Right adjoints to operadic restriction functors

arXiv:1906.12275

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Category Theory 2019

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There is an unexpected right adjoint (Templeton 2003)

$$\begin{tikzcd} \mathrm{nsOpd} & \mathbf{Opd} \arrow[l] \arrow[r, bend left, "\phi_!"] \arrow[r, bend right, dashed, red, "\phi_*"'] & \mathbf{Cyc} \arrow[l, "\phi^*" description] \end{tikzcd}$$

which may be described at an operad P by

$$(\phi_* P)(n) = \prod_{i=0}^{n} P(n) = \hom_{\Sigma_n}(\Sigma_{n+1}, P(n)).$$

When do such operadic right Kan extensions exist?

Main theorem (Monochrome version)

If P is an operad, let $\overset{=P(1)}{|P|}$ denote the underlying monoid.

Monoidal extension
An operad map $P \to Q$ is a *monoidal extension* just when

$$P \circ_{|P|} |Q| \to Q \circ_{|Q|} |Q| \cong Q$$

is an isomorphism.

Theorem (H & Drummond-Cole 2019)
Let $\phi : P \to Q$ be a map between (monochrome) operads. The restriction functor

$$\phi^* : \mathbf{Alg}(Q) \to \mathbf{Alg}(P)$$

admits a right adjoint if and only if ϕ is a monoidal extension.

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Isomorphism of underlying monoids
If $|P| \to |Q|$ is an isomorphism, then $P \to Q$ is a monoidal extension if and only if it is an isomorphism.

Standard non-example
The inclusion functor from commutative monoids to associative monoids does not admit a right adjoint.

New Example: Little Disks, Framed Little Disks

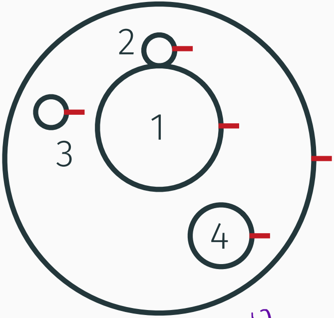
Let $\mathbb{D} \subseteq \mathbb{R}^2$ be the closed unit disk.

$$D_2(n) \subseteq D_2^{fr}(n) \subseteq \left\{ f : \coprod_{k=1}^{n} \mathbb{D} \to \mathbb{D} \right\}$$

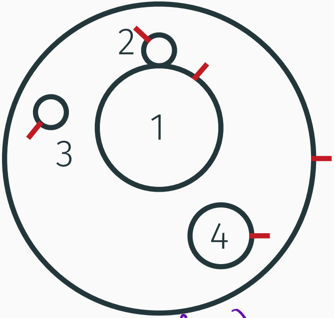
- Each $f_k : \mathbb{D} \to \mathbb{D}$ is an embedding.
- $f_k(\mathbb{D}) \cap f_j(\mathbb{D}) \subseteq f_k(\partial(\mathbb{D}))$ for $k \neq j$ $\quad a > 0$
- $f \in D_2(n)$ when each f_k is an affine map $f_k(\mathbf{x}) = a\mathbf{x} + \mathbf{b}$
- $f \in D_2^{fr}(n)$ when each f_k is a rotation followed by an affine

Observation
The inclusion $D_2 \to D_2^{fr}$ is a monoidal extension.

New Example: Little Disks, Framed Little Disks



$\in D_2(4)$



$\in D_2^{fr}(4)$

The inclusion $D_2 \to D_2^{fr}$ is a monoidal extension.

New Example: Little Disks, Framed Little Disks

If X is a D_2-algebra, then the free loop space $LX = Map(S^1, X)$ realizes the right adjoint.

- $D_2^{fr}(n) \times (LX)^{\times n} \to LX = Map(SO(2), X)$
- The adjoint to the level n action takes the form:

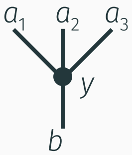
$$
\begin{tikzcd}
SO(2) \times D_2^{fr}(n) \times (LX)^{\times n} \arrow[d] & & \\
D_2^{fr}(n) \times (LX)^{\times n} & & D_2(n) \times SO(2)^{\times n} \times (LX)^{\times n} \arrow[ll, dashed, "\cong"'] \arrow[d] \\
& & D_2(n) \times X^{\times n} \arrow[d] \\
& & X
\end{tikzcd}
$$

Bicategory of colored collections

Objects: Sets named A, B, C, etc.

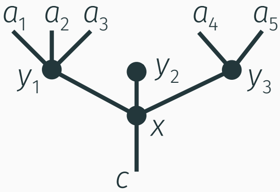
(A, B) **Collections:**

- $S_A = \{\sigma : \underline{a} = (a_1, \dots, a_n) \to (a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \underline{a}\sigma\}$
- (A, B) collection Y: functor $S_A \times B \to Set$



Horizontal Composition

- $\circ : (B, C)\text{-}Coll \times (A, B)\text{-}Coll \to (A, C)\text{-}Coll$
- Elements of $X \circ Y$



Adjoints among hom categories

- $(-) \circ Y : (B, C)\text{-}Coll \to (A, C)\text{-}Coll$ has a right adjoint (Kelly)
- $X \circ (-) : (A, B)\text{-}Coll \to (A, C)\text{-}Coll$ only has a right adjoint, denoted by $\langle X, - \rangle$, when X is concentrated in arity one

$$\langle X, Z \rangle(\underline{a}; b) = \prod_{c \in C} \mathrm{hom}\left(X(b; c), Z(\underline{a}; c)\right)$$

Colored operads

An *A-colored operad P* is a monoid in the monoidal category of (A, A)-collections:

$$\mu : P \circ P \to P \qquad\qquad \eta : \mathbf{1}_A \to P$$

Colored operads concentrated in arity one are categories.

From functions to collections

$f : A \to B$ a map of sets

Two collections concentrated in arity one:

- (A, B) collection also called f with $f(a; f(a)) = *$
- (B, A) collection called $\overline{f}$ with $\overline{f}(f(a); a) = *$

We have

- $(f \circ \overline{f})(b; b) = f^{-1}(b)$ (otherwise empty)
- if $f(a') = f(a)$, then $(\overline{f} \circ f)(a'; a) = *$ (otherwise empty)

Conclusion: $f \dashv \overline{f}$ using $\epsilon_f : f \circ \overline{f} \to \mathbf{1}_B$ and $\eta_f : \mathbf{1}_A \to \overline{f} \circ f$

Maps of colored operads

Definition

- A map of operads $\phi : (A, P) \to (B, Q)$ consists of a
  - function $f : A \to B$
  - map of monoids $P \to \bar{f} \circ Q \circ f$ in (A, A) collections
- By adjointness, the bottom is equivalent to a map $P \circ \bar{f} \to \bar{f} \circ Q$ of (B, A) collections

$|-|$ from operads to categories.

Actions

- If $\phi : (A, P) \to (B, Q)$ is a map of operads, then $\bar{f} \circ Q$ is a P-Q bimodule.

$$P \circ \bar{f} \circ Q \longrightarrow \bar{f} \circ Q \circ Q \longrightarrow \bar{f} \circ Q$$

- An *algebra* over (A, P) is nothing but an $(\varnothing, A)$-collection along with a left action by P.

Special case: $Q = |Q|$ is concentrated in arity one. Then $\bar{f} \circ |Q|$ is a $|P|$-$|Q|$ bimodule

We have an adjunction $R : \overset{\mathrm{Set}^{|P|}}{\overset{\shortparallel}{\mathbf{Alg}(|P|)}} \leftrightarrows \overset{\mathrm{Set}^{|Q|}}{\overset{\shortparallel}{\mathbf{Alg}(|Q|)}} : L$ with

$$R(-) = \mathbf{hom}_{|P|}(\bar{f} \circ |Q|, -) \subseteq \langle \bar{f} \circ |Q|, - \rangle$$

is right adjoint to

$$L(-) = (\bar{f} \circ |Q|) \circ_{|Q|} (-) \cong \bar{f} \circ (-)$$

Main theorem (Colored version)

If $\phi : (A, P) \to (B, Q)$ is a map of operads, then the composite $P \circ \bar{f} \circ |Q| \to P \circ \bar{f} \circ Q \to \bar{f} \circ Q$ descends to

$$P \circ_{|P|} (\bar{f} \circ |Q|) \to \bar{f} \circ Q \qquad (\heartsuit)$$

Definition
ϕ is a *categorical extension* when $(\heartsuit)$ is an isomorphism

Theorem (H & Drummond-Cole 2019)
Let $\phi : (A, P) \to (B, Q)$ be a map between colored operads. The restriction functor

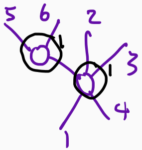
$$\phi^* : \mathbf{Alg}(Q) \to \mathbf{Alg}(P)$$

admits a right adjoint ϕ_* if and only if ϕ is a categorical extension.

Example (Operads and Cyclic Operads)

= rooted trees   = trees

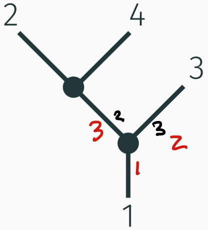
- R and T are $\mathbb{N}$-colored operads
- Operations in T are trees with total orderings on
  - set of vertices
  - vertex neighborhoods
  - boundaries
- $R \subseteq T$ consists of *rooted* trees: root of tree is first edge of boundary, root of vertex is first edge in the vertex neighborhood, and these are compatible
- $R(n; n) = \Sigma_n$ and $T(n; n) = \Sigma_{n+1}$
- $\mathbf{Alg}(R) = \mathbf{Opd}$ and $\mathbf{Alg}(T) = \mathbf{Cyc}$
- $R \subseteq T$ is a categorical extension



$R \circ_{|R|} |T| \cong T$

Non-Example (Nonsymmetric Operads and Operads)

- $P \subseteq R$ are the *planar* rooted trees.
- $P(n; n) = *$ and $R(n; n) = \Sigma_n$
- $\mathbf{Alg}(P) = \mathbf{nsOpd}$ and $\mathbf{Alg}(R) = \mathbf{Opd}$
- *Not* a categorical extension:



$\in R(2,2;3)$