The model 2-category of combinatorial model categories (Work in progress)

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Remark: I will apply this defintion also to (weak) 2-categories.

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Remark: Note the different use of "trivial / acyclic". "trivial" : characterized by a stronger weak lifting property, "acyclic" : is an equivalence.

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Note: During the talk I'll restrict myself to "*tractable*" combinatorial categories, where the generating (trivial) cofibrations have cofibrant domain. This is only to avoid some technical difficulties.

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Definition (Barwick)

A combinatorial category is a *right semi-model category* if it admit a class of equivalence $\mathcal{W}$ such that “acylic fibration with fibrant target = trivial fibration with fibrant target” and “acyclic cofibration = trivial cofibration”.

Definition (H.)

A combinatorial category is a *weak model category* if it admits a class of equivalences such that “acylic fibration with fibrant target = trivial fibration with fibrant target” and “acyclic cofibration with cofibrant domain = trivial cofibration with cofibrant domain”.

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General idea:

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General idea:

- Weak model categories are considerably easier to construct than Quillen model categories, and still allows to "do homotopy theory" as in a Quillen model categories.
- There are easy criterion to test if a weak model category is a left or right semi-model categories.
- I do not know convenient neccessary and sufficient criterion for Quillen model structures (unless we add additional assumptions like every object is (co)fibrant or properness).

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Comb *has all small (pseudo/flexible) limits and colimits. Limits are computed in the category of categories, colimits in the category of locally presentable categories.*

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There are three right semi-model structure on $\mathbf{Comb}$*, called* W, B *and* S*. They all have the same equivalences, and the identity map is a left Quillen equivalence:*

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(I'll comment later about the size problem).

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- The free combinatorial category F_* on one cofibrant object is the category of sets with cofibrations the monomorphisms and trivial cofibrations the isomorphisms.
- The free combinatorial category $F_{\hookrightarrow}$ on a cofibration with cofibrant domain is the category of presheaves of set on the category $\bullet \rightarrow \bullet$ with cofibration being the monomorphisms (end trivial cofibration the isomorphisms).

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Theorem

The W-cofibrant objects are the (retract of) categories of presheaves on a directed categories, with cofibrations being the monomorphisms.

The B-structure has one additional generating cofibration:

$$F\left(\begin{tikzcd} A \arrow[r] & B \end{tikzcd}\right) \to F\left(\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \\ C \arrow[ur] & \end{tikzcd}\right)$$

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The S-structure also has one additional generating cofibration:

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S-cofibrant objects are the (retract of) categories of models of infinitary Generalized algebraic (Cartmell) theory with no equality axioms, with their natural notion of cofibrations.

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The W-model structure is monoidal for this tensor product. The B and S model structures are not monoidal, but are enriched over the W-model structure.

Idea of the proof: One can construct a W-interval object for F_*:

$$F_* \coprod F_* \overset{(A,B)}{\hookrightarrow} F\left(\begin{tikzcd} & M & \\ A \arrow[ur, hook, "\sim"] \arrow[r, hook] & A \coprod B \arrow[u, hook] & B \arrow[ul, hook', "\sim"'] \arrow[l, hook'] \end{tikzcd}\right)$$

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Tensoring and exponentiating with this interval still gives good enough cylinder and path object functors for all three model structures, and the model structure are constructed using these functors and an appropriate modification of Cisinki-Olschok's theory.

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In particular one has three *Quillen model structures* in this case, and the underlying category is now really the category of simplicial *left semi-model categories*.

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