

The generalized Homotopy Hypothesis

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Introduction

Grothendieck n -groupoids are a globular model for weak n -groupoids, for $0 \leq n \leq \infty$. They are presented as presheaves on suitable contractible theories satisfying a Segal condition on the nose.

The generalized *Homotopy Hypothesis* states that n -groupoids (in the weak sense) are essentially the same thing as homotopy n -types i.e. spaces whose homotopy groups vanish above dimension n .

This can be made precise by asserting the existence of an equivalence between the $(\infty, 1)$ categories that they present.

This is a theorem if we encode weak n -groupoids as n -truncated Kan complexes, i.e. Kan complexes K for which $K(x, y)$ is an $n - 1$ -type.

Proving this conjecture in the case of Grothendieck n -groupoids would provide a completely algebraic model of homotopy n -types (the only known example so far being Gray-groupoids, due to Lack).

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The globe category and Θ_0

Let \mathbb{G} be the quotient of the free category on the directed graph

$$0 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} 2 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} 3 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \dots$$

under the relations $\sigma \circ \sigma = \tau \circ \sigma$ and $\sigma \circ \tau = \tau \circ \tau$.

The closure of \mathbb{G} under colimits of the form



is denoted by Θ_0 , and its objects are called *globular sums*. If we start with $\mathbb{G}_{\leq n}$, we get $\Theta_0^{\leq n}$ in the same way.

We think of them as being suitable gluings of globes (or cells) such as



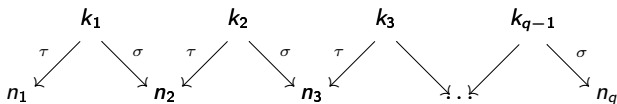
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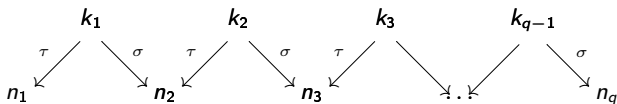
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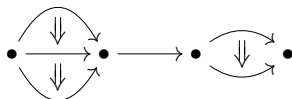
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n -Globular Theories

An n -globular theory is a pair (\mathcal{C}, F) , where \mathcal{C} is a category and $F: \Theta_0^{\leq n} \rightarrow \mathcal{C}$ is a bijective on objects functor that preserves globular sums.

An n -globular theory is called *contractible* if for every $k \leq n$ and every pair of parallel maps $f, g: D_k \rightarrow A$ the following extension problem admits a solution

$$\begin{array}{ccc} D_k & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & A \\ \sigma \downarrow & \begin{array}{c} \tau \nearrow \\ e \nearrow \end{array} & \\ D_{k+1} & & \end{array}$$

An n -globular theory \mathcal{C} is called *cellular* if there exists a cocontinuous functor $\bar{\mathcal{C}}: \gamma \rightarrow \mathbf{GTh}_n$, where γ is an ordinal, such that $\bar{\mathcal{C}}(\beta + 1)$ is obtained from $\bar{\mathcal{C}}(\beta)$ by universally adding solutions to a family of lifting problems as above and, moreover, $\bar{\mathcal{C}}(0) = \Theta_0^{\leq n}$ and $\text{colim}_{\beta \in \gamma} \bar{\mathcal{C}}(\beta) \cong \mathcal{C}$.

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n -Coherators and n -groupoids

An n -coherator is a contractible and cellular n -globular theory.

An n -groupoid (of type \mathcal{C}) is a presheaf of sets $X: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ that commutes with globular products (Segal condition). This category will be denoted by $\mathbf{Mod}(\mathcal{C})$, or simply by $n\text{-Gpd}$.

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Algebraic structure

Given an n -groupoid X , where do all the operations and coherences on the k -cells arise from? The answer is *contractibility!*

Some examples:

$$\begin{array}{ccc}
 D_0 & \xrightarrow{i_0 \circ \sigma} & D_1 \amalg_{D_0} D_1 \\
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 D_1 & &
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$$\begin{array}{ccc}
 D_1 & \xrightarrow{\tau} & D_2 \\
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e represents binary composition of 1-cells, ω witness the existence of codimension 1 inverses for 2-cells and a is the associativity constraint for two possible way of composing a triple of compatible 1-cells.

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Homotopy theory of n -groupoids

Given an n -groupoid X we set $\pi_0(X) = X_0 / \sim$, where $a \sim b$ iff $\exists f \in X_1, f: a \rightarrow b$.

Given $k \leq n$ and $g \in X_{k-1}$, we define $\pi_k(X, g) = \{h \in X_k, h: g \rightarrow g\} / \sim$, where $h \sim h'$ iff $\exists H \in X_{k+1}, H: h \rightarrow h'$.

A map $f: X \rightarrow Y$ between n -groupoids is said to be a *weak equivalence* if it induces bijections

$$\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y) \text{ and } \pi_k(f, \alpha): \pi_k(X, \alpha) \rightarrow \pi_k(Y, \alpha)$$

for every $k \leq n$ and every $\alpha \in X_{k-1}$.

The pair $(n\text{-Gpd}, \mathcal{W}_n)$, where \mathcal{W}_n is the class of equivalences we have just defined, is a relative category and thus defines an $(\infty, 1)$ -category of n -groupoids.

$$\begin{array}{ccc} n\text{-Gpd} & \begin{array}{c} \xrightarrow{|\bullet|} \\ \perp \\ \xleftarrow{\Pi_{\leq n}} \end{array} & \text{Top}_n \end{array}$$

with Top_n being the relative category of homotopy n -types and $\Pi_{\leq n}(X)_k = \text{Top}(D_k, X)$.

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The homotopy hypothesis

The Homotopy Hypothesis is the statement that

$$\Pi_{\leq n}: n\text{-}\mathcal{G}pd \rightarrow \mathbf{Top}_n$$

induces an equivalence of the associated $(\infty, 1)$ -categories.

The $(\infty, 1)$ -category \mathbf{Top}_n of homotopy n -types has a universal property: it is the free cocomplete $(\infty, 1)$ -category on an n -truncated object.

More precisely, to give a cocontinuous ∞ -functor $\mathbf{Top}_n \rightarrow \mathcal{E}$ where \mathcal{E} is cocomplete is the same as giving an object $e \in \mathcal{E}$ with $e \xrightarrow{\simeq} e^{S^{n+1}}$

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Semi-model structure

By analogy with topological spaces and the Kan-Quillen model structure on simplicial sets, it makes sense to define the set of generating cofibrations (resp. trivial cofibrations) to be given by, respectively:

$$\mathbf{I} = \{\partial_k: S^{k-1} \longrightarrow D_k\}_{k \leq n+1}$$

$$\mathbf{J} = \{\sigma_k: D_k \longrightarrow D_{k+1}\}_{k < n}$$

The hard bit is proving the pushout-lemma, i.e. that pushouts of maps in \mathbf{J} are weak equivalences.

Theorem (Henry)

If the pushout lemma holds between finitely cellular ∞ -groupoids, then the homotopy hypothesis holds true.

The strategy of the proof is to generalize the result for ∞ -groupoids of the first author to the case of n -groupoids.

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Definition

- A globular set X is called n -truncated if for each pair of k -cells x, y in X with $k \geq n$ one has:

$$X(x, y) = \begin{cases} \mathbf{1} & x = y \\ \emptyset & x \neq y \end{cases}$$

- A globular set X is called n -coskeletal if for each pair of parallel k -cells $x \parallel y$ in X with $k \geq n$ one has:

$$X(x, y) = \mathbf{1}$$

Clearly, n -truncated globular sets are $n + 1$ -coskeletal. We denote by $\mathbf{Mod}(\mathcal{C})_{\text{cosk}_n}$ the category of \mathcal{C} -models whose underlying globular set is n -coskeletal. Analogously, $\mathbf{Mod}(\mathcal{C})_{n\text{-tr}}$ will denote the category of \mathcal{C} -models whose underlying globular set is n -truncated.

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Cellularity

To adapt the machinery used in the case of ∞ -groupoids we need a cellular model for n -groupoids, i.e. one whose construction does not involve identifying operations in top dimension.

Given a coherator for ∞ -groupoids \mathcal{C} , we can consider an n -globular theory $\mathcal{C}^{\leq n}$, whose defining tower is obtained by adding the same operations that were added to get \mathcal{C} , up to dimension n .

Proposition

There are equivalence of categories of the form:

$$\mathrm{Mod}(\mathcal{C})_{\mathrm{cosk}_n} \simeq \mathrm{Mod}(\mathcal{C}^{\leq n})$$

$$\mathrm{Mod}(\mathcal{C})_{n\text{-tr}} \simeq n\text{-Gpd}$$

The first one sends an n -coskeletal model to its restriction to cells of dimension smaller than or equal to n . The second one acts by restriction on cells of dimension strictly smaller than n , and acts by quotienting n -cells by $n+1$ -cells in top dimension.

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To adapt the machinery used in the case of ∞ -groupoids we need a cellular model for n -groupoids, i.e. one whose construction does not involve identifying operations in top dimension.

Given a coherator for ∞ -groupoids \mathfrak{C} , we can consider an n -globular theory $\mathfrak{C}^{\leq n}$, whose defining tower is obtained by adding the same operations that were added to get \mathfrak{C} , up to dimension n .

Proposition

There are equivalence of categories of the form:

$$\mathbf{Mod}(\mathfrak{C})_{\text{cosk}_n} \simeq \mathbf{Mod}(\mathfrak{C}^{\leq n})$$

$$\mathbf{Mod}(\mathfrak{C})_{n\text{-tr}} \simeq n\text{-Gpd}$$

The first one sends an n -coskeletal model to its restriction to cells of dimension smaller than or equal to n . The second one acts by restriction on cells of dimension strictly smaller than n , and acts by quotienting n -cells by $n + 1$ -cells in top dimension.

The adjunction

Proposition

There is an adjunction of the form:

$$\mathbf{Mod}(\mathcal{C})_{\text{cosk}_{n+1}} \begin{array}{c} \xrightarrow{t_n} \\ \perp \\ \xleftarrow{i_n} \end{array} \mathbf{Mod}(\mathcal{C})_{n-tr} \simeq n\text{-Gpd}$$

where

$$(t_n X)_k = \begin{cases} X_k & k < n \\ X_n / \sim & k \geq n \end{cases}$$

One can define two classes $\mathcal{W}, \mathcal{W}'$ of weak equivalences in $\mathbf{Mod}(\mathcal{C})_{\text{cosk}_{n+1}}$. \mathcal{W} is the restriction of the class of weak equivalences of ∞ -groupoids, and \mathcal{W}' is simply pulled back from n -groupoids along t_n . We get the following result:

Theorem

The two classes \mathcal{W} and \mathcal{W}' coincide. Moreover, t_n is an equivalence of relative categories.



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The canonical left semi-model structure and the homotopy hypothesis

If the left semi-model structure on n -groupoids with the previously described generating (trivial) cofibrations exists, we call it the *canonical* left semi-model structure.

Theorem

Assuming the canonical left semi-model structure on n -Gpd exists, there exists a cofibrantly generated left semi-model structure on the category $\mathbf{Mod}(\mathcal{C})_{\text{cosk}_{(n+1)}}$ of $(n+1)$ -coskeletal ∞ -groupoids, and the previous adjunction is a Quillen equivalence with respect to these semi-model structures. Moreover, in this case these equivalent model structures present the $(\infty, 1)$ -category of homotopy n -types.

Corollary

Since the canonical left semi-model structure exists in dimension $n = 3$, the homotopy hypothesis holds true for $n = 3$. In particular, Grothendieck 3-groupoids model homotopy 3-types.

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Strategy for the pushout lemma

Since it all boils down to proving that pushout of maps of the form $\sigma_k: D_k \rightarrow D_{k+1}$ are weak equivalences, this problem is worth focusing on. A sufficient condition is that of (less) than a path object.

Proposition (L.)

If for every cofibrant n -groupoid X there exists a fibration $\mathbf{ev}: \mathbb{P}X \rightarrow X \times X$ such that $\mathbf{ev}_i = \pi_i \circ \mathbf{ev}$ is a trivial fibration for $i = 0, 1$, where $\pi_i: X \times X \rightarrow X$ denote the product projections, then the pushout lemma is valid and the canonical model structure on n -groupoids exists.

A good candidate is: $\mathbb{P}X_k \stackrel{\text{def}}{=} n\text{-Gpd}(\text{Cyl}(D_k), X)$, and we have the following result:

Proposition (L.)

If $\mathbb{P}X$ is endowed with an n -groupoid structure compatible with the projections to X , then $\mathbf{ev}: \mathbb{P}X \rightarrow X \times X$ is a fibration and $\mathbf{ev}_i = \pi_i \circ \mathbf{ev}$ is a trivial fibration for $i = 0, 1$.

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Thanks for your attention!