A unified framework for notions of algebraic theory

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Conceptual levels in study of algebra

1. **Algebra**
   A set (an object) equipped with an algebraic structure.
   E.g., the group $\mathfrak{S}_5$, the ring $\mathbb{Z}$.

2. **Algebraic theory**
   Specification of a type of algebras.
   E.g., the clone of groups, the operad of monoids.

3. **Notion of algebraic theory**
   Framework for a type of algebraic theories.
   E.g., \{clones\}, \{operads\}.

This talk: unified account of **notions of algebraic theory**.
Examples of notions of algebraic theory

1. **Clones/Lawvere theories** [Lawvere, 1963]
   Categorical equivalent of **universal algebra**.
   Applications to computational effects [Plotkin–Power 2002, ...].

2. **Symmetric operads, non-symmetric operads** [May, 1972]
   Originates in homotopy theory for **algebras-up-to-homotopy**.

3. **Clubs/generalised operads** [Burroni, 1971; Kelly, 1972]
   Classical approach to **categories with structure** [Kelly 1972].
   The ‘globular operad’ approach to higher categories [Batanin 1998, Leinster 2004].

4. **PROPs, PROs** [Mac Lane 1965]
   ‘Many-in, many-out’ version of (non-)symmetric operads.

5. **Monads** [Godement, 1958; Linton, 1965; Eilenberg–Moore, 1965]
   Monads on **Set** = infinitary version of clones.
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**Definition**

1. A *metatheory* is a monoidal category $\mathcal{M} = (\mathcal{M}, I, \otimes)$.
2. A *theory in* $\mathcal{M}$ is a monoid $T = (T, e, m)$ in $\mathcal{M}$. That is,
   - $T$: an object of $\mathcal{M}$;
   - $e: I \to T$;
   - $m: T \otimes T \to T$;

satisfying the associativity and unit laws.

‘*Metatheory*’ (technical term) formalises ‘*notion of algebraic theory*’ (non-technical term).
Example: clones

Definition

The category $\mathbf{F}$

- object: the sets $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{N}$;
- morphism: all functions.
Example: clones

Definition

The **metatheory of clones** is the monoidal category \(([\mathbf{F}, \mathbf{Set}], I, \bullet)\) where \(\bullet\) is the *substitution monoidal product* [Kelly–Power 1993; Fiore–Plotkin–Turi 1999].

- \(I = \mathbf{F}([1], -) \in [\mathbf{F}, \mathbf{Set}]\);
- for \(X, Y \in [\mathbf{F}, \mathbf{Set}]\),

\[
(Y \bullet X)_n = \int_{[m] \in \mathbf{F}} Y_m \times (X_n)^m.
\]
Example: clones

\[ \theta \in X_n \]

An element of \((Y \cdot X)_n = \int_{[m] \in F} Y_m \times (X_n)^m\) is:

\[ \phi \in Y_m, \theta_i \in X_n \]

modulo action of \(F\).
Example: clones

**Definition (classical; see e.g., [Taylor, 1993])**

A *clone* $C$ is given by

- $(C_n)_{n \in \mathbb{N}}$: a family of sets;
- $\forall n \in \mathbb{N}, \forall i \in \{1, \ldots, n\}$, an element $p_i^{(n)} \in C_n$;
- $\forall n, m \in \mathbb{N}$, a function

$$\circ_{(n)}^{(m)} : C_m \times (C_n)^m \rightarrow C_n$$

satisfying the associativity and the unit axioms.

(In universal algebra, people sometimes omit $C_0$.)
Example: clones

Example

\( C \): category with finite products
\( C \in C \)

The clone \( \text{End}(C) \) of **endo-multimorphisms on** \( C \) is defined by:

- \( \text{End}(C)_n = C(C^n, C) \);
- \( p_i^{(n)} \in \text{End}(C)_n \) is the \( i \)-th projection \( p_i^{(n)} : C^n \to C \);
- \( \circ_m^{(n)} : \text{End}(C)_m \times (\text{End}(C)_n)^m \to \text{End}(C)_n \) maps \( (g, f_1, \ldots, f_m) \) to \( g \circ \langle f_1, \ldots, f_m \rangle \):

\[
\begin{align*}
C^n & \xrightarrow{\langle f_1, \ldots, f_m \rangle} C^m \\
& \xrightarrow{g} C.
\end{align*}
\]

(In fact, every clone is isomorphic to \( \text{End}(C) \) for some \( C \) and \( C \in C \).

Fujii (Kyoto)
Example: clones

Proposition ([Kelly–Power, 1993; Fiore–Plotkin–Turi 1999])

There is an isomorphism of categories

\[ \text{Clo} \cong \text{Mon}([F, \text{Set}], I, \bullet). \]
Example: clones

Recall again:

**Definition**

1. A **metatheory** is a monoidal category \( \mathcal{M} \).
2. A **theory in** \( \mathcal{M} \) is a monoid \( T \) in \( \mathcal{M} \).

and:

**Definition**

The **metatheory of clones** is the monoidal category \( ([\mathbf{F}, \mathbf{Set}], I, \bullet) \).

Theories in \( ([\mathbf{F}, \mathbf{Set}], I, \bullet) \) = clones.
Example: symmetric operads

**Definition**

The category $P$
- object: the sets $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{N}$;
- morphism: all bijections.

**Definition (cf. [Kelly 2005; Curien 2012; Hyland 2014])**

The **metatheory of symmetric operads** is the monoidal category $([P, \text{Set}], /, \bullet)$.

Variables can be permuted, but cannot be copied nor discarded.

- $x_1 \cdot x_2 = x_2 \cdot x_1$; $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$.
- $x_1 \cdot x_1 = x_1$; $x_1 \cdot x_2 = x_1$. 

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Example: non-symmetric operads

**Definition**

The (discrete) category $\mathbb{N}$

- object: the sets $[n] = \{1, ..., n\}$ for all $n \in \mathbb{N}$;
- morphism: all identities.

**Definition (cf. [Kelly 2005; Curien 2012; Hyland 2014])**

The **metatheory of non-symmetric operads** is the monoidal category $([\mathbb{N}, \text{Set}], I, \bullet)$.

Variables cannot be permuted (nor discarded/copied).

- $\checkmark$ $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$;  $\phi_m(\phi_{m'}(x_1)) = \phi_{mm'}(x_1)$.
- $\times$ $x_1 \cdot x_2 \neq x_2 \cdot x_1$. 
**Definition ([Mac Lane 1965])**

A PRO is given by:

- a monoidal category $T$;
- an identity-on-objects, strict monoidal functor $J$ from the (strict) monoidal category $N = (N, [0], +)$ to $T$.

For $n, m \in \mathbf{Nat}$, an element $\theta \in T([n], [m])$ is depicted as

![Diagram](image-url)
Example: PROs

**Definition ([Bénabou 1973; Lawvere 1973])**

\[ \mathcal{A}, \mathcal{B}: \text{(small)} \]

A profunctor (\(=\) distributor \(=\) bimodule) from \(\mathcal{A}\) to \(\mathcal{B}\) is a functor

\[ H: \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}. \]

Categories, profunctors and natural transformations form a bicategory.

\(\Rightarrow\) For any category \(\mathcal{A}\), the category \([\mathcal{A}^{\text{op}} \times \mathcal{A}, \text{Set}]\) of endo-profunctors on \(\mathcal{A}\) is monoidal.

---

\(^1\)In this talk, I am going to ignore the size issues.
**Example: PROs**

**Proposition (Folklore)**

A: category

To give a monoid in \([A^{op} \times A, \text{Set}]\) is equivalent to giving a category \(B\) together with an identity-on-objects functor \(J: A \rightarrow B\).

Recall:

**Definition ([Mac Lane 1965])**

A PRO is given by:

- a monoidal category \(T\);
- an identity-on-objects, strict monoidal functor \(J\) from the (strict) monoidal category \(N = (N, [0], +)\) to \(T\).

Idea: use a monoidal version of profunctors.
Example: PROs

**Definition ([Im–Kelly 1986])**

\( \mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}}), \mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}}) \): monoidal category

A monoidal profunctor from \( \mathcal{M} \) to \( \mathcal{N} \) is a lax monoidal functor

\[
(H, h., h) : \mathcal{N}^{\text{op}} \times \mathcal{M} \longrightarrow (\textbf{Set}, 1, \times).
\]

That is:

- a functor \( H : \mathcal{N}^{\text{op}} \times \mathcal{M} \longrightarrow \textbf{Set} \);
- a function \( h. : 1 \longrightarrow H(I_{\mathcal{N}}, I_{\mathcal{M}}) \);
- a natural transformation

\[
h_{\mathcal{N}, \mathcal{N}', \mathcal{M}, \mathcal{M}'} : H(N', M') \times H(N, M) \longrightarrow H(N' \otimes_{\mathcal{N}} N, M' \otimes_{\mathcal{M}} M)
\]

satisfying the coherence axioms.
Example: PROs

Monoidal categories, monoidal profunctors and monoidal natural transformations form a bicategory.
⇒ For any monoidal category $\mathcal{M}$, the category $\text{MonCat}(\mathcal{M}^{\text{op}} \times \mathcal{M}, \text{Set})$ is monoidal.

**Proposition**

$\mathcal{M}$: monoidal category

To give a monoid in $\text{MonCat}(\mathcal{M}^{\text{op}} \times \mathcal{M}, \text{Set})$ is equivalent to giving a monoidal category $\mathcal{N}$ together with an identity-on-objects strict monoidal functor $J: \mathcal{M} \rightarrow \mathcal{N}$.

**Definition**

The metatheory of PROs is the monoidal category $\text{MonCat}(\mathcal{N}^{\text{op}} \times \mathcal{N}, \text{Set})$. 
Other examples

**Definition**

The **metatheory of PROPs** is the monoidal category $\text{SymMonCat}(\mathbf{P}^{\text{op}} \times \mathbf{P}, \text{Set})$ of symmetric monoidal endo-profunctors on $\mathbf{P}$.

**Definition**

$\mathcal{C}$: category with finite limits; $\mathcal{S}$: cartesian monad on $\mathcal{C}$

The **metatheory of clubs over $\mathcal{S}$** is the monoidal category $(\mathcal{C}/\mathcal{S}1, \eta_1, \bullet)$.

**Definition**

$\mathcal{C}$: category.

The **metatheory of monads on $\mathcal{C}$** is the monoidal category $\mathcal{E}nd(\mathcal{C}) = ([\mathcal{C}, \mathcal{C}], \text{id}_\mathcal{C}, \circ)$. 
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One theory, various models

Important feature of notions of algebraic theory (esp. of clones, operads, PROs, PROPs): **a single theory can have models in many categories.**

**Example**

A clone can have its models in **any category with finite products.** Models of the clone of groups

- in **Set**: ordinary groups;
- in **FinSet**: finite groups;
- in **Top**: topological groups;
- in **Mfd**: Lie groups;
- in **Grp**: abelian groups.

How does it work?
Given a notion of algebraic theory, ...

1. first define a **notion of model**, i.e., what it means to be a model of a theory;

2. then consider a **model** of a theory following the notion of model.

**Example**

For clones, ...

1. $C$: category with finite product
   a model in $C$ of a clone $T$ is an object $C \in C$ together with a clone morphism $T \rightarrow \text{End}(C)$;

2. find a particular model, i.e., an object $C \in C$ together with a clone morphism $T \rightarrow \text{End}(C)$. 
For **metatheories** (formalising notions of algebraic theory), we introduce **metamodels** (formalising notions of model) later.

First we look at two simple subclasses of metamodels:

- enrichment;
- (left) oplax action.
Definitions

**Definition**

\[ \mathcal{M} = (\mathcal{M}, I, \otimes): \text{metatheory};\ T = (T, e, m): \text{theory in} \ \mathcal{M}. \]

1. An **enrichment in** \( \mathcal{M} \) is a category \( \mathcal{C} \) equipped with
   - \( \langle -, - \rangle : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{M} \): a functor;
   - \( j_C : I \rightarrow \langle C, C \rangle \): a nat. tr.;
   - \( M_{A,B,C} : \langle B, C \rangle \otimes \langle A, B \rangle \rightarrow \langle A, C \rangle \): a nat. tr.

   satisfying the suitable coherence axioms.

\[(\forall C \in \mathcal{C}, \text{End}(C) = (\langle C, C \rangle, j_C, M_{C,C,C}): \text{monoid in} \ \mathcal{M}.)\]

2. A **model of** \( T \) **with respect to** \((\mathcal{C}, \langle -, - \rangle)\) is an object \( C \) of \( \mathcal{C} \) together with a monoid morphism \( T \rightarrow \text{End}(C) \). That is,
   - \( \chi : T \rightarrow \langle C, C \rangle \): a morphism in \( \mathcal{M} \)

   commuting with multiplication and unit.
Definitions

\( \mathcal{M} \): metatheory

\( T \): theory in \( \mathcal{M} \)

\( (C, \langle -, - \rangle) \): enrichment in \( \mathcal{M} \)

We obtain the category

\[ \text{Mod}(T, (C, \langle -, - \rangle)) \]

of \textit{models} and \textit{homomorphisms} together with a forgetful functor

\[ \text{Mod}(T, (C, \langle -, - \rangle)) \xrightarrow{U} C \]
Example: clones \([F, \text{Set}]\)

**Definition**

\(C\): category with finite products

The **standard** \(C\)-metamodel of clones is the enrichment
\[
\langle - , - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow [F, \text{Set}] \text{ given by}
\]

- for \(A, B \in \mathcal{C}\) and \([m] \in F\),

\[
\langle A, B \rangle_m = \mathcal{C}(A^m, B)。
\]

So a model of a theory \(T = (T, e, m)\) consists of

- an object \(C \in \mathcal{C}\);

- a nat. tr. \(\chi: T \rightarrow \langle C, C \rangle \) (w/ cond.)

- \(\forall m \in \mathbb{N}\), a function \(\chi_m: T_m \rightarrow \mathcal{C}(C^m, C)\) (w/ cond.)

- \(\forall m \in \mathbb{N}, \forall \theta \in T_m\), a morphism \([\theta]\chi: C^m \rightarrow C\) (w/ cond.).
**Example: PROs** $\mathcal{MonCat}(\mathbf{N}^{\text{op}} \times \mathbf{N}, \mathbf{Set})$

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<td>$\mathcal{C} = (\mathcal{C}, I, \otimes)$: monoidal category</td>
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The **standard $\mathcal{C}$-metamodel of PROs** is the enrichment $\langle - , - \rangle : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{MonCat}(\mathbf{N}^{\text{op}} \times \mathbf{N}, \mathbf{Set})$ given by

- for $A, B \in \mathcal{C}$ and $n, m \in \mathbf{N}$,

  $$\langle A, B \rangle([n], [m]) = \mathcal{C}(A^{\otimes m}, B^{\otimes n}).$$

There are analogous enrichments for non-symmetric operads, symmetric operads and PROPs.
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Definition

\( \mathcal{M} = (\mathcal{M}, I, \otimes) \): metatheory; \( T = (T, e, m) \): theory in \( \mathcal{M} \).

1. A **(left) oplax action of** \( \mathcal{M} \) is a category \( C \) equipped with

   - \( \ast: \mathcal{M} \times C \to C \): a functor;
   - \( \varepsilon_C: I \ast C \to C \): a nat. tr.;
   - \( \delta_{X,Y,C}: (Y \otimes X) \ast C \to Y \ast (X \ast C) \): a nat. tr.

   satisfying the suitable coherence axioms.

2. A **model of** \( T \) **with respect to** \( (C, \ast) \) is an object \( C \) of \( C \) together with a left \( T \)-action \( \gamma \) on \( C \). That is,

   - \( \gamma: T \ast C \to C \): a morphism in \( C \)

   satisfying the associativity and left unit axioms.
Definitions

\( \mathcal{M} \): metatheory
\( T \): theory in \( \mathcal{M} \)
\( (\mathcal{C}, \ast) \): oplax action of \( \mathcal{M} \)

We obtain the category

\[
\text{Mod}(T, (\mathcal{C}, \ast))
\]

of models and homomorphisms together with a forgetful functor

\[
\text{Mod}(T, (\mathcal{C}, \ast)) \xrightarrow{U} \mathcal{C}
\]
Example: monads $[C, C]$

**Definition**

$C$: category

The **standard $C$-metamodel of monads on $C$** is the action $ev_C: [C, C] \times C \to C$ given by evaluation.

So a model of a theory $T = (T, m, e)$ consists of

- an object $C \in C$;
- a morphism $\gamma: TC \to C$ in $C$

satisfying the associativity and left unit axioms. That is, an **Eilenberg–Moore algebra** of $T$.

$$\text{Mod}(T, (C, ev_C)) \cong C^T$$
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Unifying the two approaches

A unified approach?

Action
- Monad
- Generalised operad

Enrichment
- Clone
- Symmetric operad
- Non-symmetric operad
- PROP
- PRO

Action-enrichment adjunction
Unifying the two approaches via metamodels

Metamodell

- Action
  - Monad
  - Generalised operad
- Enrichment
  - Clone
  - Symmetric operad
  - Non-symmetric operad
  - PROP
  - PRO

Action-enrichment adjunction
Definition

\( \mathcal{M} = (\mathcal{M}, I, \otimes) \): metatheory; \( \mathcal{T} = (\mathcal{T}, e, m) \): theory in \( \mathcal{M} \).

1. A **metamodel of** \( \mathcal{M} \) is a category \( \mathcal{C} \) together with:
   - \( \Phi: \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set} \): a functor;
   - \( (X, A, B) \mapsto \Phi_X(A, B) \)
   - \( (\cdot)_C: 1 \rightarrow \Phi_I(C, C) \): a nat. tr.;
   - \( (\phi_{X, Y})_{A, B, C}: \Phi_Y(B, C) \times \Phi_X(A, B) \rightarrow \Phi_{Y \otimes X}(A, C) \): nat. tr.
   satisfying the suitable coherence axioms.

2. A **model of** \( \mathcal{T} \) **with respect to** \( (\mathcal{C}, \Phi) \) is \( (\mathcal{C}, \xi) \) where
   - \( C \in \mathcal{C} \);
   - \( \xi \in \Phi_T(C, C) \);
   satisfying the suitable coherence axioms.
Incorporating enrichments

Given an enrichment

\[ \langle -, - \rangle : C^{\text{op}} \times C \rightarrow \mathcal{M}, \]

define a metamodel

\[ \Phi : \mathcal{M}^{\text{op}} \times C^{\text{op}} \times C \rightarrow \text{Set} \]

by

\[ \Phi_{\mathcal{X}}(A, B) = \mathcal{M}(\mathcal{X}, \langle A, B \rangle). \]

For any theory \( T = (T, e, m) \) in \( \mathcal{M} \), we have

- a model \((C, \chi : T \rightarrow \langle C, C \rangle)\) (via enrichment)
- a model \((C, \xi \in \Phi_T(C, C))\) (via metamodel).
Incorporating oplax actions

Given an oplax action

\[ * : M \times C \rightarrow C, \]

define a metamodel

\[ \Phi : M^{\text{op}} \times C^{\text{op}} \times C \rightarrow \text{Set} \]

by

\[ \Phi_X(A, B) = C(X \ast A, B). \]

For any theory \( T = (T, e, m) \) in \( M \), we have

- a model \( (C, \gamma : T \ast C \rightarrow C) \) (via oplax action)
- a model \( (C, \xi \in \Phi_T(C, C)) \) (via metamodel).
Categories of models as hom-categories

\[ \mathcal{M}: \text{metatheory} \]

- Metamodes of \( \mathcal{M} \) form a 2-category \( \mathcal{M}\text{Mod}(\mathcal{M}) \).
- A theory \( T = (\mathcal{T}, e, m) \) in \( \mathcal{M} \) can be considered as a metamodel \( \Phi^T \) of \( \mathcal{M} \) in the terminal category \( 1 \):

  \[
  \Phi^T : \mathcal{M}^{\text{op}} \times 1^{\text{op}} \times 1 \rightarrow \text{Set} \\
  (X, *, *, *) \mapsto \mathcal{M}(X, T).
  \]

- For any theory \( T \) in \( \mathcal{M} \) and a metamodel \( (\mathcal{C}, \Phi) \) of \( \mathcal{M} \), the category of models \( \text{Mod}(T, (\mathcal{C}, \Phi)) \) is isomorphic to the hom-category

  \[ \mathcal{M}\text{Mod}(\mathcal{M})((1, \Phi^T), (\mathcal{C}, \Phi)). \]
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Morphisms of metatheories

Motivation: uniform method to relate different notions of algebraic theory.

⇒ We want a notion of morphism of metatheories, which suitably acts on metamodels.
Morphisms of metatheories

Definition (cf. [Im–Kelly 1986])

\( \mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}}), \mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}}) \): metatheories

A morphism of metatheories from \( \mathcal{M} \) to \( \mathcal{N} \), written as

\[ H = (H, h, h) : \mathcal{M} \rightarrow \mathcal{N}, \]

is a monoidal profunctor from \( \mathcal{M} \) to \( \mathcal{N} \), i.e., a lax monoidal functor

\[ (H, h, h) : \mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow (\text{Set}, 1, \times). \]

Specifically:

- a functor \( H : \mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow \text{Set}; \)
- a function \( h : 1 \rightarrow H(I_{\mathcal{N}}, I_{\mathcal{M}}); \)
- a natural transformation

\[ h_{N,N',M,M'} : H(N', M') \times H(N, M) \rightarrow H(N' \otimes_{\mathcal{N}} N, M' \otimes_{\mathcal{M}} M) \]

satisfying the coherence axioms.
Relation to lax/oplax monoidal functors

- A **lax** monoidal functor $F : \mathcal{M} \to \mathcal{N}$ induces a morphism $F_* : \mathcal{M} \to \mathcal{N}$ defined as
  \[
  F_* : \mathcal{N}^{\text{op}} \times \mathcal{M} \to \text{Set}
  \]
  \[
  (N, M) \mapsto \mathcal{N}(N, FM).
  \]

- An **oplax** monoidal functor $F : \mathcal{M} \to \mathcal{N}$ induces a morphism $F^* : \mathcal{N} \to \mathcal{M}$ defined as
  \[
  F^* : \mathcal{M}^{\text{op}} \times \mathcal{N} \to \text{Set}
  \]
  \[
  (M, N) \mapsto \mathcal{N}(FM, N).
  \]

- A **strong** monoidal functor $F : \mathcal{M} \to \mathcal{N}$ induces both $F_*$ and $F^*$, and they form an adjunction (in the bicategory of metatheories)

\[
\begin{array}{ccc}
\mathcal{M} & \\ & F_* & \\
\downarrow & & \\
F^* & \\
& \mathcal{N} & \\
\end{array}
\]
Morphisms of metatheories act on metamodules

\( \mathcal{M}, \mathcal{N} \): metatheory
\( H = (H, h, h): \mathcal{M} \to \mathcal{N} \): morphism of metatheories
\( (C, \Phi) \): metamodel of \( \mathcal{M} \)

\( \Rightarrow \) We have a metamodel \( (C, H\Phi) \) of \( \mathcal{N} \) defined as:

\[
H\Phi: \mathcal{N}^{\text{op}} \times C^{\text{op}} \times C \to \textbf{Set}
\]

\[
(N, A, B) \mapsto \int^{M \in \mathcal{M}} H(N, M) \times \Phi_M(A, B).
\]

\( \mathcal{M} Mod(-) \) extends to a pseudofunctor from the bicategory of metatheories to the 2-category of 2-categories \( 2\text{-Cat} \).
Isomorphisms between categories of models

\( \mathcal{M}, \mathcal{N} \): metatheory
\( F: \mathcal{M} \to \mathcal{N} \): **strong monoidal functor**
\( T: \text{theory in } \mathcal{M} \)
\( (C, \Phi): \text{metamodel of } \mathcal{N} \)

We can take ...

- the category of models \( \text{Mod}(F_* T, (C, \Phi)) \) (using \( \mathcal{N} \));
- the category of models \( \text{Mod}(T, (C, F^* \Phi)) \) (using \( \mathcal{M} \)).

By the 2-adjunction

\[
\begin{array}{ccc}
\mathcal{M Mod}(\mathcal{M}) & \cong & \mathcal{M Mod}(\mathcal{N}) \\
\downarrow & \swarrow_{F_*} & \searrow_{F^*}
\end{array}
\]

these two categories of models are canonically isomorphic.
Isomorphisms between categories of models

Example

\([F, \text{Set}]: \) the metatheory of clones
\([\text{Set}, \text{Set}]: \) the metatheory of monads on \(\text{Set}\)

Using the inclusion functor \(J: F \rightarrow \text{Set}\), we obtain a strong monoidal functor \(\text{Lan}_J: [F, \text{Set}] \rightarrow [\text{Set}, \text{Set}]\).

\(T: \) clone = theory in \([F, \text{Set}]\)
\((\text{Set}, \Phi): \) the standard \(\text{Set}\)-metammodel of \([\text{Set}, \text{Set}]\)

We have:

- \(\text{Lan}_J^* T: \) the finitary monad corresponding to \(T\);
- \((\text{Set}, \text{Lan}_J^* \Phi): \) the standard \(\text{Set}\)-metammodel of \([F, \text{Set}]\).

\(\Rightarrow \) The classical result on compatibility of semantics of clones (= Lawvere theories) and monads on \(\text{Set}\) [Linton, 1965].
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The category of models

\[ \mathcal{M} = (\mathcal{M}, I, \otimes) \]: metatheory; \[ T = (T, e, m) \]: theory in \( \mathcal{M} \);

\( (C, \Phi) \): metamodel of \( \mathcal{M} \)

We obtain a category

\[ \text{Mod}(T, (C, \Phi)) \] (or, \( \text{Mod}(T, C) \) for short),

a functor

\[ \text{Mod}(T, C) \]

\[ \begin{array}{c}
\downarrow U \\
C
\end{array} \]

and a natural transformation

\[ \text{Hom}_{\text{Mod}(T, C)} \]

\[ \begin{array}{c}
U
\\
\downarrow
\\
\downarrow U \\
C
\end{array} \]

\[ \Phi_T \]

\[ \rightarrow C \]
The category of models

\[ \text{Mod}(T, C) \xrightarrow{\text{Hom}} \text{Mod}(T, C) \]

\[ U \downarrow \Phi_T \downarrow \Phi_m \]

\[ \Phi_{T \otimes T} \]

\[ \text{Mod}(T, C) \xrightarrow{\text{Hom}} \text{Mod}(T, C) \xrightarrow{\text{Hom}} \text{Mod}(T, C) \]

\[ U \downarrow \Phi_T \downarrow U \]

\[ \Phi_{T, T} \]

\[ \Phi_{T \otimes T} \]
The category of models

\[ \text{Mod}(T, C) \xrightarrow{\text{Hom}} \text{Mod}(T, C) \]

\[ \Phi \]

\[ \Phi_T \]

\[ \Phi_e \]

\[ \Phi_I \]

\[ \Phi_I \]

\[ \Phi_i \]

In fact, \((\text{Mod}(T, C), U, u)\) is the universal one as such.

\[ \Rightarrow \] What is a suitable language to express this universality?
Categories of models as double limits

**Definition ([Grandis–Paré 1999])**

The pseudo double category $\mathbf{Prof}$

- object: category;
- vertical 1-cell: functor; $(G \circ H) \circ K = G \circ (H \circ K)$
- horizontal 1-cell: profunctor; $(X \circ Y) \circ Z \cong X \circ (Y \circ Z)$
- square: natural transformation.

Monoidal category $\mathcal{M}$ defines a vertically trivial (one object, one vertical 1-cell) pseudo double category $\mathbb{H}\Sigma\mathcal{M}$.

$(\mathbf{Mod}(T, (\mathcal{C}, \Phi)), U, u)$ is the **double limit** [Grandis–Paré 1999] of the lax double functor

$$
\begin{array}{c}
\mathbb{H}\Sigma(\Delta^{op}) \\
\xrightarrow{T^{op}} \\
\mathbb{H}\Sigma(\mathcal{M}^{op}) \\
\xrightarrow{\phi} \\
\mathbf{Prof}
\end{array}
$$
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Conclusion

- Unified account of various notions of algebraic theory and their semantics.
- Morphism of metatheories as a uniform method to compare different notions of algebraic theory.
  - Strong monoidal functor $\mapsto$ adjoint pair of morphisms $\mapsto$ isomorphisms of categories of models.

Future work:

- Clearer understanding of the scope of our framework.
  - In particular, intrinsic characterisation of the forgetful functors $U: \text{Mod}(T, (C, \Phi)) \rightarrow C$ arising in our framework (a Beck type theorem).
- Incorporate various constructions on algebraic theories: sums, distributive laws, tensor products, ...
The relation between action and enrichment

According to a categorical folklore [Kelly, Gordon–Power, ...]:

**Proposition**

\[ \mathcal{M} = (\mathcal{M}, I, \otimes): \text{monoidal category (metatheory)}; \quad \mathcal{C}: \text{category} \]

1. \( \ast: \mathcal{M} \times \mathcal{C} \to \mathcal{C} \): oplax left action s.t. for each \( C \in \mathcal{C} \)

\[ (\_ \ast C) \]

\[ \mathcal{M} \xrightarrow{\perp} \mathcal{C}. \]

\[ \exists \langle C, \_ \rangle \]

Then \( \langle -, - \rangle \) defines an enrichment.

2. \( \langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{M} \): enrichment s.t. for each \( C \in \mathcal{C} \)

\[ \exists (\_ \ast C) \]

\[ \mathcal{M} \xleftarrow{\perp} \mathcal{C}. \]

\[ \langle C, \_ \rangle \]

Then \( \ast \) defines an oplax left action.
The relation between action and enrichment

**Proposition**

\( \mathcal{M} = (\mathcal{M}, I, \otimes) \): metatheory; \( T = (T, e, m) \): theory in \( \mathcal{M} \)

\( (C, * : \mathcal{M} \times C \to C) \): oplax action

\( (C, \langle -,- \rangle : \mathcal{C}^{\text{op}} \times C \to \mathcal{M}) \): enrichment

If for each \( C \in \mathcal{C} \)

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow \\
\langle C,- \rangle
\end{array}
\begin{array}{c}
\to C
\end{array}
\]

(compatible with structure morphisms \( \delta, \varepsilon, M, j \)) then

\[
\begin{array}{c}
g: \mathcal{T} \times C \to C \quad \text{(via oplax action)}
\end{array}
\]

\[
\begin{array}{c}
\chi: \mathcal{T} \to \langle C,C \rangle \quad \text{(via enrichment)}.
\end{array}
\]

So \( \text{Mod}(\mathcal{T},(C, *)) \cong \text{Mod}(\mathcal{T},(C, \langle -,- \rangle)) \).
The relation between action and enrichment

Example

([F, Set], I, ⋅): the metatheory of clones

For each $S \in \text{Set}$

$$(-) \diamond S$$

$$\begin{array}{ccc}
[F, \text{Set}] & \xleftarrow{\perp} & \text{Set} \\
\langle S, - \rangle & \xrightarrow{} & \\
\end{array}$$

where

$$X \diamond S = \int_{[m] \in F} X_m \times S^m$$

and

$$\langle S, R \rangle_m = \text{Set}(S^m, R)$$