# Artin glueings of frames and toposes as semidirect products

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**Abstract**

An Artin glueing [2] of two frames $H$ and $N$ is a frame $G$ in which $H$ and $N$ are included as sublocales, with $H$ open and $N$ its closed complement. Artin glueings are not unique, but are determined by finite-meet preserving maps $f\colon H \to N$. This notion categorifies to the setting of toposes. The details of these constructions can be found in [1].

Compare this to a semidirect product $G$ of two groups $N$ and $H$. Both $N$ and $H$ are subgroups of $G$, with $N$ being normal. They satisfy that $N \cap H = \{e\}$ and $NH = G$, which if thought of in terms of the lattice of subobjects of $G$, says that $N$ and $H$ are complements. Furthermore, just as with the Artin glueing, semidirect products of groups are not unique and are similarly determined by a map $f\colon H \to \mathrm{Aut}(N)$.

In order to make this connection precise we examine a link to extension problems. It is well known that the split extensions between any two groups $N$ and $H$ are precisely the semidirect products of $N$ and $H$. We show that Artin glueings of frames are the solutions to a natural extension problem in the category **RFrm** of frames with finite-meet preserving maps, and that Artin glueings of toposes are the solutions to a natural extension problem in the category **RTopos** of toposes with finite-limit preserving maps.

Talking about extensions requires appropriate notions of kernels and cokernels. We say a chain $N \xrightarrow{m} G \xrightarrow{e} H$ is an extension when $m$ is the kernel of $e$ and $e$ is the cokernel of $m$. In the case of groups it is the split extensions that are important and these satisfy the property that if $s$ is a splitting of $e$, then the images of $m$ and $s$ together generate $G$. This is not so in **RFrm** and **RTopos** and will only occur when $s$ is the right adjoint of $e$. This motivates restricting to adjoint split extensions.

We show that there is a natural way to view Artin glueings of frames and toposes as extensions and that every extension $N \xrightarrow{m} G \xrightarrow{e} H$ can be thought of as the glueing of $H$ and $N$ along $m^*e_*$.

## References

[1] P. T. Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 1. Oxford University Press, 2002.

[2] G. Wraith. Artin glueing. *Journal of Pure and Applied Algebra*, 4(3):345–348, 1974.