarrow (G,c^e) \), and
Finally one can prove that the following diagram commutes observing that for every \((F, b) \in \text{PD}(P, R)\) and \((G, c) \in \text{PD}(R, D)\), \((GF, c \star b^e) = (GF, (c \ast b)^e)\).

\[
\begin{array}{ccc}
\text{PD}(P, R) \times \text{PD}(R, D) & \xrightarrow{e_{PRD}} & \text{PD}(P, D) \\
E_{PR \times ERD} & & \downarrow E_{PD} \\
\text{PED}(P^e, R^e) \times \text{PED}(R^e, D^e) & \xrightarrow{e_{PED^e}} & \text{PED}(P^e, D^e)
\end{array}
\]

and the same for the unit diagram.

Therefore we can conclude that \(E\) is a 2-functor.

Now we prove the 2-functor \(E: \text{PD} \rightarrow \text{PED}\) is left adjoint to the forgetful functor \(U: \text{PED} \rightarrow \text{PD}\).

Proposition 4.8. Let \(P: \text{C}^{\text{op}} \rightarrow \text{InfSL}\) be an elementary doctrine. Then

\[
(id_C, \iota_P): P \longrightarrow P^e
\]

where \(\iota_P: PA \longrightarrow P^eA\) sends \(\alpha\) into \((A \xrightarrow{id_A} A, \alpha)\) is a 1-cell. Moreover the assignment

\[
\eta: id_{\text{ED}} \longrightarrow UE
\]

where \(\eta_P := (id_C, \iota_P)\), is a 2-natural transformation.

Proof. It is easy to prove that \(\iota_P: PA \longrightarrow P^eA\) preserves all the structures. For every morphism \(f: A \longrightarrow B\) of \(\text{C}\), it one can see that the following diagram commutes

\[
\begin{array}{ccc}
P_B & \xrightarrow{P_f} & PA \\
\downarrow \iota_{PB} & & \downarrow \iota_{PA} \\
P^e_B & \xrightarrow{P^e_f} & P^eA
\end{array}
\]

Then we can conclude that \((id_C, \iota_P): P \longrightarrow P^e\) is a 1-cell of \(\text{ED}\) and it is a direct verification the proof the \(\eta\) is a 2-natural transformation.

Proposition 4.9. Let \(P: \text{C}^{\text{op}} \rightarrow \text{InfSL}\) be an existential doctrine. Then

\[
(id_C, \zeta_P): P^e \longrightarrow P
\]

where \(\zeta_P: P^eA \longrightarrow PA\) sends \((C \xrightarrow{f} A, \alpha)\) in \(\exists f(\alpha)\) is a 1-cell. Moreover the assignment

\[
\varepsilon: EU \longrightarrow id_{\text{EED}}
\]

where \(\varepsilon_P = (id_C, \zeta_P)\), is a 2-natural transformation.
Proof. Suppose \((C_1 \xrightarrow{g_1} A, \alpha_1) \leq (C_2 \xrightarrow{g_2} A, \alpha_2)\), with \(w: C_1 \longrightarrow C_2\), \(g_2w = g_1\) and \(P_w(\alpha_2) \geq \alpha_1\). Then by Beck-Chevalley we have the equality

\[ \exists g_1 \exists g_2 (\alpha_2) = P_{g_1} \exists g_2 (\alpha_2) \]

and

\[ P_{g_1} \exists g_2 (\alpha_2) = P_w P_{g_2} \exists g_2 (\alpha_2) \geq P_w(\alpha_2) \geq \alpha_1 \]

Then

\[ \exists g_1 (\alpha_1) \leq \exists g_1 \exists g_2 (\alpha_2) = \exists g_2 \exists g_1 \exists g_2 (\alpha_2) \leq \exists g_2 (\alpha_2) \]

and \(\delta_A = \zeta_A(\ A \xrightarrow{id_A} A, \ T_A)\). Now we prove the naturality of \(\zeta_P\). Let \(f: A \longrightarrow B\) be a morphism of \(C\). Then the following diagram commutes

\[
\begin{array}{ccc}
PB & \xrightarrow{Pf} & PA \\
\downarrow{\zeta_B} & & \downarrow{\zeta_A} \\
P^e B & \xrightarrow{P^e f} & P^e A
\end{array}
\]

because for every \((C \xrightarrow{g} B; \beta \in PC)\) we have \(\exists g \exists f P_r \exists g (\beta) = P_f \exists g (\beta)\) by Beck-Chevalley. Moreover it is easy to see that \(\zeta_P\) preserves left-adjoints. Then we an conclude that for every elementary existential doctrine \(P: \mathcal{C}^{op} \longrightarrow \mathsf{InfSL}\), \(\zeta_P\) is a 1-cell of \(EED\).

The proof of the naturality of \(\varepsilon\) is a routine verification. One must use the fact that we are working in \(EED\), and then for every 1-cell \((F, b)\), \(b\) preserves left-adjoints along the projections. 

\(\blacksquare\)

**Proposition 4.10.** For every primary doctrine \(P: \mathcal{C}^{op} \longrightarrow \mathsf{InfSL}\) we have

\[ \varepsilon_{P^e} \circ \eta_{P^e} = \text{id}_P \]

**Proof.** Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{C}^{op} & \xrightarrow{P^e} & \mathsf{InfSL} \\
\downarrow{id_{\mathcal{C}^{op}}} & & \downarrow{\zeta_{P^e}} \\
\mathcal{C}^{op} & \xrightarrow{(P^e)^e} & \mathsf{InfSL} \\
\downarrow{id_{\mathcal{C}^{op}}} & & \downarrow{\zeta_{(P^e)^e}} \\
C & \xrightarrow{P^e} & \mathsf{InfSL}
\end{array}
\]

16
and let \(( C \xrightarrow{g} A, \alpha \in PA)\) be an element of \(P^e A\). Then
\[
\iota_{P^e A}( C \xrightarrow{g} A, \alpha \in PC) = ( A \xrightarrow{id_A} A, ( C \xrightarrow{g} A, \alpha \in PC) \in P^e A)
\]
and
\[
\zeta_{P^e A}( A \xrightarrow{id_A} A, ( C \xrightarrow{g} A, \alpha \in PC) \in P^e A) = \exists^{e}_{id_A}( C \xrightarrow{g} A, \alpha \in PC)
\]
By definition of \(\exists^e\) we have
\[
\exists^{e}_{id_A}( C \xrightarrow{g} A, \alpha \in PC) = ( C \xrightarrow{g} A, \alpha \in PC)
\]
Then we can conclude that for every \(P: C^{op} \to \inf\) we have \(\varepsilon_P \circ \eta_P = id_P\).

**Corollary 4.11.**
\[
\varepsilon E \circ E \eta = id_E
\]

**Proposition 4.12.** For every existential doctrine \(P: C^{op} \to \inf\) we have
\[
\varepsilon_P \circ \eta_P = id_P
\]

**Proof.** One can check it directly.

**Corollary 4.13.**
\[
U \varepsilon \circ \eta U = id_U
\]

**Theorem 4.14.** The 2-functor \(E\) is 2-adjoint to the 2-functor \(U\).

5  The 2-monad \(T_e\)

In this section we construct a 2-monad \(T_e: PD \to PD\), and we prove that every existential doctrine can be seen as an algebra for this 2-monad. Finally we show that the 2-monad \(T_e\) is lax-idempotent.

We define:

- \(T_e: ED \to ED\) the 2-functor \(T = U \circ E\);
- \(\eta: id_{ED} \to T_e\) is the 2-natural transformation defined in Proposition 4.8;
- \(\mu: T^2_e \to T_e\) is the 2-natural transformation \(\mu = U \varepsilon E\);

**Proposition 5.1.** \(T_e\) is a 2-monad.
Proof. One can easily check that the following diagrams commute

\[
\begin{array}{c}
\begin{array}{ccc}
T^3_e & \xrightarrow{\mu T_e} & T^2_e \\
\downarrow{T_e,\mu} & & \downarrow{\mu} \\
T^2_e & \xrightarrow{\mu} & T_e
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\text{id}_{ED} \circ T_e & \xrightarrow{\eta T_e} & T^2_e \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
T_e & \xrightarrow{\mu} & T_e
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
T^2_e & \xrightarrow{T_e \eta} & T_e \circ \text{id}_{ED} \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
T_e & \xrightarrow{\text{id}} & T_e
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
T^3_e & \xrightarrow{\mu T_e} & T^2_e \\
\downarrow{T_e,\mu} & & \downarrow{\mu} \\
T^2_e & \xrightarrow{\mu} & T_e
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\text{id}_{ED} \circ T_e & \xrightarrow{\eta T_e} & T^2_e \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
T_e & \xrightarrow{\mu} & T_e
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
T^2_e & \xrightarrow{T_e \eta} & T_e \circ \text{id}_{ED} \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
T_e & \xrightarrow{\text{id}} & T_e
\end{array}
\end{array}
\]

Remark 5.2. Observe that $T^2_e \simeq T_e$, and $\mu_P$ is an isomorphism.

Proposition 5.3. Let $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ be an existential doctrine. Then $(P, \varepsilon_P)$ is a $T_e$-algebra.

Proof. It is a direct verification.

Proposition 5.4. Let $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ be a primary doctrine, and let $(P, (F, a))$ be a $T_e$-algebra. Then $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ is existential, $F = \text{id}_{\mathcal{C}}$ and $a = \varepsilon_P$.

Proof. By the identity axiom for $T_e$-algebras, we know that $F$ must be the identity functor, and $a_{A \! \! \to A} = \text{id}_{PA}$. 

Now for every morphism $f : A \to B$ of $\mathcal{C}$, where $f$ is a projection, we claim that 

\[
\exists_f(\alpha) := a_B \exists_f a_A(\alpha)
\]

is left adjoint to $P_f$. Let $\alpha \in PA$ and $\beta \in PB$, and suppose that $\alpha \leq P_f(\beta)$. Then we have that

\[
( A \xrightarrow{f} B , \alpha) \leq ( B \xrightarrow{id_B} B , \beta) \]

in $P^e B$ and $( A \xrightarrow{f} B , \alpha) = \exists_f( A \xrightarrow{id_A} A , \alpha)$. Therefore, by definition of $\iota$, we have

\[
\exists_f a_A(\alpha) \leq \iota_B(\beta)
\]
and then
\[ a_B \mathfrak{F}_f \iota_A(\alpha) \leq a_B \iota_B(\beta) = \beta \]

Now suppose that \( \mathfrak{F}_f(\alpha) \leq \beta \). Then
\[ a_B( A \xrightarrow{f} B, \alpha) \leq \beta \]
so
\[ P_f a_B( A \xrightarrow{f} B, \alpha) \leq P_f(\beta) \]

Now we use the naturality of \( a \), and we have
\[ P_f a_B( A \xrightarrow{f} B, \alpha) = a_A P_f( A \xrightarrow{f} B, \alpha) \]

Now observe that \( P_f( A \xrightarrow{f} B, \alpha) \geq ( A \xrightarrow{id} A, \alpha) = \iota_A(\alpha) \). Therefore we can conclude that
\[ \alpha = a_A \iota_A(\alpha) \leq P_f a_B( A \xrightarrow{f} B, \alpha) \leq P_f(\beta) \]

Now we prove that Bech-Chevalley holds. Consider the following pullback

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & A' \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{g} & A
\end{array}
\]

and \( \alpha \in PX \). Then we have
\[ \mathfrak{F}_g P_f'(\alpha) = a_{A'} \mathfrak{F}_g' \iota_{X'}(P_f'\alpha) = a_{A'}( X' \xrightarrow{g'} A', P_f'(\alpha)) \]

Observe that
\[ ( X' \xrightarrow{g'} A', P_f'(\alpha)) = P_f'( X \xrightarrow{g} A, \alpha) \]
and since \( a \) is a natural transformation, we have
\[ a_{A'} P_f'( X \xrightarrow{g} A, \alpha) = P_f a_A( X \xrightarrow{g} A, \alpha) \]

Finally we can conclude that Bech-Chevalley holds because
\[ P_f \mathfrak{F}_g(\alpha) = P_f a_A \mathfrak{F}_g \iota_X(\alpha) = P_f a_A( X \xrightarrow{g} A, \alpha) \]
and then
\[ \mathfrak{F}_g P_f'(\alpha) = P_f \mathfrak{F}_g(\alpha) \]
Now consider a projection \( f: A \to B \), an two elements \( \beta \in PB \) and \( \alpha \in PA \). We want to prove that the Frobenius reciprocity holds.

\[
\exists_f(P_f(\beta) \land \alpha) = a_B \exists_f \iota_A(P_f(\beta) \land \alpha) = a_B( A \xrightarrow{f} B, P_f(\beta) \land \alpha)
\]

and

\[
\beta \land \exists_f(\alpha) = a_B \iota_B(\beta) \land a_B( A \xrightarrow{f} B, \alpha)
\]

and

\[
a_B \iota_B(\beta) \land a_B( A \xrightarrow{f} B, \alpha) = a_B(( B \xrightarrow{id_B} B, \beta) \land ( A \xrightarrow{f} B, \alpha))
\]

We can observe that

\[
a_B(( B \xrightarrow{id_B} B, \beta) \land ( A \xrightarrow{f} B, \alpha)) = a_B( A \xrightarrow{f} B, P_f(\beta) \land \alpha)
\]

Then we can conclude that

\[
\exists_f(P_f(\beta) \land \alpha) = \beta \land \exists_f(\alpha)
\]

Therefore the primary doctrine \( P: C^{\text{op}} \to \InfSL \) is existential. Finally we can observe that

\[
a_A( C \xrightarrow{g} A, \alpha) = a_A \exists_f^R( C \xrightarrow{id_C} C, \alpha) = a_A \exists_f^R \iota_C(\alpha) = \exists_g(\alpha)
\]

\[\square\]

**Proposition 5.5.** Let \((P, \varepsilon_P)\) and \((R, \varepsilon_R)\) be two \( T_e \)-algebras. If \((F, b): (P, \varepsilon_P) \to (R, \varepsilon_R)\) is a morphism of \( T_e \)-algebras, then \((F, b)\) is a 1-cell of \( \PED \). Moreover every 1-cell of \( \PED \) induces a morphism of \( T_e \)-algebras.

**Proof.** By definition of morphism of \( T_e \)-algebras, the following diagram commutes

\[
\begin{array}{ccc}
P^e & \xrightarrow{(F, b)^e} & R^e \\
\varepsilon_P & & \varepsilon_R \\
P & \xrightarrow{(F, b)} & R \\
\end{array}
\]

then for every object \(( C \xrightarrow{g} A, \alpha \in PC)\) of \( P^e A \) we have

\[
\exists^R_{Fg} b_C(\alpha) = b_A \exists^R_{g}(\alpha)
\]

and this means that for every projection \(( g: C \to A)\) the following diagram commutes

\[
\begin{array}{ccc}
PC & \xrightarrow{\exists^R_g} & PA \\
b_C & & b_A \\
RFC & \xrightarrow{\exists^R_{Fg}} & RFA \\
\end{array}
\]

20
We can prove the converse using the same arguments.

**Corollary 5.6.** We have the following isomorphism\[ T_{e}\text{-Alg} \cong PED \]

**Proof.** It follows from Proposition 5.5 and Proposition 5.4.

**Proposition 5.7.** Let \( P : C^{op} \longrightarrow \text{InfSL} \) be a primary doctrine, and let \( (P,(F,a)) \) be a pseudo-\( T_{e}\)-algebra. Then \( P : C^{op} \longrightarrow \text{InfSL} \) is existential.

**Proof.** Let \( (P,(F,a)) \) be a pseudo-algebra, then there exists an invertible 2-cell

\[
\begin{array}{ccc}
P & \xrightarrow{\eta_A} & P_{e} \\
\downarrow{id_P} & \searrow{\beta_{a_P}} & \downarrow{(F,a)} \\
P & \downarrow{(F,a)} & P
\end{array}
\]

and by definition, it is a natural transformation \( a_{\eta} : F \longrightarrow \text{id}_{C} \), and for every \( A \in C \) and \( \alpha \in PA \) we have \( a_{A}(\alpha) = P_{a_{\eta}(\alpha)} \).

Now consider a morphism \( f : A \longrightarrow B \) in \( C \) and \( \alpha \in PA \). We define

\[
\exists_{f}(\alpha) := P_{a_{\eta}(\alpha)\beta_{a_B}a_{\eta}(\alpha)}
\]

Using the same argument of Proposition 5.4 we can conclude that the elementary doctrine \( P : C^{op} \longrightarrow \text{InfSL} \) is existential.

**Proposition 5.8.** \( \lambda_{P} : \text{id}_{P^{e_{c}}} \Longrightarrow \eta_{P^{e_{c}}} \mu_{P} \) defined as \( \lambda_{P} := \text{id}_{C} \) is a 2-cell in \( ED \).

**Proof.** It is clearly a natural transformation. We must check that for every \( \alpha \in (P^{e_{c}})^{e}A \)

\[
\alpha \leq \iota_{P^{e_{c}}A} \zeta_{P^{e_{c}}A}(\alpha)
\]

Let \( \alpha := (C \xrightarrow{g} A, (D \xrightarrow{f} C, \beta \in PD)) \). Then we have

\[
\iota_{P^{e_{c}}A} \zeta_{P^{e_{c}}A}(\alpha) = \iota_{P^{e_{c}}A}(D \xrightarrow{gf} A, \beta \in PD) = (A \xrightarrow{id_{A}} A, (D \xrightarrow{gf} A, \beta \in PD))
\]

Now we want to prove that

\[
P_{g}(D \xrightarrow{gf} A, \beta \in PD) \geq (D \xrightarrow{f} C, \beta \in PD)
\]

To see this inequality we can observe that the following diagram commutes, since every square is a pullback
and then $P_w(P_m(\beta)) = \beta$.

**Corollary 5.9.** We can define a modification $\lambda: \text{id}_{T^2} \to \eta T, \mu$, where $\lambda_P$ is defined as above.

**Theorem 5.10.** The 2-natural transformation $\mu$ is left adjoint to $\eta T, \epsilon$, where the unit of the adjunction is $\lambda$ and the counit is the identity.

**Proof.** It follows from the fact that for every $P: C \to \text{InfSL}$, the first component of the 1-cells $\mu_P, \eta T, \epsilon$ are the identity functor, and since $\lambda_P$ is the identity natural transformation, when we look at the conditions of adjoint 1-cell in the 2-category $\text{Cat}$, we can observe that all the components are identities.

**Corollary 5.11.** The 2-monad $T_e: PD \to PD$ is lax-idempotent.

**Proof.** It is a direct consequence of [9, Theorem 6.2] and Theorem 5.10.

Observe that we can prove that the 2-monad $T_e$ is lax-idempotent directly.

**Proposition 5.12.** Let $(P, \varepsilon_P)$ and $(R, \varepsilon_R)$ be $T_e$ algebras, and let $(F, b): P \to R$ be a 1-cell of $PD$. Then $(F, b, \text{id}_F)$ is lax-morphism of algebras, and $\text{id}_F: \varepsilon_R(F, b^e) \Rightarrow (F, b)\varepsilon_P$ is the unique 2-cell making $(\text{id}_F, (F, b))$ a lax-morphism.

**Proof.** Consider the following diagram

We must prove that for every object $A$ of $C$ and every $(C \to A, \alpha)$ in $P^e A$

$$\exists_F b_C(\alpha) \leq b_A \exists_f^R(\alpha)$$

22
but the previous holds if and only if
\[ b_C(\alpha) \leq R_F b_A \mathfrak{A}^P_f(\alpha) = b_C P_f \mathfrak{A}^P_f(\alpha) \]
and this holds since \( \alpha \leq P_f \mathfrak{A}^P_f(\alpha) \).

Finally it is easy to see that \( \text{id}_F : \varepsilon_R(F, b^e) \Rightarrow (F, b)^e_P \) satisfies the coherence conditions for lax-\( T_e \)-morphisms.

Now suppose there exists another 2-cell \( \theta : \varepsilon_R(F, b^e) \Rightarrow (F, b)^e_P \) such that \( ((F, b), \theta) \) is a lax-morphism

\[
\begin{array}{c}
P^e \\
\downarrow \varepsilon_P \\
(\varepsilon, \eta) \\
\downarrow \theta \\
P \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
R \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
\downarrow \varepsilon_R \\
R \\
\end{array}
\]

Then it must satisfy the following condition

\[
\begin{array}{c}
P \\
\downarrow \varepsilon_P \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
R \\
\downarrow \varepsilon_R \\
(\varepsilon, \eta) \\
\downarrow \varepsilon_R \\
R \\
\end{array}
\]

and this means that \( \theta = \text{id}_F \).

6 Exact completion for elementary doctrine

It is proved in [18] that there is a bi-adjunction between the category \( \text{EED} \to \text{Xct} \) given by the composition of the following 2-functors: the first is the left biadjoint to the inclusion of \( \text{CEED} \) into \( \text{EED} \), see [18, Theorem 3.1]. The second is the biequivalence between \( \text{CEED} \) and the 2-category \( \text{LFS} \) of categories with finite limits and a proper stable factorization system, see [7]. The third is provided in [8], where it is proved that the inclusion of the 2-category \( \text{Reg} \) of regular categories (with exact functors) into \( \text{LFS} \) has a left bi-adjoint. The last functor is the bi-adjoint to the forgetful functor from the 2-category \( \text{Xct} \) into \( \text{Reg} \), see [5].

In this section we combine these results with the existential completion for elementary doctrine, proving that the completion presented in Section 4 preserves the elementary doctrines, in the sense that if \( P : C^{\text{op}} \to \text{InfSL} \) is an elementary doctrine, then \( P^e : C^{\text{op}} \to \text{InfSL} \) is an
elementary existential doctrine.

Let $P: C^{\text{op}} \longrightarrow \text{InfSL}$ be an elementary doctrine, and consider its existential completion $P^e: C^{\text{op}} \longrightarrow \text{InfSL}$. Given two objects $A$ and $C$ of $C$ we define

$$\mathfrak{X}_{A \times \text{id}_C}: P^e(A \times C) \longrightarrow P^e(A \times A \times C)$$

on $\alpha := (A \times C \times D \xrightarrow{\text{pr}} A \times C, \alpha \in P(A \times C \times D))$ as

$$\mathfrak{X}_{A \times \text{id}_C}(\alpha) := (A \times A \times C \times D \xrightarrow{\text{pr}} A \times A \times C, \mathfrak{X}_{A \times \text{id}_C}(\alpha) \in P(A \times A \times C \times D))$$

**Remark 6.1.** We can prove that $\mathfrak{X}_{A \times \text{id}_C}$ is a well defined functor for every $A$ and $C$: consider two elements of $P^e(A \times C)$

$$\overline{\alpha} := (A \times C \times D \xrightarrow{\text{pr}} A \times C, \alpha \in P(A \times C \times D))$$

and

$$\overline{\beta} = (A \times C \times B \xrightarrow{\text{pr}'} A \times C, \beta \in P(A \times C \times B))$$

and suppose that $\overline{\alpha} \leq \overline{\beta}$. By definition there exists a morphism $f: A \times C \times D \longrightarrow B$ such that the following diagram commutes

\[
\begin{array}{ccc}
A \times C \times D & \xrightarrow{\langle \text{pr}_{A \times C}, f \rangle} & A \times C \\
\downarrow \text{pr}_{A \times C} & & \downarrow \text{pr}_{A \times C} \\
A \times C \times B & \xrightarrow{\text{pr}'_{A \times C}} & A \times C
\end{array}
\]

and $P_{\langle \text{pr}_{A \times C}, f \rangle}(\beta) \geq \alpha$. Since the doctrine $P: C^{\text{op}} \longrightarrow \text{InfSL}$ is elementary we have

$$\beta \leq P_{\Delta \times \text{id}_{C \times B}} \mathfrak{X}_{\Delta \times \text{id}_{C \times B}}(\beta)$$

and then

$$\alpha \leq P_{\langle \text{pr}_{A \times C}, f \rangle}(P_{\Delta \times \text{id}_{C \times B}} \mathfrak{X}_{\Delta \times \text{id}_{C \times B}}(\beta))$$

Now observe that $(\Delta \times \text{id}_{C \times B})(\langle \text{pr}_{A \times C}, f \rangle) = ((\text{pr}_{A \times A \times C}, f \text{pr}_{A \times C \times D}))(\Delta \times \text{id}_{C \times D})$, and this implies

$$\alpha \leq P_{\Delta \times \text{id}_{C \times D}}(P_{\langle \text{pr}_{A \times A \times C}, \text{pr}_{A \times C \times D} \rangle} \mathfrak{X}_{\Delta \times \text{id}_{C \times B}}(\beta))$$

Therefore we conclude

$$\mathfrak{X}_{\Delta \times \text{id}_{C \times D}}(\alpha) \leq P_{\langle \text{pr}_{A \times A \times C}, \text{pr}_{A \times C \times D} \rangle} \mathfrak{X}_{\Delta \times \text{id}_{C \times B}}(\beta).$$

It is easy to observe that the last inequality implies

$$\mathfrak{X}_{\Delta \times \text{id}_{C \times D}}(\overline{\alpha}) \leq \mathfrak{X}_{\Delta \times \text{id}_{C \times B}}(\overline{\beta}).$$
Proposition 6.2. With the notation used before the functor
\[ \mathfrak{A}^e_{\Delta \times \text{id}_C} : P^e(A \times C) \longrightarrow P^e(A \times A \times C) \]
is left adjoint to the functor
\[ P^e_{\Delta \times \text{id}_C} : P^e(A \times A \times C) \longrightarrow P^e(A \times C) \]

Proof. Consider an element \( \overline{\alpha} \in P^e(A \times C) \),
\[
\overline{\alpha} := (A \times C \xrightarrow{pr} A \times C, \alpha \in P(A \times C \times B))
\]
and an element \( \overline{\beta} \in P^e(A \times A \times C) \),
\[
\overline{\beta} := (A \times A \times C \xrightarrow{pr'} A \times A \times C, \beta \in P(A \times A \times C \times D))
\]
and suppose that
\[
\mathfrak{E}^e_{\Delta \times \text{id}_C}(\overline{\alpha}) \leq \overline{\beta}
\]
which means that there exists \( f : A \times A \times C \times B \longrightarrow D \)
such that \( \mathfrak{E}^e_{\Delta \times \text{id}_C}(\alpha) \leq P^{(pr_{A \times A \times C})}_{A \times A \times C}(\beta) \). Therefore we have
\[
\alpha \leq P_{\Delta \times \text{id}_{C \times B}} P^{(pr_{A \times A \times C})}_{A \times A \times C}(\beta)
\]
and since
\[
((pr_{A \times A \times C}))((\Delta_A \times \text{id}_{C \times B}) = (\Delta_A \times \text{id}_{C \times D}) pr_{A \times C \times D}((pr_{A \times A \times C})((\Delta_A \times \text{id}_{C \times B})) \]
we can conclude that
\[
\alpha \leq P_{pr_{A \times C \times D}}(pr_{A \times A \times C}((\Delta_A \times \text{id}_{C \times B}))(P_{\Delta \times \text{id}_{C \times D}}(\beta))
\]
and then
\[
\overline{\alpha} \leq P^e_{\Delta \times \text{id}_C}(\overline{\beta})
\]
because
\[
P^e_{\Delta \times \text{id}_C}(\overline{\beta}) = (A \times C \times D \xrightarrow{pr_{A \times C}} A \times C, P_{\Delta \times \text{id}_{C \times D}}(\beta))
\]
In the same way we can prove that \( \overline{\alpha} \leq P^e_{\Delta \times \text{id}_C}(\overline{\beta}) \) implies \( \mathfrak{E}^e_{\Delta \times \text{id}_C}(\overline{\alpha}) \leq \overline{\beta}. \) □
Proposition 6.3. For every $A$ and $C$, $\exists^e_{\Delta A \times \text{id}_C}$ satisfies the Frobenius condition.

Proof. Consider $\alpha \in P^e(A \times A \times C),\alpha = (A \times A \times C \times D \xrightarrow{\text{pr}_{A \times A \times C}} A \times A \times C, \alpha \in P(A \times A \times C \times D))$

and $\beta \in P^e(A \times C),\beta = (A \times C \times B \xrightarrow{\text{pr}_{A \times C}} A \times C, \beta \in P(A \times C \times B))$

We can observe that

$$P^e_{\Delta A \times \text{id}_C}(\alpha) = (A \times C \times D \xrightarrow{\text{pr}_{A \times C}} A \times C, P_{\Delta A \times \text{id}_{C \times D}}(\alpha))$$

and

$$P^e_{\Delta A \times \text{id}_C}(\alpha) \wedge \beta = (A \times C \times D \times B \xrightarrow{\text{pr}_{A \times C}} A \times C, P_{(\text{pr}_{A \times C} \circ \text{pr}_{D})} P_{\Delta A \times \text{id}_{C \times D}}(\alpha) \wedge P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta))$$

Moreover we can observe that $(\Delta_A \times \text{id}_{C \times D})(\text{pr}_{A}, \text{pr}_{C}, \text{pr}_{D}) = \text{pr}_{A \times A \times C \times D}(\Delta_A \times \text{id}_{C \times D \times B})$. Therefore we have

$$\exists^e_{\Delta A \times \text{id}_C}(P^e_{\Delta A \times \text{id}_C}(\alpha) \wedge \beta)$$

is equal to

$$(A \times A \times C \times D \times B \xrightarrow{\text{pr}} A \times A \times C, \exists^e_{\Delta A \times \text{id}_{C \times D \times B}}(P_{(\Delta_A \times \text{id}_{C \times D})(\text{pr}_{A \times C} \circ \text{pr}_{D})}(\alpha) \wedge P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta)))$$

Now we can observe that

$$\exists^e_{\Delta A \times \text{id}_{C \times D \times B}}(P_{(\Delta_A \times \text{id}_{C \times D})(\text{pr}_{A \times C} \circ \text{pr}_{D})}(\alpha) \wedge P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta))$$

is by definition

$$\exists^e_{\Delta A \times \text{id}_{C \times D \times B}}(P_{(\Delta_A \times \text{id}_{C \times D \times B}) P_{\text{pr}_{A \times A \times C \times D}}(\alpha) \wedge P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta)).$$

Since the doctrine $P : C^{op} \longrightarrow \text{InfSL}$ is elementary, it holds the Frobenius reciprocity for $\exists$, and therefore the previous is equal to

$$P_{\text{pr}_{A \times A \times C \times D}}(\alpha) \wedge \exists^e_{\Delta A \times \text{id}_{C \times D \times B}} P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta).$$

Then we have that

$$\exists^e_{\Delta A \times \text{id}_C}(P^e_{\Delta A \times \text{id}_C}(\alpha) \wedge \beta)$$

is equal to

$$(A \times A \times C \times D \times B \xrightarrow{\text{pr}_{A \times A \times C}} A \times A \times C, P_{\text{pr}_{A \times A \times C \times D}}(\alpha) \wedge \exists^e_{\Delta A \times \text{id}_{C \times D \times B}} P_{(\text{pr}_{A \times C} \circ \text{pr}_{B})}(\beta)).$$
Now we look for \( \forall \xi \land \exists_{\Delta \times \text{id}_C} (\beta) \). It is straightforward to prove that the previous is equal to

\[
(A \times A \times C \times D \times B \xrightarrow{\text{pr}_{A \times A \times C \times D}} A \times A \times C , \quad P_{\text{pr}_{A \times A \times C \times D}}(\alpha) \land P_{\text{pr}_{A \times A \times C \times D}}(\beta))
\]

Since \( P : C^{\text{op}} \to \text{InfSL} \) is elementary we know that

\[
\exists_{\Delta \times \text{id}_C \times B}(\beta) = P_{(\text{pr}_{A \times A \times C \times B})}(\beta) \land P_{(\text{pr}_{A \times A \times C \times B})}(\delta_A)
\]

where \( \text{pr}'_A : A \times A \times C \times B \to A \) is the projection on the second component. By a direct computation we have

\[
P_{(\text{pr}_{A \times A \times C \times B})}(\beta) \land P_{(\text{pr}_{A \times A \times C \times B})}(\delta_A) = P_{(\text{pr}'_A \times C \times B)}(\beta) \land P_{(\text{pr}'_A \times C \times B)}(\delta_A)
\]

and

\[
\exists_{\Delta \times \text{id}_C \times D \times B}(\beta) = P_{(\text{pr}'_A \times C \times D \times B)}(\beta) \land P_{(\text{pr}'_A \times C \times D \times B)}(\delta_A).
\]

It is direct to verify that

\[
P_{(\text{pr}'_A \times C \times D \times B)}(\beta) \land P_{(\text{pr}'_A \times C \times D \times B)}(\delta_A) = P_{(\text{pr}'_A \times C \times D \times B)}(\beta) \land P_{(\text{pr}'_A \times C \times D \times B)}(\delta_A).
\]

Therefore the Frobenius condition is satisfied.

\[\square\]

**Corollary 6.4.** For every elementary doctrine \( P : C^{\text{op}} \to \text{InfSL} \), the doctrine \( P^e : C^{\text{op}} \to \text{InfSL} \) is elementary and existential.

We combine the existential completion for elementary doctrines with the completions stated at the begin of this section, obtaining a general version of the exact completion described in [15,18]. We can summarise this operation with the following diagram

\[
\text{PED} \longrightarrow \text{EED} \longrightarrow \text{CEED} \longrightarrow \text{LFS} \longrightarrow \text{Reg} \longrightarrow \text{Xct}.
\]

Given an elementary existential doctrine \( P : C^{\text{op}} \to \text{InfSL} \), the completion \( \text{EED} \to \text{Xct} \) produces an exact category denoted by \( T_P \) and this category is defined following the same idea used to define a topos from a tripos. See [6,15,18,22].

We conclude giving a complete description of the exact category \( T_{P^e} \) obtained from an elementary doctrine \( P : C^{\text{op}} \to \text{InfSL} \).

Given an elementary doctrine \( P : C^{\text{op}} \to \text{InfSL} \), consider the category \( T_{P^e} \), called **exact completion of the elementary doctrine** \( P \), whose **objects** are pair \((A, \rho)\) such that \( \rho \) is in \( P(A \times A \times C) \) for some \( C \) and satisfies:

1. there exists a morphism \( f : A \times A \times C \to C \) such that

\[
\rho \leq P_{(\text{pr}_2, f)}
\]

in \( P(A \times A \times C) \) where \( \text{pr}_1, \text{pr}_2 : A \times A \times C \to A \);
2. there exists a morphism \( g: A \times A \times A \to C \) such that
\[
P_{(pr_1,pr_2,pr_4)}(\rho) \land P_{(pr_2,pr_3,pr_4)}(\rho) \leq P_{(pr_1,pr_3,g)}(\rho)
\]
where \( pr_1, pr_2, pr_3: A \times A \times A \to A \);

**a morphism** \( \phi: (A, \rho) \to (B, \sigma) \), where \( \rho \in P(A \times A \times C) \) and \( \sigma \in P(B \times B \times D) \), is an object \( \phi \) in \( P(A \times B \times E) \) for some \( E \) such that

1. there exists a morphism \( \langle f_1, f_2 \rangle: A \times B \times E \to C \times D \) such that
\[
\phi \leq P_{(pr_1,pr_1,f_1)}(\rho) \land P_{(pr_2,pr_2,f_2)}(\sigma)
\]
where the \( pr_i \)'s are the projections from \( A \times B \times E \);

2. there exists a morphism \( h: A \times A \times B \times C \times E \to E \) such that
\[
P_{(pr_1,pr_2,pr_4)}(\rho) \land P_{(pr_2,pr_3,pr_4)}(\phi) \leq P_{(pr_1,pr_3,h)}(\phi)
\]
where the \( pr_i \)'s are the projections from \( A \times A \times B \times C \times E \);

3. there exists a morphism \( k: A \times B \times B \times D \times E \to E \) such that
\[
P_{(pr_2,pr_3,pr_4)}(\sigma) \land P_{(pr_1,pr_2,pr_4)}(\phi) \leq P_{(pr_1,pr_3,k)}(\phi)
\]
where the \( pr_i \)'s are the projections from \( A \times B \times B \times D \times E \);

4. there exists a morphism \( l: A \times B \times B \times E \to D \) such that
\[
P_{(pr_1,pr_2,pr_4)}(\phi) \land P_{(pr_1,pr_3,pr_4)}(\phi) \leq P_{(pr_2,pr_3,l)}(\sigma)
\]
where the \( pr_i \)'s are the projections from \( A \times B \times B \times E \);

5. there exists a morphism \( \langle g_1, g_2 \rangle: A \times C \to B \times E \) such that
\[
P_{(pr_1,pr_1,pr_2)}(\rho) \leq P_{(pr_1,g_1,g_2)}(\phi)
\]
where the \( pr_i \)'s are the projections from \( A \times C \).

The composition of two morphisms is defined following the same structure of the tripos to topos.

Therefore we conclude with the following theorem which generalized the exact completion for an elementary existential doctrine to an arbitrary elementary doctrine.

**Theorem 6.5.** The 2-functor \( Xct \to PED \) that sends an exact category to the elementary doctrine of its subobjects has a left bi-adjoint which associates the exact category \( T_{P^e} \) to an elementary doctrine \( P: C^{op} \to \text{InfSL} \).
References


