A Constructive Kan–Quillen Model Structure

Nicola Gambino Christian Sattler Karol Szumiło

April 30, 2019

The Kan–Quillen model structure refers to the model structure on the category of simplicial sets, introduced by Quillen, comprising weak homotopy equivalences, Kan fibrations and monomorphisms. It has become one of the foundation stones of modern homotopy theory and many different proofs and treatments have been developed. None of them was fully constructive, however, which became a major obstacle for its use as a model of Homotopy Type Theory.

In joint research with Nicola Gambino and Christian Sattler, we have proven the following theorem (first obtained independently by Simon Henry).

Theorem (constructive logic)**.** *The category of simplicial sets carries a cofibrantly generated, cartesian, proper model structure where*

- *weak equivalences are weak homotopy equivalences;*
- *fibrations are Kan fibrations;*
- *cofibrations are Reedy decidable inclusions.* □

The fundamental reason why the standard approaches fail to be constructive is that they all rely on the classically trivial statement "a simplex of a simplicial set is either degenerate or non-degenerate".

In order to adapt this theory to the constructive setting, we modify the notion of a cofibration. A *decidable inclusion* is, in categorical terms, a map of sets $i \colon A \to B$ such that there is a map $C \to B$ that together with i exhibits B as the coproduct of A and C. In logical terms, "$x \in A$" (where x is a variable of type B) is a decidable proposition. Then a cofibration is a *Reedy decidable inclusion*, i.e., a simplicial map $A \to B$ such that for all $m \in \mathbb{N}$, the relative latching map $A_m \sqcup_{L_m A} L_m B \to B_m$ is a decidable inclusion. Logically, this means that the proposition "$x \in A$ or x is degenerate" (for x varying over B_m) is decidable. Replacing monomorphisms with cofibrations in this sense fixes the constructivity issues of the standard approaches, but it introduces new difficulties since not all simplicial sets are cofibrant any more. In particular, weak homotopy equivalences need to be defined with great care.

In our work we have obtained two different proofs of the theorem above. One is inspired by type theoretic developments and using techniques such as the "Forbenius property" and the "equivalence extension property". The other is based on classical methods of simplicial homotopy theory, including Kan's Ex functor and Quillen's Theorem A which need to be redeveloped carefully to ensure the constructivity of all arguments. Both treatments are given in a categorical language which has an advantage that it is potentially interpretable in categories other than that of simplicial sets. In particular, the basic theory of decidable maps relies solely on the fact the category of sets is extensive. In this talk I will present the latter approach.