Relative Partial Combinatory Algebras over Heyting Categories
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Abstract

A partial combinatory algebra (PCA) is an abstract model of computation that generalizes the classical notion of computability on the set of natural numbers. More precisely, it is a nonempty set equipped with a binary partial operation that satisfies an abstract version of the Smn-theorem. These models can be studied from the point of view of category theory. Every PCA \( A \) gives rise to a category of assemblies \( \text{Asm}(A) \), which may be viewed as the category of all data types that can be implemented in \( A \). A category of the form \( \text{Asm}(A) \) is always a quasitopos, and quasitoposes are closed under slicing. However, categories of assemblies for a PCA are not in general closed under slicing. Therefore, we wish to investigate what categories of the form \( \text{Asm}(A)/X \), where \( X \) is an assembly, look like.

The answer to this question is provided by W. Stekelenburg’s PhD thesis [Ste13], which generalizes the notion of a PCA in two ways. Firstly, the category of sets, which plays a crucial role in the construction of \( \text{Asm}(A) \), is replaced by a general Heyting category \( \mathcal{H} \). Secondly, the notion of computability is relativized by selecting a set of privileged subobjects of \( A \) that count as ‘realizing sets’. We therefore call these objects relative PCAs constructed over a Heyting category, or HPCAs for short. The construction of \( \text{Asm}(A) \) can be generalized to HPCAs \( A \), and if \( X \) is an assembly, then \( \text{Asm}(A)/X \) is of the form \( \text{Asm}(A') \) for some HPCA \( A' \). We describe this \( A' \) explicitly in terms of \( A \) and \( X \) and use this description to compute a number of examples of categories of the form \( \text{Asm}(A)/X \), where \( A \) is a PCA.

PCAs are the objects of a preorder-enriched category (see, e.g., [Lon94] and [vO08]). Similarly, HPCAs constructed over a given Heyting category \( \mathcal{H} \) can be made into a preorder-enriched category \( \text{PCA}_\mathcal{H} \). In this talk, we construct a larger 2-category \( \text{PCA} \), which is the total category of an opfibration over the category of Heyting categories and whose fiber above a Heyting category \( \mathcal{H} \) is precisely \( \text{PCA}_\mathcal{H} \). We investigate the structure of these categories, showing that \( \text{PCA} \) has small products, and that each of the fibers \( \text{PCA}_\mathcal{H} \) has finite (pseudo-)coproducts. Moreover, we extend the construction of \( \text{Asm}(A) \) to a 2-functor from \( \text{PCA} \) into the category of categories. We characterize the image of this 2-functor, thereby generalizing Longley’s work from [Lon94].

References

