Baez, Pollard: *A compositional framework for reaction networks*

Rosebrugh, Sabadini, Walters: *Calculating colimits compositionally*

Bonchi, Sobocinski, Zanasi: *A categorical semantics of signal flow graphs*

Spivak: *The operad of wiring diagrams*
\[
\left\{ \text{Cospan}_T \xrightarrow{\text{lax monoidal}} \text{Set} \right\}
\]

\[
\uparrow \downarrow
\]

\[
\left\{ \text{hypergraph categories} \right\}
\]
The category of cospan algebras is equivalent to the category of objectwise-free hypergraph categories.
\[ \left\{ \text{Cospan}_T \xrightarrow{\text{lax monoidal}} \text{Set} \right\} \]

\[ \uparrow \downarrow \]

\[ \left\{ \text{hypergraph categories} \right\} \]
\{ \text{Cospan}_T \xrightarrow{\text{lax monoidal}} \text{Set} \}\n
\{ \text{categories} \}
\[ \{ \text{Cospan}_T \xrightarrow{\text{lax monoidal}} \text{Poset} \} \]

\[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \]

\[ \{ \text{categories} \} \]
\{ \text{Cospan}^\text{co}_T \xrightarrow{\text{lax monoidal}} \text{Poset} \}
\begin{align*}
\{ \text{Cospan}^\text{co}_T \text{ right ajax monoidal} \rightarrow \text{Poset} \} \\
\text{subobject lattices} \uparrow \quad \downarrow \quad \text{syntactic category} \\
\{ \text{regular categories} \}
\end{align*}
Key idea: Regular calculi present regular categories.
Outline

I. Motivation
II. The theorem
III. Proof sketch
II. The theorem
A regular category is a category with finite limits and pullback stable image factorisations.

A regular functor is a functor between regular categories that preserves finite limits and image factorisations.

Examples: FinSet, FinSet\(^{\text{op}}\), Set, Set\(^{\text{op}}\), FDVect, Vect, abelian categories, toposes, any category monadic over Set, . . .

Given a regular category \(\mathcal{R}\), we may construct its relations bicategory \(\text{Rel}_{\mathcal{R}}\) with the same objects, but where 1-morphisms are jointly-monic spans.

\[
\begin{array}{ccc}
X & \xleftarrow{f} & A & \xrightarrow{g} & Y
\end{array}
\]
Ajax functors

A right ajax (monoidal) functor is a lax monoidal functor $P: \mathbb{C} \to \mathbb{D}$ in which the laxators are right adjoints.

Example: a right ajax functor $P: 1 \to \mathbb{Poset}$ is a meet semilattice.
A regular calculus $(T, P)$ is a set $T$ and a right ajax functor

$$P: \text{Cospan}^\text{co}_T \rightarrow \text{Poset}.$$ 

A morphism $(F, F^\#): (T, P) \rightarrow (T', P')$ of regular calculi is a function $F$ and a monoidal natural transformation $F^\#$:
Theorem
We have an adjunction

\[
\text{RgCalc} \xRightarrow{\text{syn}} \Rightarrow \xleftarrow{\text{prd}} \text{RgCat}.
\]

where \text{prd} is fully faithful, and for any regular category \( \mathcal{R} \), the counit map \( \text{syn}(\text{prd}(\mathcal{R})) \to \mathcal{R} \) is an equivalence.
\[
\left\{ \text{Cospan}_{T}^{\text{co}} \right\} \xrightarrow{\text{right ajax monoidal}} \text{Poset}
\]

subobject lattices \uparrow \quad \text{syntactic category} \downarrow

\left\{ \text{regular categories} \right\}

**Key idea:** Regular calculi present regular categories.
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III. Proof sketch
Given a regular category $\mathcal{R}$, we construct the regular calculus

$$\text{prd}(\mathcal{R}): \text{Cospan}^\text{co}_{\text{Ob}\mathcal{R}} \xrightarrow{\text{Frob}} \text{Rel}_{\mathcal{R}} \xrightarrow{\text{Rel}_{\mathcal{R}}(1,-)} \text{Poset}$$

where Frob is given by the hypergraph structure on $\text{Rel}_{\mathcal{R}}$.

Given a regular calculus $P: \text{Cospan}^\text{co}_T \to \text{Poset}$, we may construct the bicategory $\text{Rel}_{\text{syn}}(P)$:

- **objects**: $\{ (\Gamma, s) \mid \Gamma \in \text{Cospan}^\text{co}_T, s \in P(\Gamma) \}$
- **hom-posets**: $\text{Hom}((\Gamma, s), (\Gamma', s')) = P(\Gamma \oplus \Gamma') - \leq_{\rho(s,s')}$

The **syntactic category** $\text{syn}(P)$ is the category of left adjoints in $\text{Rel}_{\text{syn}}(P)$. 
Given any regular calculus, we may draw and interpret diagrams such as those below. The properties of regular calculi give ‘deduction rules’:
One might view this as a graphical regular logic, where regular logic is the fragment of first order logic given by $=, \top, \land, \exists.$

$$\psi(u, v, w, x, y, z) =$$

$$\exists a, b. \theta_1(u, b, y, y) \land \theta_2(a, b, u) \land \theta_3(b, x, y) \land (v = w) \land (z = z).$$
Pullback:

\[
\left( \Gamma_1 \oplus \Gamma_2, \xrightarrow{\theta_1} \xrightarrow{\theta_2} \right) \to (\Gamma_2, \varphi_2)
\]
\[
\downarrow \quad \downarrow \theta_2
\]
\[
(\Gamma_1, \varphi_1) \xrightarrow{\theta_1} (\Gamma, \varphi)
\]

Equaliser:

\[
\left( \Gamma, \xrightarrow{\theta} \xrightarrow{\theta'} \right) \to (\Gamma, s) \xrightarrow{\theta} (\Gamma', s')
\]

Epi-mono factorisation:
Theorem
We have an adjunction

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\text{RgCalc} \xleftarrow{\text{syn}} \Rightarrow \xrightarrow{\text{prd}} \text{RgCat}.
\]

where \text{prd} is fully faithful, and for any regular category \( \mathcal{R} \), the counit map \( \text{syn}(\text{prd}(\mathcal{R})) \to \mathcal{R} \) is an equivalence.