UNIFORM KAN FIBRATIONS IN SIMPLICIAL SETS

BENNO VAN DEN BERG

One of the cornerstones of homotopy type theory is the construction by Voevodsky of a model of Martin-Loef type theory with a univalent universe in which the types are the ∞-groupoids. (Very) roughly speaking, to build a model of type theory we need a category and a distinguished class of maps (the fibrations) to model the type dependencies. For his model, Voevodsky chose simplicial sets and the Kan fibrations and he heavily exploited the Kan-Quillen model structure on simplicial sets.

After the work by Voevodsky, much time and energy has been spent of trying to prove the same result in a constructive metatheory. Bezem, Coquand and Parmann identified several obstacles, which led Coquand and collaborators to turn to cubical sets. There they could define a suitable notion of fibration for which they managed to prove constructively that it gave them a model of Martin-Loef type theory with a univalent universe. Later, Sattler managed to prove (constructively) that there is a model structure as well. Many variations on these results have since appeared (by using different variants of cubical sets and different notions of fibration, etc), but, as far as I can see, so far none of these model structures have been shown to be equivalent to the standard Kan-Quillen model structure on simplicial sets.

In a way, Coquand et al made two changes: they changed the underlying category and they started adding uniformity conditions to the notion of a fibration (compatibility conditions for the fillers). Building on ideas by Coquand et al, Gambino and Sattler and Sattler returned to simplicial sets and showed that one can get quite far by adding similar uniformity conditions in simplicial sets. In fact, they almost managed to recover the Kan-Quillen model structure in a constructive metatheory. One key difficulty is that (constructively) it is not clear whether their notion of Kan fibration is "local", in the sense that it is not clear that a map will be a uniform Kan fibration if all its pullbacks with representable codomain are. This is important for constructing universes.

In joint work with Eric Faber I have starting investigating two notions of a uniform Kan fibration, both of which are stronger than the one used by Gambino and Sattler and both of which are local. This is all still very much work in progress, but right now I am convinced that one obtains with both our notions of a uniform fibration and within a constructive metatheory (say, Aczel's CZF with some universes) a model of type theory with universes (including identity and Pi-types) and a model structure on the fibrant objects. We are still working on univalence and extending the model structure to the entire category.

Besides emphasising that this is still joint work in progress with Eric Faber, I should also stress that, while we try to do all our maths constructively, our project forces us to prove results which are new and (I believe) interesting in the classical setting as well.