Duality, definability and conceptual completeness for \( \kappa \)-pretoposes

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The \( \kappa \)-geometric toposes introduced in [2], for regular \( \kappa \) such that \( \kappa^{<\kappa} = \kappa \) (or any regular \( \kappa \) if the Generalized Continuum Hypothesis holds), are associated to \( \kappa \)-geometric logic, an extension of geometric logic in which conjunction of less than \( \kappa \) many formulas as well as quantification of less than \( \kappa \) many variables are possible. They in fact occur as \( \kappa \)-classifying toposes, i.e., toposes with the obvious universal property applied to \( \kappa \)-geometric morphisms (those whose inverse image preserve all \( \kappa \)-small limits). When these toposes are in addition \( \kappa \)-separable, they turn out to have enough \( \kappa \)-points. We prove here that this completeness theorem has the surprising consequence that for \( \lambda > \kappa \), the \( \lambda \)-classifying topos of any theory with at most \( \kappa \) axioms, expressible in \( \kappa \)-geometric logic, is in fact the topos of presheaves over the category of \( \lambda \)-presentable models of the theory.

As applications we get positive results on definability theorems for infinitary logic, conceptual completeness for \( \kappa \)-pretoposes, infinitary versions of Joyal’s completeness theorem for infinitary intuitionistic logic, a Stone type duality in the form of a biequivalence arising from a syntax-semantics adjunction, the descent theorem for \( \kappa \)-pretoposes, and a characterization of categoricity for models of infinitary sentences. Time permitting we will show how this latter result provides a topos-theoretic approach to Shelah’s eventual categoricity conjecture.

1 References