HOMOTOPY-COHERENT ALGEBRAS AND POLYNOMIAL MONADS

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²Center for Geometry and Physics Institute for Basic Science Pohang, Republic of Korea *Homotopy coherent algebras*: identities are replaced by an infinite hierarchy of compatible coherence equivalences.

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Let \mathbb{F}_* = category of pointed finite sets and let \mathcal{S} = ∞ -category of spaces. A special Γ -space is a functor

$$F \colon \mathbb{F}_* \to \mathcal{S}$$

satisfying the Segal condition $(F(\langle n \rangle) \xrightarrow{\sim} \prod_{i=1}^{n} F(\langle 1 \rangle))$.

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- $\cdot \, \infty \text{-} properads: \, \Gamma^{\mathrm{op}}$ of Hackney, Robertson and Yau

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But in which generality can we talk about "Segal conditions"?

We observe that all the examples have certain key features in common. This leads us to the following definition.

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An algebraic pattern \mathcal{P} is an ∞ -category equipped with

- \cdot an inert-active factorization system ($\mathcal{P}^{\text{int}}, \mathcal{P}^{\text{act}}$),
- · a full subcategory $\mathcal{P}^{el} \subseteq \mathcal{P}^{int}$ (its objects are called *elementary*).

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Let \mathcal{P} be an algebraic pattern and let $\mathcal{P}_{X/}^{el} := \mathcal{P}^{el} \times_{\mathcal{P}^{int}} \mathcal{P}_{X/}^{int}$.

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Then a Segal \mathcal{P} -space or object is a functor $F: \mathcal{P} \to \mathcal{S}$ (or $\mathcal{C} = Cat_{\infty}, \infty$ -topos, ...) which satisfies the "Segal condition", i.e.

$$F(X) \xrightarrow{\sim} \lim_{E \in \mathcal{P}_{X/}^{\mathrm{el}}} F(E).$$

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- \cdot elementary objects \xrightarrow{F} building blocks
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- · active maps \xrightarrow{F} algebraic operations (multiplications, compositions,...)

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- · Ξ = category of unrooted trees defined by HRY. \Rightarrow Cyclic ∞-operads = Segal Ξ^{op} -spaces.
- $\begin{array}{l} \cdot \ \mathcal{O} = \text{Lurie's (generalized)} \ \infty \text{-operad} \Rightarrow \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) = \text{Segal} \\ \mathcal{O}\text{-objects in } \mathcal{C}. \end{array}$

Remark

A functor $f: \mathcal{P} \rightarrow \mathcal{Q}$ induces adjunctions

 $f_!$: Fun $(\mathcal{P}, \mathcal{S}) \rightleftharpoons$ Fun $(\mathcal{Q}, \mathcal{S})$: f^* and f^* : Fun $(\mathcal{Q}, \mathcal{S}) \rightleftharpoons$ Fun $(\mathcal{P}, \mathcal{S})$: f_* .

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Under certain *checkable* conditions f^* , $f_!$ and f_* restrict to functors

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 \mathcal{P} is extendable if $i_!$: $\operatorname{Fun}(\mathcal{P}^{\operatorname{el}}, \mathcal{S}) \simeq \operatorname{Seg}_{\mathcal{P}^{\operatorname{int}}}(\mathcal{S}) \to \operatorname{Seg}_{\mathcal{P}}(\mathcal{S})$ is given by restriction, where $i: \mathcal{P}^{\operatorname{int}} \hookrightarrow \mathcal{P}$ is the inclusion.

Applications

By checking conditions

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By checking conditions

- \cdot for right Kan extensions f_* : The forgetful functor ${\rm CycOpd}_\infty\to {\rm Opd}_\infty$ has a right adjoint.
- for left Kan extensions *f*₁: We recover the formula for operadic left Kan extensions in the sense of Lurie.
- for extendability: Θ_n^{op} , $\Delta^{n,\text{op}}$ and Ω^{op} are extendable. \Rightarrow Formula for free (∞, n) -categories, free *n*-fold ∞ -categories and free ∞ -operads.

Proposition

For every \mathcal{P} the adjunction $\operatorname{Fun}(\mathcal{P}^{\operatorname{el}}, \mathcal{S}) \rightleftarrows \operatorname{Seg}_{\mathcal{P}}(\mathcal{S})$ is monadic.

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Questions

Which monads on presheaf ∞ -categories can be described as the free Segal \mathcal{P} -space monad for an extendable pattern \mathcal{P} ? How far is this correspondence from being an equivalence?

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 - \cdot where an object is a polynomial monadic right adjoints over some functor category Fun(\mathcal{I}, \mathcal{S}),

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Notation

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 - where an object is a polynomial monadic right adjoints over some functor category $Fun(\mathcal{I}, \mathcal{S})$, (a monad is *polynomial* if it is cartesian and a local right adjoint)
 - where a morphism is a commutative square whose mate transformation is cartesian.

If \mathcal{P} is extendable, then the associated monad $T_{\mathcal{P}}$ on $Fun(\mathcal{P}^{el}, \mathcal{S})$ is polynomial. Hence, we have a functor

 $\mathfrak{M}\colon \mathsf{ExtPatt}\to\mathsf{PolyMnd}.$

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We now want to prove that this functor is essentially surjective. For a given polynomial monad we want to construct the associated algebraic pattern.

Remark

The main input for the proof for the essential surjectivity is an ∞ -categorical version of work of Berger, Melliés and Weber and it is closely related to the "nerve theorem" studied by Leinster, Kock and Weber.

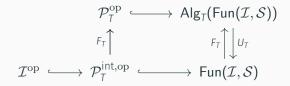
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The difference between cartesian monads (induced by Σ -free operads) and *weakly* cartesian monads in the 1-categorical world vanishes by going to that of ∞ -categories.

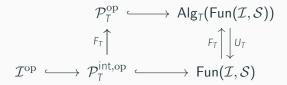
Construction

Given a polynomial monad T on the presheaf ∞ -category Fun(\mathcal{I}, \mathcal{S}), define \mathcal{P}_T by



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The construction defines an algebraic pattern \mathcal{P}_T with a factorization system ($\mathcal{P}_T^{\text{int}}, \mathcal{P}_T^{\text{act}}$) and $\mathcal{P}_T^{\text{el}} = \mathcal{I}$.

The assignment $T \mapsto \mathcal{P}_T$ gives a functor

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Moreover, this functor is fully faithful and $\mathfrak{MP}\simeq\mathsf{id}.$

Examples

• Free operad monad on $\operatorname{Fun}(\mathcal{I}, \mathcal{S})$ (\mathcal{I} = the category of trees with at most one vertex) $\stackrel{\mathfrak{P}}{\mapsto} \Omega^{\operatorname{op}}$.

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- · Free (∞, n) -category monad $\stackrel{\mathfrak{P}}{\mapsto} \Theta_n^{\operatorname{op}}$.

What is the essential image of the fully faithful functor $\mathfrak{P}\colon \mathsf{PolyMnd}\to\mathsf{ExtPatt}?$

An extendable algebraic patterns lies in the essential image of \mathfrak{P} iff it is *nice*, i.e. every object $X \in \mathcal{P}$ admits an active map $X \to E, E \in \mathcal{P}^{\text{el}}$ and $\operatorname{Map}_{\mathcal{P}^{\operatorname{int}}}(X, -) \in \operatorname{Seg}_{\mathcal{P}^{\operatorname{int}}}(\mathcal{S})$.

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Many algebraic patterns are nice: $\Delta^{n, \text{op}}, \Theta_n^{\text{op}}, \Omega^{\text{op}}, \dots$

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Many algebraic patterns are nice: $\Delta^{n, \text{op}}, \Theta_n^{\text{op}}, \Omega^{\text{op}}, \dots$ \mathbb{F}_* is not nice, $\mathfrak{PM}(\mathbb{F}_*) \simeq \text{Span}^{\text{inj}}(\mathbb{F}).$ ExtPatt_n = the full subcategory of ExtPatt spanned by nice algebraic patterns.

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Corollary

The adjunction $\mathfrak{M}:\mathsf{ExtPatt}\rightleftarrows\mathsf{PolyMnd}:\mathfrak{P}$ restricts to an equivalence

ExtPatt_n \simeq PolyMnd.

In particular, PolyMnd is a localization of ExtPatt.