The generalized Homotopy Hypothesis

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Introduction

Grothendieck n-groupoids are a globular model for weak n-groupoids, for $0 \leq n \leq \infty$. They are presented as presheaves on suitable contractible theories satisfying a Segal condition on the nose.

The generalized *Homotopy Hypothesis* states that n-groupoids (in the weak sense) are essentially the same thing as homotopy n-types i.e. spaces whose homotopy groups vanish above dimension n.
This can be made precise by asserting the existence of an equivalence between the $(\infty, 1)$ categories that they present.
This is a theorem if we encode weak n-groupoids as n-truncated Kan complexes, i.e. Kan complexes K for which $K(x, y)$ is an $n-1$-type.
Proving this conjecture in the case of Grothendieck n-groupoids would provide a completely algebraic model of homotopy n-types (the only known example so far being Gray-groupoids, due to Lack).

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The globe category and Θ_0

Let $\mathbb{G}$ be the quotient of the free category on the directed graph

$$0 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} 1 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} 2 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} 3 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \dots$$

under the relations $\sigma \circ \sigma = \tau \circ \sigma$ and $\sigma \circ \tau = \tau \circ \tau$.

The closure of $\mathbb{G}$ under colimits of the form

$$\begin{tikzcd}
& k_1 \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & & k_2 \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & & k_3 \arrow[dl, "\tau"'] \arrow[dr] & & k_{q-1} \arrow[dl] \arrow[dr, "\sigma"] & \\
n_1 & & n_2 & & n_3 & & \dots & & n_q
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is denoted by Θ_0, and its objects are called *globular sums*. If we start with $\mathbb{G}_{\leq n}$, we get $\Theta_0^{\leq n}$ in the same way.

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\bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet
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$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=A}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, ""'{name=C}] \arrow[from=A, to=B, phantom, "\Downarrow"] \arrow[from=B, to=C, phantom, "\Downarrow"] & \bullet \arrow[r] & \bullet \arrow[r, bend left, ""{name=D}] \arrow[r, bend right, ""'{name=E}] \arrow[from=D, to=E, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

n-Globular Theories

An n-globular theory is a pair $(\mathfrak{C}, F)$, where $\mathfrak{C}$ is a category and $F \colon \Theta_0^{\leq n} \longrightarrow \mathfrak{C}$ is a bijective on objects functor that preserves globular sums.

An n-globular theory is called *contractible* if for every $k \leq n$ and every pair of parallel maps $f, g \colon D_k \longrightarrow A$ the following extension problem admits a solution

$$\begin{tikzcd} D_k \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & A \\ D_{k+1} \arrow[ur, dashed, "\varrho"'] & \end{tikzcd}$$

An n-globular theory $\mathfrak{C}$ is called *cellular* if there exists a cocontinuous functor $\bar{\mathfrak{C}} \colon \gamma \longrightarrow \mathbf{GlTh}_n$, where γ is an ordinal, such that $\bar{\mathfrak{C}}(\beta + 1)$ is obtained from $\bar{\mathfrak{C}}(\beta)$ by universally adding solutions to a family of lifting problems as above and, moreover, $\bar{\mathfrak{C}}(0) = \Theta_0^{\leq n}$ and $\operatorname{colim}_{\beta \in \gamma} \bar{\mathfrak{C}}(\beta) \cong \mathfrak{C}$.

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n-Coherators and n-groupoids

An *n-coherator* is a contractible and cellular n-globular theory.

An n-groupoid (of type $\mathfrak{C}$) is a presheaf of sets $X\colon \mathfrak{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ that commutes with globular products (Segal condition). This category will be denoted by $\mathbf{Mod}(\mathfrak{C})$, or simply by n-$\mathcal{G}$pd.

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Algebraic structure

Given an n-groupoid X, where do all the operations and coherences on the k-cells arise from? The answer is *contractibility!*

Some examples:

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\begin{tikzcd}
D_0 \arrow[r, shift left, "i_0 \circ \sigma"] \arrow[r, shift right, "i_1 \circ \tau"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & D_1 \coprod_{D_0} D_1 \\
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ϱ represents binary composition of 1-cells, ω witness the existence of codimension 1 inverses for 2-cells and a is the associativity constraint for two possible way of composing a triple of compatible 1-cells.

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Homotopy theory of n-groupoids

Given an n-groupoid X we set $\pi_0(X) = X_0/\sim$, where $a \sim b$ iff $\exists f \in X_1,\ f\colon a \longrightarrow b$.
Given $k \leq n$ and $g \in X_{k-1}$, we define $\pi_k(X,g) = \{h \in X_k,\ h\colon g \longrightarrow g\}/\sim$, where $h \sim h'$ iff $\exists H \in X_{k+1},\ H\colon h \longrightarrow h'$.
A map $f\colon X \longrightarrow Y$ between n-groupoids is said to be a *weak equivalence* if it induces bijections

$$\pi_0(f)\colon \pi_0(X) \longrightarrow \pi_0(Y) \text{ and } \pi_k(f,\alpha)\colon \pi_k(X,\alpha) \longrightarrow \pi_k(Y,\alpha)$$

for every $k \leq n$ and every $\alpha \in X_{k-1}$.
The pair $(n\text{-}\mathcal{G}\mathrm{pd}, \mathcal{W}_n)$, where $\mathcal{W}_n$ is the class of equivalences we have just defined, is a relative category and thus defines an $(\infty,1)$-category of n-groupoids.

$$\begin{tikzcd} n\text{-}\mathcal{G}\mathrm{pd} \arrow[r, bend left, "|\bullet|"] \arrow[r, phantom, "\perp"] & \mathbf{Top}_n \arrow[l, bend left, "\Pi_{\leq n}"] \end{tikzcd}$$

with $\mathbf{Top}_n$ being the relative category of homotopy n-types and $\Pi_{\leq n}(X)_k = \mathbf{Top}(D_k, X)$.

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The homotopy hypothesis

The Homotopy Hypothesis is the statement that

$$\Pi_{\leq n} \colon n\text{-}\mathcal{G}\mathrm{pd} \to \mathbf{Top}_n$$

induces an equivalence of the associated $(\infty, 1)$-categories.

The $(\infty, 1)$-category $\mathbf{Top}_n$ of homotopy n-types has a universal property: it is the free cocomplete $(\infty, 1)$-category on an n-truncated object.

More precisely, to give a cocontinuous ∞-functor $\mathbf{Top}_n \to \mathscr{E}$ where $\mathscr{E}$ is cocomplete is the same as giving an object $e \in \mathscr{E}$ with $e \xrightarrow{\simeq} e^{S^{n+1}}$

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Semi-model structure

By analogy with topological spaces and the Kan-Quillen model structure on simplicial sets, it makes sense to define the set of generating cofibrations (resp.trivial cofibrations) to be given by, respectively:

$$\mathbf{I} = \{\partial_k \colon S^{k-1} \longrightarrow D_k\}_{k \leq n+1}$$

$$\mathbf{J} = \{\sigma_k \colon D_k \longrightarrow D_{k+1}\}_{k < n}$$

The hard bit is proving the pushout-lemma, i.e. that pushouts of maps in $\mathbf{J}$ are weak equivalences.

Theorem (Henry)

If the pushout lemma holds between finitely cellular ∞-groupoids, then the homotopy hypothesis holds true.

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Truncated and coskeletal models

Definition

- A globular set X is called n-truncated if for each pair of k-cells x, y in X with $k \geq n$ one has:

$$X(x,y) = \begin{cases} \mathbf{1} & x = y \\ \emptyset & x \neq y \end{cases}$$

- A globular set X is called n-coskeletal if for each pair of parallel k-cells $x \parallel y$ in X with $k \geq n$ one has:

$$X(x,y) = \mathbf{1}$$

Clearly, n-truncated globular sets are $n+1$-coskeletal. We denote by $\mathbf{Mod}(\mathfrak{C})_{cosk_n}$ the category of $\mathfrak{C}$-models whose underlying globular set is n-coskeletal. Analogously, $\mathbf{Mod}(\mathfrak{C})_{n-tr}$ will denote the category of $\mathfrak{C}$-models whose underlying globular set is n-truncated.

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Cellularity

To adapt the machinery used in the case of ∞-groupoids we need a cellular model for n-groupoids, i.e. one whose construction does not involve identifying operations in top dimension.

Given a coherator for ∞-groupoids $\mathfrak{C}$, we can consider an n-globular theory $\mathfrak{C}^{\leq n}$, whose defining tower is obtained by adding the same operations that were added to get $\mathfrak{C}$, up to dimension n.

Proposition

There are equivalence of categories of the form:

$$\mathbf{Mod}(\mathfrak{C})_{cosk_n} \simeq \mathbf{Mod}(\mathfrak{C}^{\leq n})$$

$$\mathbf{Mod}(\mathfrak{C})_{n-tr} \simeq n\text{-}\mathcal{G}pd$$

The first one sends an n-coskeletal model to its restriction to cells of dimension smaller than or equal to n. The second one acts by restriction on cells of dimension strictly smaller than n, and acts by quotienting n-cells by n + 1-cells in top dimension.

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There is an adjunction of the form:

$$\begin{tikzcd} \mathbf{Mod}(\mathfrak{C})_{cosk_{n+1}} \arrow[r, bend left, "t_n"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathfrak{C})_{n-tr} \simeq n\text{-}\mathcal{G}pd \arrow[l, bend left, "i_n"] \end{tikzcd}$$

where

$$(t_n X)_k = \begin{cases} X_k & k < n \\ X_n/\sim & k \geq n \end{cases}$$

One can define two classes $\mathcal{W}, \mathcal{W}'$ of weak equivalences in $\mathbf{Mod}(\mathfrak{C})_{cosk_{n+1}}$. $\mathcal{W}$ is the restriction of the class of weak equivalcences of ∞-groupoids, and $\mathcal{W}'$ is simply pulled back from n-groupoids along t_n. We get the following result:

Theorem

The two classes $\mathcal{W}$ and $\mathcal{W}'$ coincide. Moreover, t_n is an equivalence of relative categories.

The adjunction

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$$\begin{tikzcd} \mathbf{Mod}(\mathfrak{C})_{cosk_{n+\mathbf{1}}} \arrow[r, bend left, "t_n"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathfrak{C})_{n-tr} \simeq n\text{-}\mathcal{G}pd \arrow[l, bend left, "i_n"] \end{tikzcd}$$

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The canonical left semi-model structure and the homotopy hypothesis

If the left semi-model structure on n-groupoids with the previously described generating (trivial) cofibrations exists, we call it the *canonical* left semi-model structure.

Theorem

Assuming the canonical left semi-model structure on n-$\mathcal{G}$pd exists, there exists a cofibrantly generated left semi-model structure on the category $\mathbf{Mod}(\mathscr{C})_{cosk_{(n+1)}}$ of $(n+1)$-coskeletal ∞-groupoids, and the previous adjunction is a Quillen equivalence with respect to these semi-model structures. Moreover, in this case these equivalent model structures present the $(\infty, 1)$-category of homotopy n-types.

Corollary

Since the canonical left semi-model structure exists in dimension $n = 3$, the homotopy hypothesis holds true for $n = 3$. In particular, Grothendieck 3-groupoids model homotopy 3-types.

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Strategy for the pushout lemma

Since it all boils down to proving that pushout of maps of the form $\sigma_k \colon D_k \to D_{k+1}$ are weak equivalences, this problem is worth focusing on. A sufficient condition is that of (less) than a path object.

Proposition (L.)

If for every cofibrant n-groupoid X there exists a fibration $\mathbf{ev} \colon \mathbb{P}X \to X \times X$ such that $\mathbf{ev}_i = \pi_i \circ \mathbf{ev}$ is a trivial fibration for $i = 0, 1$, where $\pi_i \colon X \times X \to X$ denote the product projections, then the pushout lemma is valid and the canonical model structure on n-groupoids exists.

A good candidate is: $\mathbb{P}X_k \overset{def}{=} n\text{-}\mathcal{G}pd(\mathbf{Cyl}(D_k), X)$, and we have the following result:

Proposition (L.)

If $\mathbb{P}X$ is endowed with an n-groupoid structure compatible with the projections to X, then $\mathbf{ev} \colon \mathbb{P}X \to X \times X$ is a fibration and $\mathbf{ev}_i = \pi_i \circ \mathbf{ev}$ is a trivial fibration for $i = 0, 1$.

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Thanks for your attention!