Double quandle coverings

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? Double quandle coverings ?

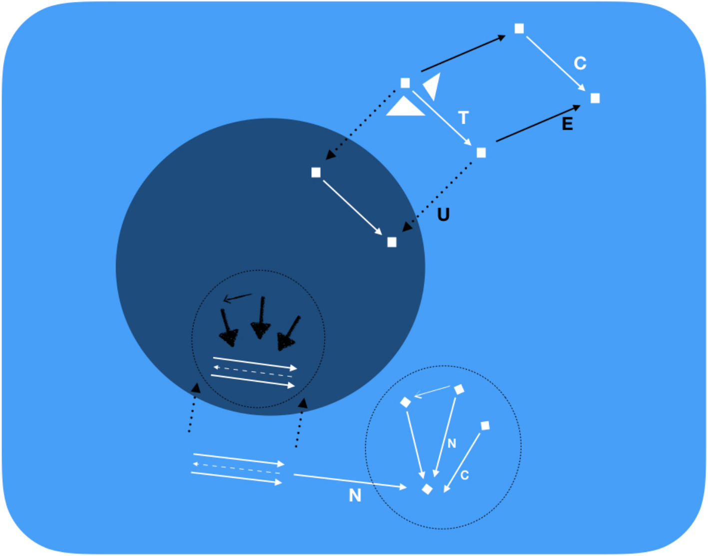
Previous work:

- D.E. Joyce (1979) (Supervised by Peter J. Freyd)
  An algebraic approach to symmetry and applications in knot theory
- M. Eisermann (2007)
  Quandle coverings and their Galois correspondence
- V. Even (2014)
  A Galois-Theoretic Approach to the Covering Theory of Quandles

A motivation: Illustrate, in algebra, an instance of Galois theory with geometrical intuition – display homotopical information...
also in higher dimensions

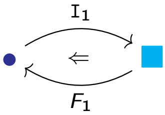
A first step: "What are double central extensions *for quandles* ?"
[Same question *for groups* in 1991: R.Brown asks G.Janelidze]

Categorical Galois theory [G.Janelidze 1990]



Higher categorical Galois theory

Start: Galois structure in dimension 1



consider the category of extensions Ext:

$$f_A \xrightarrow{\alpha} f_B \quad \leftrightarrow \quad \begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_B} \\ A_0 & \xrightarrow[\alpha_0]{} & B_0 \end{array}$$

Get: Galois structure in dimension 2

$$\mathsf{CExt} \underset{F_2}{\overset{\mathtt{I}_2}{\rightleftarrows}} \mathsf{Ext} \qquad (\Leftarrow)$$

good notion of double extensions:

$$\begin{array}{ccccc} A_1 & & \xrightarrow{\alpha_1} & & B_1 \\ & \searrow & & \nearrow & \\ {\scriptstyle f_A}\downarrow & & A_0 \times_{B_0} B_1 & & \downarrow{\scriptstyle f_B} \\ & \swarrow & & & \\ A_0 & & \xrightarrow[\alpha_0]{} & & B_0 \end{array}$$

Question: "What are double central extensions ?"

context – Racks [Conway and Wraith 1959]

Definition:

A set X equipped with:

symmetries/inner-automorphisms assigned to each point

$$X \underset{\mathsf{S}^{-1}}{\overset{\mathsf{S}}{\rightrightarrows}} X^X$$

two inverse and self distributive binary operations

$$X \times X \underset{\triangleleft^{-1}}{\overset{\triangleleft}{\rightrightarrows}} X,$$

$$\mathsf{S}_y(x) \Leftrightarrow x \triangleleft y$$

$$\text{(R1)} \quad (x \triangleleft y) \triangleleft^{-1} y = x = (x \triangleleft^{-1} y) \triangleleft y$$

$$\text{(R2)} \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

context – Racks [Conway and Wraith 1959]

Definition:

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$$S_y(x) \Leftrightarrow x \triangleleft y$$

(R1) $x \triangleleft y \triangleleft^{-1} y = x = x \triangleleft^{-1} y \triangleleft y$

(R2') $x \triangleleft (y \triangleleft z) = x \triangleleft^{-1} z \triangleleft y \triangleleft z$

Examples – Quandles

A rack X is a *quandle* if moreover (idempotency)

$$\text{(Q1)}\quad x \triangleleft x = x$$

For instance:

1. Sets $\qquad y \triangleleft x = x \qquad\qquad \mathrm{I}\colon \mathsf{Set} \to \mathsf{Qnd} \to \mathsf{Rac}$
2. Groups

$$\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Qnd} \to \mathsf{Rac}$$

$$(G, \cdot, e) \mapsto (G, \triangleleft, \triangleleft^{-1})$$

$$x \triangleleft y := y^{-1}xy$$

3. Knot quandles
4. Symmetric spaces [O. Loos 1969]

Connected components adjunction

$$\mathsf{Set} \underset{\pi_0}{\overset{\mathtt{I}}{\rightleftarrows}} \mathsf{Rac} \qquad (\Leftarrow)$$

Define: Elements x and y in a rack X are *connected* $(x \sim_X y)$

if there is a *primitive path* from x to y : $\quad y = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n$

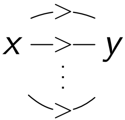
$$x \xrightarrow{{a_1}^{\delta_1} \cdots {a_n}^{\delta_n}} y$$

$\dashrightarrow$ Send X to $\pi_0(X) \coloneqq X/\sim_X$ it's *set of connected components*

Primitive paths – Observations

– Invert / concatenate primitive paths:

– A lot of different prim. paths from x to y



– Prim. paths which could be *equivalent* ?

- Using axiom (R1)

$$x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n \triangleleft^{-1} z \triangleleft z$$

- Using axiom (R2), say $a_i = y \triangleleft z$

$$x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_i} (y \triangleleft z) \cdots \triangleleft^{\delta_n} a_n = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{-\delta_i} z \triangleleft y \triangleleft^{\delta_i} z \cdots \triangleleft^{\delta_n} a_n$$

The group of paths – homotopy equivalent prim. paths

Define the functor

$$\mathsf{Rac} \xrightarrow{\mathsf{Pth}} \mathsf{Grp} \qquad \mathsf{Pth}(A) \coloneqq \mathsf{F_g}(A)/\langle (x \triangleleft a)^{-1} a^{-1} x a | a, x \in A \rangle$$

Representatives of the symmetries: $\mathrm{pth}_A \colon a \in A \mapsto a \in \mathsf{Pth}(A)$

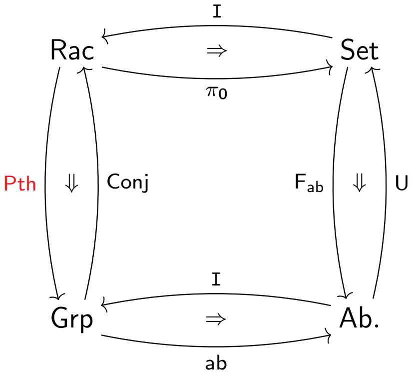
Action by inner-automorphisms: given $g = {a_1}^{\delta_1} \dots \; {a_n}^{\delta_n}$ in $\mathsf{Pth}(A)$

$$x.g = x.({a_1}^{\delta_1} \dots \; {a_n}^{\delta_n}) = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n$$

$$x \overset{g}{\longrightarrow} x.g$$

The group of paths is left adjoint to $\mathsf{Conj} \colon \mathsf{Grp} \to \mathsf{Rac}$

Commutative square of adjunctions



The free rack [R. Fenn and C. Rourke 1991]

Given a set A the free rack is

$$\mathsf{F_r}(A) := A \rtimes \mathsf{F_g}(A)$$

elements are pairs $\quad (a, g) \quad a \overset{g}{\rightarrowtail} << a.g >>$

A path acts on another « with its codomain »

$$(a, g) \triangleleft (b, h) = (a, gh^{-1}bh)$$

Unit: $\quad A \to \mathsf{F_r}(A): \; a \mapsto (a, e)$

★ The group of paths $\mathsf{Pth}(\mathsf{F_r}(A)) = \mathsf{F_g}(A)$ acts freely on $\mathsf{F_r}(A)$:

$$g = {g_1}^{\delta_1} \cdots {g_n}^{\delta_n} \; \in \mathsf{F_g}(A)$$

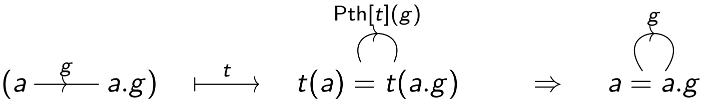
$$(a, h).g = (a, h) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_n} (g_n, e) = (a, h{g_1}^{\delta_1} \cdots {g_n}^{\delta_n}) = (a, hg)$$

Trivial extensions

Definition:

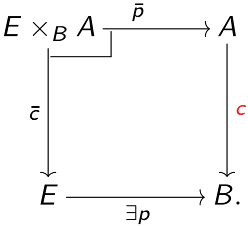
$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \pi_0(A) \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_0(t)} \\ B & \xrightarrow[\eta_B]{} & \pi_0(B). \end{array}$$

Characterization: A path sent to a loop was already a loop



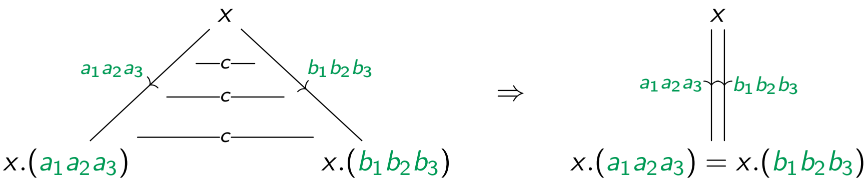
Characterization of central extensions ?

Objective: condition on extention c s.t. there is p such that $\bar{c}$ is trivial



Condition [Eisermann]: c is a covering if $\quad c(a) = c(b) \;\Rightarrow\; x \triangleleft a = x \triangleleft b$

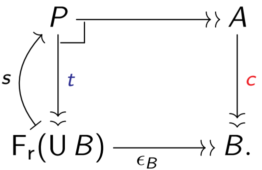
Geometric interpretation:



Characterization of central extensions [V.Even 2014] – new proof

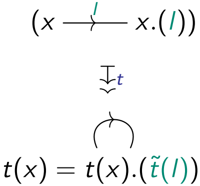
Objective:

c a covering $\Rightarrow$ t trivial

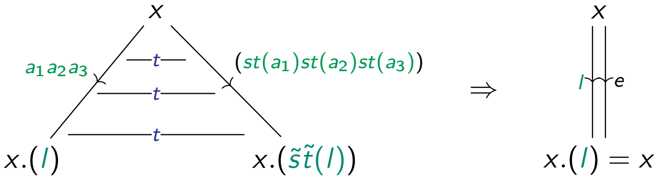


Test if t is trivial:

if t sends a path l to a loop $\tilde{t}(l)$

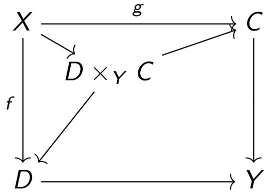


1. Downstairs: paths act freely $\Rightarrow$ loops are trivial $\Rightarrow$ $\tilde{t}(l) = e$
2. Send trivial loop back up via splitting s: $\tilde{s}\tilde{t}(l) = e$
3. Upstairs: path l and loop $\tilde{s}\tilde{t}(l)$ act the same because t is a covering.



Towards higher dimensions

Double extension

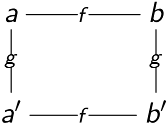


Condition for double covering ?

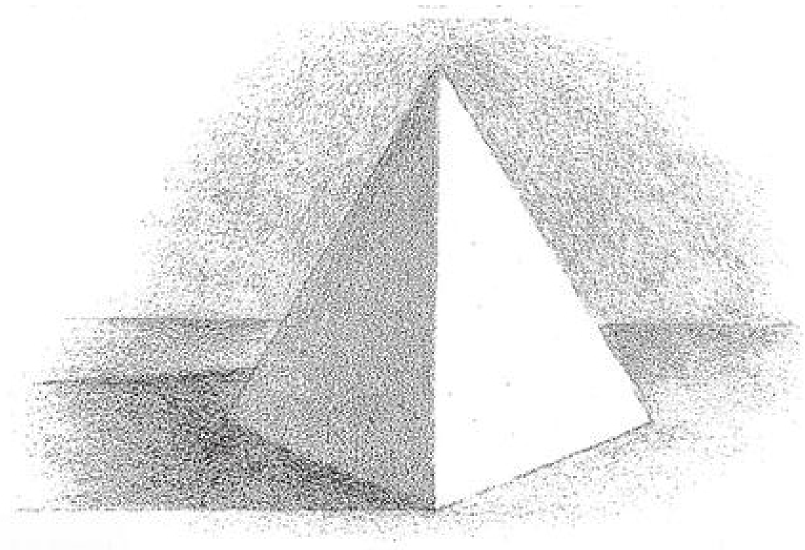
1-dimensional covering : act on $x \in X$ with 1-dimensional data

$$a \text{ —}f\text{— } b$$

2-dimensional covering : act on $x \in X$ with 2-dimensional data



Double covering



Commutator condition

Given:

- quandle X
- congruences R and S

Define: $[R, S]$ the congruence generated by the pairs

$$(x \lhd a \lhd^{-1} b,\ x \lhd c \lhd^{-1} d)$$

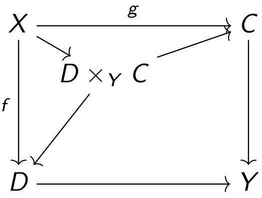
for any x, a, b, c and d in X such that

$$\begin{array}{ccc} a & \overset{R}{\text{———}} & b \\ S \big| & & \big| S \\ c & \overset{R}{\text{———}} & d. \end{array}$$

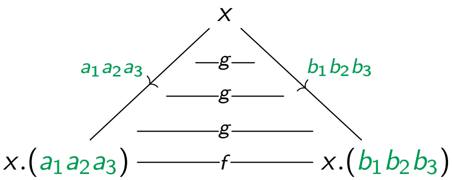
1. $[R, S] = [S, R] \subset R \cap S$
2. $[X \times X, X \times X] \ = \sim_X$ i.e. connectedness
3. 1-dimensional centrality $\Leftrightarrow ([\mathrm{Eq}(f), X \times X] = \Delta_X)$
4. 2-dimensional centrality $\Leftrightarrow ([\mathrm{Eq}(f), \mathrm{Eq}(g)] = \Delta_X)$

Double trivial coverings

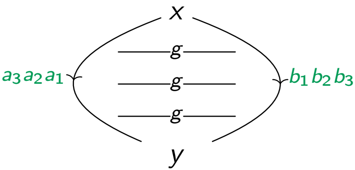
A double extension



is trivial iff in X:



$\Rightarrow$

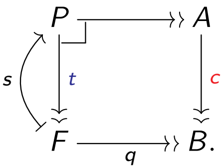


$$y = x.(a_1 a_2 a_3) = x.(b_1 b_2 b_3)$$

Characterization of double central extensions

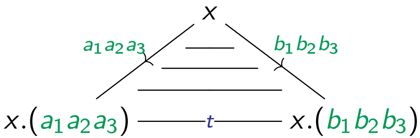
Objective:

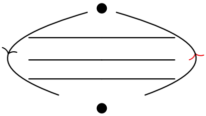
c a double covering $\Rightarrow$ t trivial ?



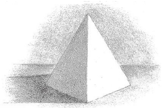
Test if t is trivial:

if open membr. $\overset{t}{\mapsto}$ closed membr.





1. closed membrane = trivial loop in the domain of F
2. obtain trivial loop in P via splitting s
3. closed membrane above the open membrane, fitting into a cone...



References

- M. Eisermann. Quandle coverings and their Galois correspondence. *Fund. Math.*, 225(1):103–168, 2014.
- V. Even. A Galois-theoretic approach to the covering theory of quandles. *Appl. Categ. Structures*, 22(5–6):817–831, 2014.
- R. Fenn and C. Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1(4):343–406, 1992.
- G. Janelidze. Pure Galois Theory in Categories. *J. Algebra*, 132:270–286, 1990.
- G. Janelidze, What is a double central extension? (The question was asked by Ronald Brown), *Cah. Top. Géom. Diff. Catég.* XXXII (1991), no. 3, 191–201.
- D. Joyce. An Algebraic Approach to Symmetry With Applications to Knot Theory. *PhD thesis, University of Pennsylvania*, 1979.

Thank you