$\mathcal{M}$-coextensivity and the strict refinement property

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Unique factorization

The general question

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For integers, there is the *fundamental theorem*:

Euclid

Given prime numbers $p_1, p_2, ..., p_n$ and $q_1, q_2, ..., q_m$ such that

$$p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$$

then $n = m$ and there is a permutation $\sigma \in S_n$ such that $p_i = q_{\sigma(i)}$.

Direct product decompositions of groups

- Every finitely generated abelian group is uniquely represented as a product of cyclic groups (L. Kronecker 1870).
- **The Krull-Schmidt Theorem**: In R-Mod, every module of finite height can be uniquely represented as a direct-sum of indecomposable ones.
- In 1909, J. Wedderburn asked if any finite group can be uniquely decomposed as a product of directly-irreducible ones. It was shown by R. Remak in 1911 that they do. In 1925, this result was generalized by W. Krull and O. Schmidt, where they showed that any group whose normal subgroup lattice has finite height has UFP.

Refinement properties

Refinement for integers

Suppose that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m$, then there exists a family of integers $c_{i,j}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ such that

$$a_i = \prod_j c_{i,j} \text{ and } b_j = \prod_i c_{i,j}$$

.

The refinement property for objects in a category

RP

An object X is said to have the (finite) *refinement property* if for any two (finite) product diagrams $(X \xrightarrow{a_i} A_i)_{i\in I}$ and $(X \xrightarrow{b_j} B_i)_{j\in J}$, there exist morphisms $A_i \xrightarrow{\alpha_{i,j}} C_{i,j}$ and $B_j \xrightarrow{\beta_{i,j}} C_{i,j}$ indexed by $i \in I$ and $j \in J$ such that the diagrams $(A_i \xrightarrow{\alpha_{i,j}} C_{i,j})_{j\in J}$ and $(B_j \xrightarrow{\beta_{i,j}} C_{i,j})_{i\in I}$ are product diagrams.

The strict refinement property

C. Chang, B. Jonsson, A. Tarski (1964)

An object X is said to have the (finite) *strict refinement property* if for any two (finite) product diagrams $(X \xrightarrow{a_i} A_i)_{i\in I}$ and $(X \xrightarrow{b_j} B_j)_{j\in J}$, there exist morphisms $A_i \xrightarrow{\alpha_{i,j}} C_{i,j}$ and $B_j \xrightarrow{\beta_{i,j}} C_{i,j}$ indexed by $i \in I$ and $j \in J$, such that the diagrams $(A_i \xrightarrow{\alpha_{i,j}} C_{i,j})_{j\in J}$ and $(B_j \xrightarrow{\beta_{i,j}} C_{i,j})_{i\in I}$ are product diagrams, and such that the square

$$\begin{tikzcd} X \arrow[r, "a_i"] \arrow[d, "b_j"'] & A_i \arrow[d, "\alpha_{i,j}"] \\ B_j \arrow[r, "\beta_{i,j}"'] & C_{i,j} \end{tikzcd}$$

commutes.

Examples of structures with the strict refinement property

Listed below are examples of structures which have strict refinements.

- Every lattice (in the category of lattices). More generally, every non-empty algebra in a congruence distributive (universal) algebra has the strict refinement property.
- Every unitary ring.
- Every centerless/perfect group has the strict refinement property.
- Every poset with a bottom element (G. Birkhoff 1940).
- Every connected poset (J. Hashimoto 1951).
- Every connected graph (J. Walker 1987).
- Every object in $\mathbf{Top}^{\mathrm{op}}, \mathbf{Pos}^{\mathrm{op}}, \mathbf{Grph}^{\mathrm{op}}, G - \mathbf{Set}^{\mathrm{op}}$ has the strict refinement property, or more generally any object in a coextensive category (with finite products) has the (finite) strict refinement property.

Coextensive categories

Theorem (Carboni, Lack, Walters)

A category with binary products is coextensive if and only if it has pushouts along product projections and in every commutative diagram

$$\begin{tikzcd} A_1 \arrow[d, "f_1"'] & A_1 \times A_2 \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] \arrow[d, "f"'] & A_2 \arrow[d, "f_2"] \\ X_1 & X \arrow[l, "x_2"] \arrow[r, "x_1"'] & X_2 \end{tikzcd}$$

the bottom row is a product diagram if and only if the two squares are pushouts.

The relationship between the strict refinement property and coextensivity can be seen through the notion of an $\mathcal{M}$-coextensive object in a category $\mathbb{C}$, where $\mathcal{M}$ is a distinguished class of morphisms from $\mathbb{C}$. As we will see, if $\mathcal{M}$ is the class of product projections in a regular category $\mathbb{C}$, then $\mathcal{M}$-coextensivity is precisely the strict refinement property.

$\mathcal{M}$-pushouts

Let $\mathcal{M}$ be a class of morphisms in $\mathbb{C}$. An *$\mathcal{M}$-pushout* is a pushout square in $\mathbb{C}$, in which the pushout inclusions are morphisms in $\mathcal{M}$.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "\mathcal{M}"] \\
\bullet \arrow[r, "\mathcal{M}"'] & \bullet
\end{tikzcd}$$

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Convention

We will assume that $\mathcal{M}$ is actually a subcategory of $\mathbb{C}$, which is closed under binary products in $\mathbb{C}$, and closed under composition with isomorphisms in $\mathbb{C}$.

$\mathcal{M}$-coextensive objects

Definition

An object X in a category $\mathbb{C}$ with binary products is $\mathcal{M}$-coextensive if it admits $\mathcal{M}$-pushouts of morphisms in $\mathcal{M}$ along its product projections, and in every commutative diagram

$$\begin{tikzcd}
X_1 \arrow[d, "\mathcal{M}"] & X \arrow[l] \arrow[r] \arrow[d, "\mathcal{M}"] & X_2 \arrow[d, "\mathcal{M}"] \\
A_1 & A \arrow[l] \arrow[r] & A_2
\end{tikzcd}$$

where the top row is a product diagram and the vertical morphisms are in $\mathcal{M}$, the bottom row is a product diagram if and only if the two squares are $\mathcal{M}$-pushouts.

Projection-coextensive objects

Definition

An object in a category $\mathbb{C}$ is *projection-coextensive* if it is $\mathcal{M}$-coextensive with $\mathcal{M}$ the class of all product projections in $\mathbb{C}$.

Proposition

If X is a projection-coextensive object in a category with products, then X has the strict refinement property.

Proof Sketch.

Given two product diagrams $(X \xrightarrow{a_i} A_i)_{i\in I}$ and $(X \xrightarrow{b_j} B_j)_{j\in J}$ diagrams in $\mathbb{C}$, we form the pushouts:

$$\begin{tikzcd} X \arrow[r, "a_i"] \arrow[d, "b_j"'] & A_i \arrow[d, "\alpha_{i,j}"] \\ B_j \arrow[r, "\beta_{i,j}"'] & C_{i,j} \end{tikzcd}$$

Then the $\alpha_{i,j}$ and $\beta_{i,j}$ form the strict refinement for the two product diagrams. □

A characterization of strict refinement property

Theorem (Chang, Jonsson, Tarski)

Let X be an algebra in variety, then X has the strict refinement property if and only if the factor congruences of X form a Boolean lattice.

A characterization of projection-coextensivity

Terminology

If $X \xrightarrow{p_1} A$ is product projection, then a *complement* for p_1 is a morphism $X \xrightarrow{p_2} B$ such that the diagram

$$A \xleftarrow{p_1} X \xrightarrow{p_2} B$$

is a product diagram.

A characterization of projection-coextensivity

Let X be a projection-coextensive object in a category with finite products.

- Any product projection of X is an epimorphism.
- The full subcategory of $(X \downarrow \mathbb{C})$ consisting of product projections of X forms a preorder $\mathrm{Proj}_{\mathbb{C}}(X)$.
- The posetal-reflection of $\mathrm{Proj}_{\mathbb{C}}(X)$ will be denoted by $\mathrm{Proj}(X)$. Note that $[p_1] \leqslant [p_2]$:

$$
\begin{tikzcd}
& X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A \arrow[rr, dotted] & & B
\end{tikzcd}
$$

Some basic properties of $\mathrm{Proj}(X)$

- The join of two members $[p], [q] \in \mathrm{Proj}(X)$ exists, and is represented by the diagonal of any pushout of p along q.
- $\mathrm{Proj}(X)$ admits a top element, and it is represented by a terminal morphism $X \to 1$.
- For any product projection $\pi_1 : X \to A$ there exists a unique (up to isomorphism) complement ,i.e., a unique $\pi_2 : X \to B$ making the diagram

$$A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$$

  a product diagram.

The orthocomplement $\mathrm{Proj}(X)$

We have a well-defined map sending an $x \in \mathrm{Proj}(X)$ to the element $x^\perp$ represented by a complement of any representative of x.

The orthocomplement

The map $\mathrm{Proj}(X) \to \mathrm{Proj}(X)$ defined by $x \mapsto x^\perp$, is order-reversing, i.e., it satisfies:

$$x \leqslant y \implies y^\perp \leqslant x^\perp.$$

This map turns $\mathrm{Proj}(X)$ into a lattice where meets are given by

$$x \wedge y = (x^\perp \vee y^\perp)^\perp.$$

Proof that $x \mapsto x^{\perp}$ is order-reversing.

The proof amounts to showing that in the diagram

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, dotted] & X \arrow[l, "\pi_1"'] \arrow[u, "\pi_1'"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_2'"] & A' \\
 & B' \arrow[ur, dashed] &
\end{tikzcd}$$

where the central column and central row are product diagrams, if the dotted arrow exists making triangle commute, then the dashed arrow exists making the triangle commute. $\square$

Proof continued...

Consider the diagram below, where each square is a pushout.

$$
\begin{tikzcd}
C_1 & B \arrow[l, "\gamma"'] \arrow[r] & C_2 \\
A \arrow[u] \arrow[d] \arrow[ur, dotted] & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "\pi_1'"'] \arrow[d, "\pi_2'"] & A' \arrow[u] \arrow[d, "\beta"] \\
C_3 & B' \arrow[l] \arrow[r, "\alpha"'] & C_4
\end{tikzcd}
$$

Since π_1 is an epimorphism, the upper left-hand triangle commutes, and hence γ is an isomorphism. This implies that C_2 is a terminal object, which implies that β is an isomorphism. $\square$

$\mathrm{Proj}(X)$ is a Boolean lattice

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- $\mathrm{Proj}(X)$ is a bounded orthocomplemented lattice where $x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}$. Moreover, $\mathrm{Proj}(X)$ is distributive, and hence a Boolean lattice.

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Theorem

Let $\mathbb{C}$ be a category with finite products, and suppose X is an object in $\mathbb{C}$. Then the following are equivalent.

1. *X is projection-coextensive.*
2. *X has epimorphic product projections, admits Proj-pushouts, and $\mathrm{Proj}(X)$ is a Boolean lattice.*

Projection-coextensivity in regular and exact categories

Theorem

Let $\mathbb{C}$ be a Barr exact category, and suppose that X is an object in $\mathbb{C}$ with global support. Then the following are equivalent:

1. *X has the strict refinement property*
2. *X is projection-coextensive.*
3. *$\mathrm{Proj}(X)$ is a Boolean lattice, where joins are given by pushout.*
4. *$F(X)$ is a Boolean lattice under the operations $\circ$ and $\cap$.*
5. *For any two factor relations F, G on X we have $F \circ G = G \circ F$ and*

$$q(G) \cap q(G^{\perp}) = \Delta_X$$

where $q : X \to X/F$ is a canonical quotient

Strict refinement for regular categories

Proposition

An object with global-support in a regular category $\mathbb{C}$ has the strict refinement property if and only if for any diagram

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, dotted] \arrow[r, dotted] & \bullet \\
\bullet \arrow[u, dotted] \arrow[d, dotted] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u, dotted] \arrow[d, dotted] \\
\bullet & \bullet \arrow[l, dotted] \arrow[r, dotted] & \bullet
\end{tikzcd}$$

where the central vertical column and central horizontal row are product diagrams, the dotted arrows exist making the diagram commute, and making each edge a product diagram.

Majority categories and strict refinements

Majority categories

Majority categories are a class of categories that correspond to varieties that admit a *majority term* (i.e. a ternary term $m(x, y, z)$ satisfying $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$) in a similar way as Mal'tsev categories correspond to Mal'tsev varieties. These categories allow for a categorical treatment of various properties of the category **Lat** of lattices.

Concluding remarks

- In a finitely complete Mal'tsev category, it is possible to define what a *centerless* object is, and for Barr-exact Mal'tsev categories every centerless object is projection-coextensive.
- When $\mathcal{M}$ is the class of regular epimorphisms in a regular category $\mathbb{C}$, then the $\mathcal{M}$-coextensive objects are precisely those objects that have *factorable congruences*.
- It was proved by A. Iskander that if a universal algebra $X \neq \emptyset$ has factorable congruences, then X has the strict refinement property.

Thank you for listening

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