Tangent Categories from the Coalgebras of Differential Categories

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The Differential Category World - How It’s All Connected

Differential Categories
Blute, Cockett, Seely - 2006

Cartesian Differential Categories
Blute, Cockett, Seely - 2009

Restriction Differential Categories
Cockett, Cruttwell, Gallagher - 2011

Tangent Categories
Rosicky - 1984
Cockett, Cruttwell - 2014

coKleisli
⊗-Representation

Total Maps

co-Eilenberg-Moore

Today’s Story

Differential Objects

Manifold Completion
A **tangent category** is a category $\mathbf{X}$ which comes equipped with:

- An endofunctor $T : \mathbf{X} \to \mathbf{X}$ called the tangent functor

  ![Diagram](image)

  Today’s Story

- A natural transformation $p : T \Rightarrow 1_\mathbf{X}$ such that all pullbacks of $p$ along itself $n$-times exists:

  $$\begin{array}{ccc}
  T_n(M) & \xrightarrow{p} & M \\
  \downarrow & & \downarrow \\
  T(M) & \xleftarrow{p} & T(M)
  \end{array}$$

  Plus other natural transformations and certain limits, such that various coherences hold which capture the essential properties of the **tangent bundle functor** for smooth manifolds.

**Example**

- The category of finite dimensional smooth manifolds is a tangent category with the tangent functor which maps a smooth manifold $M$ to its tangent bundle $T(M)$.

- Any category with finite biproducts $\oplus$ is a tangent category with the tangent functor defined on objects as $T(A) := A \oplus A$ (While trivial: very important for later)

- Let $k$ be a field. The category of commutative $k$-algebras, $\text{CALG}_k$, is a tangent category with the tangent functor which maps a commutative $k$-algebra $A$ to its ring of dual numbers:

  $$T(A) = A[\epsilon] = \{a + b\epsilon \mid a, b \in A \text{ and } \epsilon^2 = 0\} = \frac{A[x]}{(x^2)}$$
A **representable tangent category** is a tangent category with finite products $\times$ such that $T \cong (-)^D$ for some object $D$, that is, $T$ is the right adjoint to $- \times D$:

$$
\begin{align*}
M \times D &\to N \\
M &\to T(N)
\end{align*}
$$

The object $D$ is called an **infinitesimal object**.

**Example**

- Every tangent category embeds into a representable tangent category. (Garner 2018)

- The subcategory of infinitesimally and vertically linear objects of any model of synthetic differential geometry is a representable tangent category with infinitesimal object $D = \{x \in R \mid x^2 = 0\}$, where $R$ is the line object.

- Let $k$ be a field. $\text{CALG}^{op}_k$ is a representable tangent category with infinitesimal object $k[\epsilon]$, the ring of dual numbers over $k$. For a commutative $k$-algebra $A$, $A^{k[\epsilon]}$ (in $\text{CALG}^{op}_k$) is defined as the symmetric $A$-algebra of the Kähler module of $A$.

**TODAY’S GOAL:** Showing the following:

- The Eilenberg-Moore category of a codifferential category is a tangent category;

- The coEilenberg-Moore category of a differential category is a representable tangent category.
A **codifferential category** consists of:

- A (strict) symmetric monoidal category \((\mathcal{X}, \otimes, K, \tau)\);

- Which is enriched over commutative monoids: so each hom-set is a commutative monoid with an addition operation \(+\) and a zero \(0\), such that the additive structure is preserved by composition \(^1\) and \(\otimes\).

- An **algebra modality**, which is a monad \((S, \mu, \eta)\) equipped with two natural transformations:
  
  \[
  m : S(A) \otimes S(A) \to S(A) \quad u : K \to S(A)
  \]

  such that \(S(A)\) is a commutative monoid and \(\mu\) is a monoid morphism.

- And equipped with a **deriving transformation**, which is a natural transformation:
  
  \[
  d : S(A) \to S(A) \otimes A
  \]

  which satisfies certain equalities which encode the basic properties of differentiation such as the chain rule, product rule, etc.

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\(^1\) Composition is written diagramatically throughout this presentation: so \(fg\) is \(f\) then \(g\).
Codifferential Categories - Examples

Example

Let \( k \) be a field and \( \text{VEC}_k \) the category \( k \)-vector spaces.

Define the algebra modality \( \text{Sym} \) on \( \text{VEC}_k \) as follows: for a \( k \)-vector space \( V \), let \( \text{Sym}(V) \) be the free commutative \( k \)-algebra over \( V \), also known as the free symmetric algebra on \( V \). In particular if \( X = \{x_1, x_2, \ldots \} \) is a basis of \( V \), then \( \text{Sym}(V) \cong k[X] \).

The deriving transformation can be described in terms of polynomials as follows:

\[
d : \mathbb{K}[X] \rightarrow \mathbb{K}[X] \otimes V
\]

\[
p(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i
\]

So \( \text{VEC}_k \) is a codifferential category, that is, \( \text{VEC}_k^{\text{op}} \) is a differential category.

- Cofree cocommutative coalgebras also give rise to a differential category structure on \( \text{VEC}_k \).
- Free \( C^\infty \)-rings give rises to a codifferential category structure on \( \text{VEC}_\mathbb{R} \) via differentiating smooth functions.
- Categorial models of differential linear logic (such as REL, convenient vector spaces, etc.) are differential categories.
CALGₖ was a tangent category where $T(A) = A[\epsilon]$.

Any category with biproducts $⊕$ is a tangent category with $T(A) = A ⊕ A$. So $\text{VEC}_k$ is a tangent category.

Notice that the underlying $k$-vector space of $A[\epsilon]$ is precisely $A ⊕ A$.

Turns out that the tangent structure on CALGₖ is really just a lifting of the biproduct tangent structure on $\text{VEC}_k$.

CALGₖ is equivalent to the Eilenberg-Moore category of Sym from the previous slide, and in particular the Eilenberg-Moore category of a codifferential category!

This example will be our inspiration.
A tangent monad on a tangent category is a monad \((S, \eta, \mu)\) equipped with a distributive law:

\[
\lambda_M : S(T(M)) \to T(S(M))
\]

such that \(\lambda\) satisfies the necessary conditions which makes the Eilenberg-Moore category of \(S\) a tangent category such that the forgetful functor preserves the tangent structure strictly.

\[
\begin{array}{ccc}
X^S & \xrightarrow{T} & X^S \\
\downarrow U & & \downarrow U \\
X & \xrightarrow{T} & X
\end{array}
\]

\[
\overline{T}(A, S(A) \xrightarrow{\nu} A) := (T(A), \ ST(A) \xrightarrow{\lambda_A} TS(A) \xrightarrow{T(\nu)} T(A))
\]
Let $\mathbb{X}$ be a codifferential category with algebra modality $(S, \eta, \mu, \nabla, u)$ and deriving transformation $d$, and suppose that $\mathbb{X}$ admits finite biproducts $\oplus$.

**Proposition**

Define the natural transformation $\lambda_A : S(A \oplus A) \to S(A) \oplus S(A)$ as:

$$
\begin{array}{c}
S(A \oplus A) \\ S(\pi_0) \\
\downarrow \lambda_A \\
S(A) \leftarrow \pi_0 \\
S(A) \oplus S(A) \\
\end{array}
\xrightarrow{\sigma} 
\begin{array}{c}
S(A \oplus A) \otimes (A \oplus A) \\
S(\pi_0) \otimes \pi_1 \\
\downarrow 1_{S(A)} \otimes \eta_A \\
S(A) \otimes S(A) \\
\end{array}
\xrightarrow{\nabla_A}
S(A)
$$

Then $(S, \mu, \eta, \lambda)$ is a tangent monad on $\mathbb{X}$ (with respect to the biproduct tangent structure).

**Theorem**

The EM category of a codifferential category with finite biproducts is a tangent category.

$$
\overline{T}(A, S(A) \xrightarrow{\nu} A) := (A \oplus A, S(A \oplus A) \xrightarrow{\lambda_A} S(A) \oplus S(A) \xrightarrow{\nu \oplus \nu} A \oplus A)
$$

In a certain sense, $\overline{T}(A, \nu)$ is the ring of dual numbers of an $S$-algebra $(A, \nu)$. 

When Tangent Functors have Adjoint}

To show that the coEilenberg-Moore category of a differential category is a representable tangent category, we want to make use of the following:

**Proposition (Cockett and Cruttwell)**

If $X$ is a tangent such that its tangent functor $T$ has a left adjoint $P$, and each of the $T_n$ has a left adjoint $P_n$, then $X^{op}$ has a tangent structure with tangent functor $P$.

**Corollary**

If $X$ is a representable tangent category with $T := (−)^D$, then $X^{op}$ is a tangent category with tangent functor $− \times D$.

- Coproduct of $\text{CALG}_k$ is given by the tensor product $\otimes$ (so a product in $\text{CALG}^{op}_k$)

- Cockett and Cruttwell first showed that $\text{CALG}^{op}_k$ was a representable tangent category with infinitesimal object $D = \mathbb{N}[\epsilon]$, and then used the corollary to obtain that $\text{CALG}_k$ was a tangent category with tangent functor $− \otimes \mathbb{N}[\epsilon]$, which gives

$$A \otimes \mathbb{N}[\epsilon] \cong A[\epsilon]$$

- We’re going to do the opposite! Use the proposition to instead go from the tangent structure on $\text{CALG}_k$ to $\text{CALG}^{op}_k$ (or rather for Eilenberg-Moore categories of codifferential categories).
An adjoint lifting theorem

In a category with biproducts, the tangent functor is its own adjoint:

\[
\begin{align*}
A &\rightarrow B \oplus B \\
A \oplus A &\rightarrow B
\end{align*}
\]

Somehow we would like lift this adjoint to the Eilenberg-Moore category. However in the Eilenberg-Moore category, \(\overline{T}\) is not necessarily its own adjoint (rarely is!). We can’t use adjoint lifting theorems on the nose. Instead we require a specialized version of an adjoint existence theorem of Butler’s, which can be found in Barr and Well’s TTT book \(^2\):

**Proposition**

Let \(\lambda\) be a distributive law of a functor \(R : X \rightarrow X\) over a monad \((S, \mu, \eta)\), and suppose that \(R\) has a left adjoint \(L\). If \(X^S\) admits reflexive coequalizers then the lifting of \(R\), \(\overline{R} : X^S \rightarrow X^S\), has a left adjoint \(G : X^S \rightarrow X^S\) such that \(G(S(A), \mu_A) = (SL(A), \mu_{L(A)})\).

\(^2\)Special thanks to Steve Lack for pointing this out to us and avoiding us doing extra work!
Proposition

Let \( \mathcal{X} \) be a codifferential category with algebra modality \( S \) and suppose that \( \mathcal{X} \) admits finite biproducts and \( \mathcal{X}^S \) admits reflexive coequalizers. Then for each \( n \in \mathbb{N} \), \( \overline{T}_n : \mathcal{X}^S \to \mathcal{X}^S \) has a left adjoint. And so \( (\mathcal{X}^S)^{op} \) is a tangent category.

Theorem

If the coEilenberg-Moore category of a differential category with finite biproducts admits coreflexive equalizers (the dual of reflexive coequalizers), then the coEilenberg-Moore category is a tangent category.

But we would like a representable tangent functor! And for this we need at least products...

So how do we get products in the coEilenberg-Moore category of a differential category? (or how do we get coproducts in the Eilenberg-Moore category of a codifferential category?)
Seely Isomorphisms

In a codifferential category with biproducts, the biproduct tangent functor can be written out as:

\[ A \oplus A \cong A \otimes (K \oplus K) \]

We would like to turn the tensor product into a coproduct in the Eilenberg-Moore category.

A well-known (dual) result from the categorical semantics of linear logic is that the Eilenberg-Moore category \( X^S \) has finite coproducts if and only if \( S \) has the Seely isomorphisms:

\[ S(A \oplus B) \cong S(A) \otimes S(B) \quad S(0) \cong K \]

The \( \otimes \) of \( X \) becomes a coproduct in \( X^S \).

**Example**

The algebra modality \( \text{Sym} \) on \( \text{VEC}_k \) has the Seely isomorphisms. Therefore, the tensor product of \( \text{VEC}_k \) becomes a coproduct in \( \text{VEC}_{k}^{\text{Sym}} \cong \text{CALG}_k \).

Furthermore, there is a map \( n_K : S(K) \to K \) which makes \( (K, n_K) \) an \( S \)-algebra and we have that:

\[ \overline{T}(A, \nu) \cong (A, \nu) \otimes \overline{T}(K, n_K) \]

So if we have reflexive equalizers, \( (\_ \_ \otimes \overline{T}(K, n_K) \) has a left adjoint, which in the dual case gives...
Theorem

If the coEilenberg-Moore category of a differential category (whose coalgebra modality has the Seely isomorphisms) admits coreflexive equalizers, then the coEilenberg-Moore category is a representable tangent category.

Let $\mathcal{X}$ be a differential category with finite biproducts and coalgebra modality $!$ (dual of an algebra modality), which has the Seely isomorphisms: $!(A \oplus B) \cong !A \otimes !B$ and $!(0) \cong K$.

Then there is a map $m_K : K \to !(K)$ that makes $(K, m_K)$ into a $!$-coalgebra.

The infinitesimal object is $(K \oplus K, m_K^\#)$, where $m_K^\# : K \oplus K \to !(K \oplus K)$ is defined as:

\[
\begin{array}{ccccccc}
K & \xrightarrow{\iota_0} & K \oplus K & \xleftarrow{\iota_1} & K & \xrightarrow{=} & K \otimes K \\
m_K & \downarrow & m_K^\# & \downarrow & m_K \otimes 1_K & \downarrow & !((\iota_0) \otimes \iota_1) \\
!(K) & \xrightarrow{!(\iota_0)} & !(K \oplus K) & \xleftarrow{d_{K \oplus K}} & !(K \oplus K) \otimes (K \oplus K)
\end{array}
\]
Now that we have a (representable) tangent category, lots of things we can do and study!

- Vector Fields (Answer: Generalized Differential Algebras)
- Various type of Line Objects (How close do we get to SDG?)
- Differential Objects (Euclidean Spaces/Cartesian Differential Categories)
- etc.

Study these constructions for other well-known differential categories (ex. convenient vector spaces) and construct new examples.

Does the tangent bundle functor on the coEM category of a differential category have a more explicit construction? Does \((-)^{(K \oplus K, m^k_k)}\) ever have a nice form?

For example in general, for cofree !-coalgebras we have that:

$$T(! (A), \delta_A) = (! (A \oplus A), \delta_{A \oplus A})$$

whether \(T\) is representable or not.
The Differential Category World: It’s all connected!

Hope you enjoyed it! Thanks for listening! Merci!

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⊗-Representation

Manifold Completion

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