# An Enriched Perspective on Differentiable Stacks

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# Differentiable Stacks

## Definition

A *split differentiable stack* is a (2,1)-sheaf

$$\mathcal{X} : \mathbf{Man} \to \mathbf{Gpd}$$

with respect to the open cover topology on **SMan** with a morphism $y(M) \to \mathcal{X}$ such that

1. For all $y(N) \to \mathcal{X}$, $y(N) \times_{\mathcal{X}} y(M)$ is a manifold.
2. For all $y(N) \to \mathcal{X}$, $y(N) \times_{\mathcal{X}} y(M) \to y(N)$ is a submersion

There is an embedding of smooth manifolds into the category of stacks, using the Yoneda lemma for (2,1)-categories.

# Tangent Bundle of a Differentiable Stack

There is a tangent bundle construction on the category of differentiable stacks, due to Hepworth. It is constructed via a Kan extension:

$$\begin{tikzcd}
\mathbf{SMan} \arrow[r, "T"] \arrow[dr, hook] & \mathbf{SMan} \arrow[r, hook] & \mathbf{DStack} \\
& \mathbf{DStack} \arrow[ur, dotted, "T^*"'] &
\end{tikzcd}$$

This has the property that $y \circ T \cong T^* \circ y$.

# Problems

These Kan extension definitions of the tangent bundle can be quite challenging to work with.

- Kan extension isn't a monoidal functor (so $T^* T^*$ need not equal $(TT)^*$)
- Addition of tangent vectors is not well defined in general.
- It's not clear whether symmetry of partial derivatives holds.

**Possible approach**: Identify a full subcategory of *microlinear* stacks

## Goal

Refine the notion of a differentiable stack based on enriched category theory so that it has a well-behaved tangent bundle (in the sense of tangent categories).

## Definition (Rosicky, Cockett&Cruttwell)

A *tangent category* is a category $\mathbb{X}$ is given by:

- A natural additive bundle $(T, p, 0, +)$, where pullback powers of $p$ are preserved by $T$.
- Natural transformations $c : T^2 \Rightarrow T^2$, $\ell : T \Rightarrow T^2$.

satisfying some coherences.

The flip $c$ represents symmetry of mixed partial derivatives $\frac{\partial^2 f(x,y)}{\partial x \partial y}(a, b) \cdot (u, v)$.
The map $\ell$ is universal, and represents linearity of the vector argument $\frac{\partial^2 f(x)}{\partial x}(a) \cdot (v)$.

## Examples of tangent categories

- The category of smooth manifolds
- The microlinear objects of a model of Synthetic Differential Geometry
- Examples arising from computer science (e.g. the coKleisli category, or as JS will tell you, the co-Eilenberg-Moore category of a monoidal differential category).

## Some Successes of Tangent Categories

- Very clear description of Sector Form cohomology, leading to some new observations. (Cruttwell & Lucyshyn-Wright)
- New observations on connections and affine manifolds.
- Related to the semantics of differentiable programming languages.

# Weil Algebras

R-Weil algebras: *infinitesimal thickening* of $R$, $(R[x]/x^2)$

## Definition

The category of *Weil algebras* is the full subcategory of $R\textbf{Alg}/R$ of $\pi : W \to R$ such that:

- $ker(\pi)$ is nilpotent.
- The underlying $R$-module of $W$ is $R^n$

## Proposition

- Every Weil algebra may be written $R[x_i]/I$
- Coproducts: $R[x_i]/I \otimes R[y_j]/J = R[x_i, y_j]/(I \cup J)$
- Products: $R[x_i]/I \times R[y_j]/J = R[x_i, y_j]/(I \cup J \cup \{x_i y_j\})$
- $R$ is a zero object

## Proposition (Leung)

Let $W := R[x]/x^2$. The category of Weil algebras is a tangent category, with $T(-) := W \otimes -$.

We can restrict our attention to powers of $W$ to construct the *free tangent category*:

## Definition (Leung)

The category $\mathbf{Weil}_1$ is the full subcategory of $\mathbb{N} - \mathbf{Weil}$ whose objects are of the form: $W^{n_1} \otimes \cdots \otimes W^{n_k}$
Note that this category has binary pullbacks, and they are preserved by $W \otimes -$.

## Remark

We regard $(\mathbf{Weil}_1, \otimes, R)$ as a monoidal category.

**Theorem**

*The following are equivalent.*

1. *A tangent category* $\mathcal{X}$
2. *A monoidal functor* $\mathbf{Weil}_1 \to [\mathcal{X}, \mathcal{X}]$ *sending binary pullbacks to pointwise limits (Leung)*
3. *An actegory* $\mathbf{Weil}_1 \times \mathcal{X} \to \mathcal{X}$ *preserving binary pullbacks in* $\mathbf{Weil}_1$ *(Leung)*
4. *A category enriched in* $\mathcal{E} := \mathbf{Mod}(\mathbf{Weil_1})$ *with powers by representable functors (Garner).*

(3) to (4) follows by a theorem due to Wood.

# Two things

We need two generalizations to move forwards:

## Sheaves

The sheaf condition is at the core of the classical definition of a differentiable stack, is already an enriched concept. How can we generalize this?

## **Strict** Tangent (2,1)-categories

We want a definition of 2-category with tangent structure

The following theorem is from Borceux and Quinteiro

**Theorem**

*The following are equivalent for $\mathcal{C}$ enriched in a regular, finitely presented $\mathcal{V}$*

- *Grothendieck topologies on $\mathcal{C}$.*
- *Left-exact idempotent monads on $[\mathcal{C}, \mathcal{V}]$.*
- *Universal closure operations on $[\mathcal{C}, \mathcal{V}]$.*

But the category $\mathcal{E}$ is not regular!

**Definition (Tangent sheaf)**

A tangent sheaf on a tangent category $\mathcal{C}$ is an EM-algebra of a left-exact idempotent monad $M$ on $[\mathcal{C}, \mathcal{E}]$.

We may apply the following theorem due to Wolff:

**Theorem (Wolff)**

*Sheaves commute with models of enriched sketches.*

Using forthcoming work, we have:

**Corollary (Gallagher, Lucyshyn-Wright, M.)**

*The category of differential objects in $Sh(M)$ is equivalent to a category of sheaves into differential objects of $\mathcal{E}$.*

## Definition

A *strict tangent 2-category* is a category enriched in

$$\hat{\mathcal{E}} := Mod(Weil_1 \otimes TGpd, Set)$$

with powers by representable functors $Weil_1 \to Set \hookrightarrow Gpd$.

## Slogan

A strict tangent structure on a (2,1)-category is *property* of the tangent structure on the underlying category.

# Tangent 2-categories as an actegory

**Proposition**

For every 2-functor $\mathbf{Weil}_1 \times \mathcal{X} \to \mathcal{X}$ which satisfies the coherences of an actegory *on the nose*, there is a corresponding category enriched in $Mod(Weil_1 \otimes Gpd)$ with powers by representables $Weil_1 \to Set \hookrightarrow Gpd$,

For the implication, the new hom is defined the same way:

$$\mathcal{X}(A, B)(V) := \mathcal{X}(A, V \propto B) \in \mathbf{Gpd}$$

note that we can identify a functor $\mathbf{Weil}_1$ into $\mathbf{Gpd}$ as a 1-category with a 2-functor where we treat $\mathbf{Weil}_1$ as a 2-category.

# Tangent (2,1)-Monad

**Question:** Why is it insufficient to have a (2,1)-category whose underlying category is a tangent category?
**Answer:** Consider the underlying (2,1)-category of a tangent (2,1)-category $\mathcal{K}$, there is the *tangent 2-monad*

$$y(R[x,y]/x^2, y^2) \pitchfork M \xrightarrow{x,y \mapsto z} y(R[z]/z^2) \pitchfork M$$

$$M \xrightarrow{0} y(R[z]/z^2) \pitchfork M$$

By the following theorem we may regard being a (2,1)-monad (or 2-monad) as a property of the underlying monad.

**Theorem (Power)**

*If $\mathcal{C}$ is a (2,1)-category with powers and copowers by $\rightarrow$, then* **any** *1-monad on $U(\mathcal{C})$ has at most one enrichment.*

We also see that a tangent (2,1)-category has a 2-commutative monoid of vector spaces.

## Definition

A map $X : 1 \to \mathcal{C}(A, y(x^2) \pitchfork A)$ that is a section of $p_A$ on the nose is a *geometric vector field* - these form a commutative monoid. Note that $X$ is an "object" of $\mathcal{C}(A, y(x^2) \pitchfork A) : Gpd(\mathcal{E})$, and given 2-cells $\gamma : X \Rightarrow X', \psi : Y \Rightarrow Y'$, we may also form $\psi + \gamma : X + Y \Rightarrow X' + Y'$.

$$\begin{tikzcd} A \arrow[r, bend left, "X+Y"] \arrow[r, bend right, "X'+Y'"'] \arrow[r, phantom, "\psi+\gamma"] & TA \end{tikzcd} := \begin{tikzcd} A \arrow[r, bend left, "{(X,Y)}"] \arrow[r, bend right, "{(X',Y')}"'] \arrow[r, phantom, "{(\psi,\gamma)}"] & T_2A \arrow[r, "+"] & TA \end{tikzcd}$$

## Examples

- Lie groupoids in a tangent category.
- Restriction tangent categories is a tangent 2-category (the 2-cells are $\leq$).
- A 2-category with 2-biproducts.

## Non-Examples

Lex with $T = Mod(ABun, -)$. The addition is given by fibered biproducts of additive bundles, so addition is only associated *up to a coherent isomorphism*.

**Remark**

There is a functor $I : \mathrm{TangCat} \hookrightarrow$ Tang-(2,1)-Cat by lifting sets up to discrete groupoids.

**Definition**

Let $\mathcal{X}$ be a tangent 1-category, and $M$ be an left-exact idempotent $\hat{\mathcal{E}}$-monad on $[I(\mathcal{X}), \hat{\mathcal{E}}]$. An EM-algebra of $M$ is a *tangent stack* over $M$.

**Theorem**

The (2,1)-category of tangent stacks on a tangent category is a (2,1)-tangent category.

# Conclusions and Future Work

We now have a notion of "tangent stack" (and geometric tangent stack) that has a well behaved tangent bundle.
What can we do with this/what is left to do?

- How do tangent stacks relate to tangent fibrations?
- How can we weaken our definition of tangent 2-category, and what is the relevent coherence theorem.
- Sector form cohomology works on tangent stacks without any significant modification (differential forms on stacks are hard!).