Tangent Categories from the Coalgebras of Differential Categories

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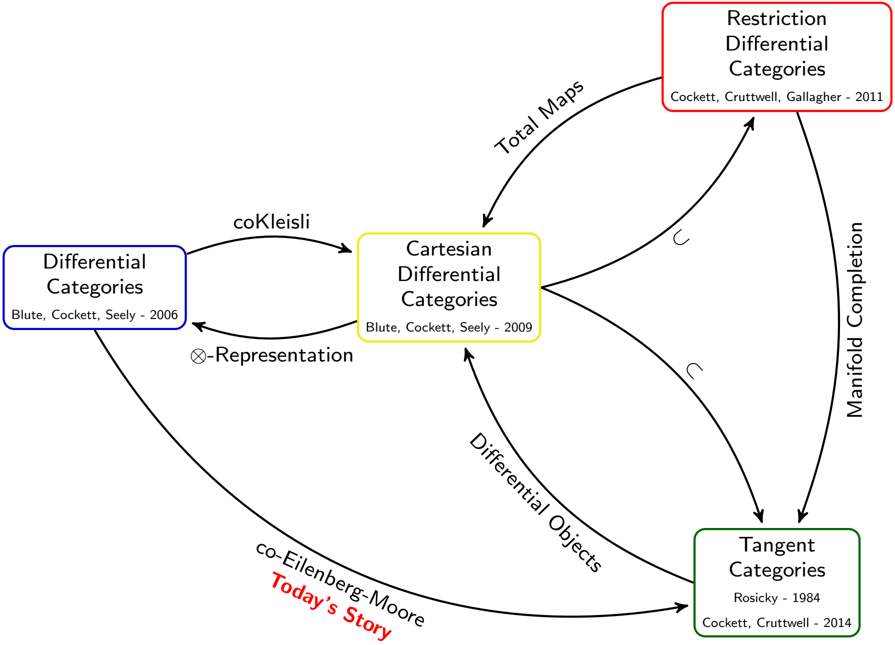
The Differential Category World - How It's All Connected



Tangent Categories - Rosicky (1984) , Cockett and Cruttwell (2014)

A **tangent category** is a category $\mathbb{X}$ which comes equipped with:

- $\boxed{\text{An endofunctor } \mathsf{T} : \mathbb{X} \to \mathbb{X} \text{ called the tangent functor}}$ ← **Today's Story**
- A natural transformation $\mathsf{p} : \mathsf{T} \Rightarrow 1_{\mathbb{X}}$ such that all pullbacks of p along itself n-times exists:

$$\begin{tikzcd} & \mathsf{T}_n(M) \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathsf{T}(M) \arrow[dr, "\mathsf{p}"'] & \ldots \arrow[d, "\mathsf{p}"] & \mathsf{T}(M) \arrow[dl, "\mathsf{p}"] \\ & M & \end{tikzcd}$$

- Plus other natural transformations and certain limits, such that various coherences hold which capture the essential properties of the tangent bundle functor for smooth manifolds.

Example

- The category of finite dimensional smooth manifolds is a tangent category with the tangent functor which maps a smooth manifold M to its tangent bundle $\mathsf{T}(M)$.
- Any category with finite biproducts $\oplus$ is a tangent category with the tangent functor defined on objects as $\mathsf{T}(A) := A \oplus A$ (While trivial: very important for later)
- Let k be a field. The category of commutative k-algebras, CALG_k, is a tangent category with the tangent functor which maps a commutative k-algebra A to its ring of dual numbers:

$$\mathsf{T}(A) = A[\epsilon] = \{a + b\epsilon |\ a, b \in A \text{ and } \epsilon^2 = 0\} = A[x]/(x^2)$$

Representable Tangent Categories: The Link to SDG

A **representable tangent category** is a tangent category with finite products $\times$ such that $\mathsf{T} \cong (-)^D$ for some object D, that is, T is the right adjoint to $- \times D$:

$$\frac{M \times D \to N}{\overline{M \to \mathsf{T}(N)}}$$

The object D is called an **infinitesimal object**.

Example

- Every tangent category embeds into a representable tangent category. (Garner 2018)
- The subcategory of infinitesimally and vertically linear objects of any model of synthetic differential geometry is a representable tangent category with infinitesimal object $D = \{x \in R |\ x^2 = 0\}$, where R is the line object
- Let k be a field. CALG_k^{op} is a representable tangent category with infinitesimal object $k[\epsilon]$, the ring of dual numbers over k. For a commutative k-algebra A, $A^{k[\epsilon]}$ (in CALG_k^{op}) is defined as the symmetric A-algebra of the Kähler module of A.

TODAY'S GOAL: Showing the following:

- The Eilenberg-Moore category of a codifferential category is a tangent category;
- The coEilenberg-Moore category of a differential category is a representable tangent category.

Codifferential Categories - Blute, Cockett, Seely (2006)

A **codifferential category** consists of:

- A (strict) symmetric monoidal category $(\mathbb{X}, \otimes, K, \tau)$;
- Which is enriched over commutative monoids: so each hom-set is a commutative monoid with an addition operation $+$ and a zero 0, such that the additive structure is preserves by composition [1] and $\otimes$.
- An **algebra modality**, which is a monad (S, μ, η) equipped with two natural transformations:

  $$\mathsf{m} : \mathsf{S}(A) \otimes \mathsf{S}(A) \rightarrow \mathsf{S}(A) \qquad \mathsf{u} : K \rightarrow \mathsf{S}(A)$$

  such that $\mathsf{S}(A)$ is a commutative monoid and μ is a monoid morphism.
- And equipped with a **deriving transformation**, which is a natural transformation:

  $$\mathsf{d} : \mathsf{S}(A) \rightarrow \mathsf{S}(A) \otimes A$$

  which satisfies certain equalities which encode the basic properties of differentiation such as the chain rule, product rule, etc.

[1] Composition is written diagramaticaly throughout this presentation: so fg is f then g.

Codifferential Categories - Examples

Example

- Let k be a field and VEC_k the category k-vector spaces.

  Define the algebra modality Sym on VEC_k as follows: for a $\mathbb{K}$-vector space V, let $\mathrm{Sym}(V)$ be the free commutative $\mathbb{K}$-algebra over V, also known as the free symmetric algebra on V. In particular if $X = \{x_1, x_2, \dots\}$ is a basis of V, then $\mathrm{Sym}(V) \cong k[X]$.

  The deriving transformation can be described in terms of polynomials as follows:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$

$$\mathsf{p}(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \dots, x_n) \otimes x_i$$

  So VEC_k is a codifferential category, that is, VEC_k^{op} is a differential category.

- Cofree cocommutative coalgebras also give rise to a differential category structure on VEC_k.
- Free $\mathcal{C}^\infty$-rings give rises to a codifferential category structure on $\mathrm{VEC}_{\mathbb{R}}$ via differentiating smooth functions.
- Categorial models of differential linear logic (such as REL, convenient vector spaces, etc.) are differential categories.

A closer look at the tangent structure of CALG_k

- CALG_k was a tangent category where $\mathsf{T}(A) = A[\epsilon]$.
- Any category with biproducts $\oplus$ is a tangent category with $\mathsf{T}(A) = A \oplus A$. So VEC_k is a tangent category.
- Notice that the underlying k-vector space of $A[\epsilon]$ is precisely $A \oplus A$.
- Turns out that the tangent structure on CALG_k is really just a **lifting** of the biproduct tangent structure on VEC_k.
- CALG_k is equivalent to the Eilenberg-Moore category of Sym from the previous slide, and in particular the Eilenberg-Moore category of a codifferential category!
- This example will be our inspiration.

Lifting Tangent Structure

A **tangent monad** on a tangent category is a monad (S, η, μ) equipped with a distributive law:

$$\lambda_M : \mathsf{S}(\mathsf{T}(M)) \to \mathsf{T}(\mathsf{S}(M))$$

such that λ satisfies the necessary conditions which makes the Eilenberg-Moore category of S a tangent category such that the forgetful functor preserves the tangent structure strictly.

$$\begin{tikzcd} \mathbb{X}^{\mathsf{S}} \arrow[r, "\overline{\mathsf{T}}"] \arrow[d, "\mathsf{U}"'] & \mathbb{X}^{\mathsf{S}} \arrow[d, "\mathsf{U}"] \\ \mathbb{X} \arrow[r, "\mathsf{T}"'] & \mathbb{X} \end{tikzcd}$$

$$\overline{\mathsf{T}}(A,\ \mathsf{S}(A) \xrightarrow{\nu} A\ ) := (\mathsf{T}(A),\ \mathsf{ST}(A) \xrightarrow{\lambda_A} \mathsf{TS}(A) \xrightarrow{\mathsf{T}(\nu)} \mathsf{T}(A))$$

Eilenberg-Moore Category of a Codifferential Category

Let $\mathbb{X}$ be a codifferential category with algebra modality $(\mathsf{S}, \eta, \mu, \nabla, \mathsf{u})$ and deriving transformation d, and suppose that $\mathbb{X}$ admits finite biproducts $\oplus$.

Proposition

Define the natural transformation $\lambda_A : \mathsf{S}(A \oplus A) \to \mathsf{S}(A) \oplus \mathsf{S}(A)$ *as:*

$$\begin{tikzcd}
& \mathsf{S}(A \oplus A) \arrow[r, "\mathsf{d}"] \arrow[dl, "\mathsf{S}(\pi_0)"'] \arrow[d, dashed, "\lambda_A"] & \mathsf{S}(A \oplus A) \otimes (A \oplus A) \arrow[r, "\mathsf{S}(\pi_0) \otimes \pi_1"] & \mathsf{S}(A) \otimes A \arrow[r, "1_{\mathsf{S}(A)} \otimes \eta_A"] & \mathsf{S}(A) \otimes \mathsf{S}(A) \arrow[d, "\nabla_A"] \\
\mathsf{S}(A) & \mathsf{S}(A) \oplus \mathsf{S}(A) \arrow[l, "\pi_0"] \arrow[rrr, "\pi_1"'] & & & \mathsf{S}(A)
\end{tikzcd}$$

Then $(\mathsf{S}, \mu, \eta, \lambda)$ *is a tangent monad on* $\mathbb{X}$ *(with respect to the biproduct tangent structure).*

Theorem

The EM category of a codifferential category with finite biproducts is a tangent category.

$$\overline{\mathsf{T}}(A,\ \mathsf{S}(A) \xrightarrow{\nu} A\ ) := (A \oplus A,\ \mathsf{S}(A \oplus A) \xrightarrow{\lambda_A} \mathsf{S}(A) \oplus \mathsf{S}(A) \xrightarrow{\nu \oplus \nu} A \oplus A)$$

In a certain sense, $\overline{\mathsf{T}}(A, \nu)$ is the ring of dual numbers of an S-algebra (A, ν).

When Tangent Functors have Adjoints

To show that the coEilenberg-Moore category of a differential category is a representable tangent category, we want to make use of the following:

Proposition (Cockett and Cruttwell)

If $\mathbb{X}$ is a tangent such that its tangent functor T has a left adjoint P, and each of the T_n has a left adjoint P_n, then $\mathbb{X}^{op}$ has a tangent structure with tangent functor P.

Corollary

If $\mathbb{X}$ is a representable tangent category with $\mathsf{T} := (-)^D$, then $\mathbb{X}^{op}$ is a tangent category with tangent functor $- \times D$.

- Coproduct of CALG_k is given by the tensor product $\otimes$ (so a product in CALG_k^{op})
- Cockett and Cruttwell first showed that CALG_k^{op} was a representable tangent category with infinitesimal object $D = \mathbb{N}[\epsilon]$, and then used the corollary to obtain that CALG_k was a tangent category with tangent functor $- \otimes \mathbb{N}[\epsilon]$, which gives

$$A \otimes \mathbb{N}[\epsilon] \cong A[\epsilon]$$

- We're going to do the opposite! Use the proposition to instead go from the tangent structure on CALG_k to CALG_k^{op} (or rather for Eilenberg-Moore categories of codifferential categories).

An adjoint lifting theorem

In a category with biproducts, the tangent functor is its own adjoint:

$$\frac{A \to B \oplus B}{\overline{\overline{A \oplus A \to B}}}$$

Somehow we would like lift this adjoint to the Eilenberg-Moore category. However in the Eilenberg-Moore category, $\overline{\mathsf{T}}$ is not necessarily its own adjoint (rarely is!). We can't use adjoint lifting theorems on the nose. Instead we require a specialized version of an adjoint existence theorem of Butler's, which can be found in Barr and Well's TTT book [2]:

Proposition

Let λ be a distributive law of a functor $\mathsf{R} : \mathbb{X} \to \mathbb{X}$ over a monad (S, μ, η), and suppose that R has a left adjoint L. If $\mathbb{X}^{\mathsf{S}}$ admits reflexive coequalizers then the lifting of R, $\overline{\mathsf{R}} : \mathbb{X}^{\mathsf{S}} \to \mathbb{X}^{\mathsf{S}}$, has a left adjoint $\mathsf{G} : \mathbb{X}^{\mathsf{S}} \to \mathbb{X}^{\mathsf{S}}$ such that $\mathsf{G}(\mathsf{S}(A), \mu_A) = (\mathsf{SL}(A), \mu_{\mathsf{L}(A)})$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{X}^{\mathsf{S}} \arrow[r, shift left=2, "\mathsf{G}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "\mathsf{U}"] \arrow[d, phantom, "\dashv"] & \mathbb{X}^{\mathsf{S}} \arrow[l, shift left=2, "\overline{\mathsf{R}}"] \arrow[d, shift left=2, "\mathsf{U}"] \arrow[d, phantom, "\dashv"] \\
\mathbb{X} \arrow[u, shift left=2, "\mathsf{S}"] \arrow[r, shift left=2, "\mathsf{L}"] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[u, shift left=2, "\mathsf{S}"] \arrow[l, shift left=2, "\mathsf{R}"]
\end{tikzcd}$$

[2] Special thanks to Steve Lack for pointing this out to us and avoiding us doing extra work!

Proposition

Let $\mathbb{X}$ be a codifferential category with algebra modality S and suppose that $\mathbb{X}$ admits finite biproducts and $\mathbb{X}^{\mathsf{S}}$ admits reflexive coequalizers. Then for each $n \in \mathbb{N}$, $\overline{\mathsf{T}}_n : \mathbb{X}^{\mathsf{S}} \to \mathbb{X}^{\mathsf{S}}$ has a left adjoint. And so $(\mathbb{X}^{\mathsf{S}})^{op}$ is a tangent category.

Theorem

If the coEilenberg-Moore category of a differential category with finite biproducts admits coreflexive equalizers (the dual of reflexive coequalizers), then the coEilenberg-Moore category is a tangent category.

But we would like a representable tangent functor! And for this we need at least products...

So how do we get products in the coEilenberg-Moore category of a differential category?

(or how do we get coproducts in the Eilenberg-Moore category of a codifferential category?)

Seely Isomorphisms

In a codifferential category with biproducts, the biproduct tangent functor can be written out as:

$$A \oplus A \cong A \otimes (K \oplus K)$$

We would like to turn the tensor product into a coproduct in the Eilenberg-Moore category.

A well-known (dual) result from the categorical semantics of linear logic is that the Eilenberg-Moore category $\mathbb{X}^{\mathsf{S}}$ has finite coproducts if and only if S has the Seely isomorphisms:

$$\mathsf{S}(A \oplus B) \cong \mathsf{S}(A) \otimes \mathsf{S}(B) \qquad \mathsf{S}(0) \cong K$$

The $\otimes$ of $\mathbb{X}$ becomes a coproduct in $\mathbb{X}^{\mathsf{S}}$.

Example

The algebra modality Sym on VEC_k has the Seely isomorphisms. Therefore, the tensor product of VEC_k becomes a coproduct in $\mathsf{VEC}_k^{\mathsf{Sym}} \cong \mathsf{CALG}_k$.

Furthermore, there is a map $\mathsf{n}_K : \mathsf{S}(K) \to K$ which makes (K, n_K) an S-algebra and we have that:

$$\overline{\mathsf{T}}(A, \nu) \cong (A, \nu) \otimes \overline{\mathsf{T}}(K, \mathsf{n}_K)$$

So if we have reflexive equalizers, $(-) \otimes \overline{\mathsf{T}}(K, \mathsf{n}_K)$ has a left adjoint, which in the dual case gives...

Representable Tangent Category from Coalgebras

Theorem

If the coEilenberg-Moore category of a differential category (whose coalgebra modality has the Seely isomorphisms) admits coreflexive equalizers, then the coEilenberg-Moore category is a representable tangent category.

Let $\mathbb{X}$ be a differential category with finite biproducts and coalgebra modality $!$ (dual of an algebra modality), which has the Seely isomorphisms: $!(A \oplus B) \cong !A \otimes !B$ and $!(0) \cong K$.

Then there is a map $\mathsf{m}_K : K \to !(K)$ that makes (K, m_K) into a $!$-coalgebra.

The infinitesimal object is $(K \oplus K, \mathsf{m}^\sharp_K)$, where $\mathsf{m}^\sharp_K : K \oplus K \to !(K \oplus K)$ is defined as:

$$\begin{tikzcd}
K \arrow[r, "\iota_0"] \arrow[dd, "\mathsf{m}_K"'] & K \oplus K \arrow[dd, dashed, "\mathsf{m}^\sharp_K"] & K \arrow[l, "\iota_1"'] \arrow[r, "\cong"] & K \otimes K \arrow[d, "\mathsf{m}_K \otimes 1_K"] \\
 & & & !(K) \otimes K \arrow[d, "!(\iota_0) \otimes \iota_1"] \\
!(K) \arrow[r, "!(\iota_0)"'] & !(K \oplus K) & & !(K \oplus K) \otimes (K \oplus K) \arrow[ll, "\mathsf{d}_{K \oplus K}"]
\end{tikzcd}$$

Future Work - Lots to do!

- Now that we have a (representable) tangent category, lots of things we can do and study!
  - Vector Fields (Answer: Generalized Differential Algebras)
  - Various type of Line Objects (How close do we get to SDG?)
  - Differential Objects (Euclidean Spaces/Cartesian Differential Categories)
  - etc.

- Study these constructions for other well-known differential categories (ex. convenient vector spaces) and construct new examples.

- Does the tangent bundle functor on the coEM category of a differential category have a more explicit construction? Does $(-)^{(K \oplus K, \mathsf{m}_K^\sharp)}$ ever have a nice form?

  For example in general, for cofree !-coalgebras we have that:

  $$\mathsf{T}(!(A), \delta_A) = (!(A \oplus A), \delta_{A \oplus A})$$

  whether T is representable or not.

The Differential Category World: It's all connected!

Hope you enjoyed it!
Thanks for listening!
Merci!