Split extension classifiers in the category of cocommutative Hopf algebras

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Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras

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“Abelian” versus “semi-abelian”

Definition
A category $\mathbb{C}$ is abelian if

- $\mathbb{C}$ has a 0-object
- $\mathbb{C}$ has finite products
- any arrow f in $\mathbb{C}$ has a factorisation $f = i \circ p$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"', two heads] & & Y \\ & I \arrow[ur, "i"', tail] & \end{tikzcd}$$

where p is a *normal epi* and i is a *normal mono*.

Ab is the typical example of abelian category :

- Ab has a 0-object : the trivial group $\{0\}$
- Ab has finite products
- any homomorphism f in Ab has a factorisation $f = i \circ p$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\
& f(X) \arrow[ur, "i"'] &
\end{tikzcd}$$

where p is a *surjective homomorphism* (= normal epi) and i is an inclusion as a *normal subgroup* (= normal mono).

Grp is not abelian :

- Grp has a 0-object : the trivial group
- Grp has finite products
- Problem : an arrow f in Grp does not have a factorisation $f = i \circ p$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & f(X) \arrow[ur, hook, "i"'] & \end{tikzcd}$$

with p a *surjective homomorphism* and i an inclusion as a *normal* subgroup.

Question : is there a list of simple axioms to develop a unified treatment of the categories Grp, Rng, Lie_K,... ?

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S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)

Several proposals of “non-abelian contexts” for

radical theory :

S. A. Amitsur (1954), A.G. Kurosh (1959)

non-abelian homological algebra :

A. Frölich (1961), M. Gerstenhaber (1970), G. Orzech (1972)

commutator theory :

P. Higgins (1956), S.A. Huq (1968), etc.

Definition (G. Janelidze, L. Márki, W. Tholen, JPAA, 2002)

A finitely complete category $\mathbb{C}$ is semi-abelian if

- $\mathbb{C}$ has a 0-object
- $\mathbb{C}$ has $A + B$
- $\mathbb{C}$ is (Barr)-exact
- $\mathbb{C}$ is (Bourn)-protomodular :

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, shift right, "f"'] \arrow[d, "v"] & B \arrow[l, shift right] \arrow[d, "w"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift right, "f'"'] & B' \arrow[l, shift right]
\end{tikzcd}
$$

u, w isomorphisms $\Rightarrow$ v isomorphism.

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Grp, Rng, Lie_K, XMod (more generally, any variety of Ω-groups)

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[$\mathbb{C}$ is abelian] $\Leftrightarrow$ [$\mathbb{C}$ and $\mathbb{C}^{op}$ are semi-abelian] !

Many new connections have been discovered between semi-abelian (co)homology and commutator theory in universal algebra.

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A description in the case of Hopf algebras

Let K be a field.

Bialgebras

A *K-bialgebra* $(A, m, u, \Delta, \epsilon)$ is both a K-algebra (A, m, u) and a K-coalgebra (A, Δ, ϵ), where m, u, Δ, ϵ are linear maps such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1_A \otimes m"] \arrow[d, "m \otimes 1_A"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes K \arrow[r, "1_A \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] & K \otimes A \arrow[l, "u \otimes 1_A"'] \arrow[dl, "l_A"] \\
& A &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
A \otimes A \arrow[r, "1_A \otimes \Delta"'] & A \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes K & A \otimes A \arrow[l, "1_A \otimes \epsilon"'] \arrow[r, "\epsilon \otimes 1_A"] & K \otimes A \\
& A \arrow[ul, "r_A^{-1}"] \arrow[u, "\Delta"] \arrow[ur, "l_A^{-1}"'] &
\end{tikzcd}$$

commute, and m and u are K-coalgebra morphisms.

A Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ is a K-bialgebra with an antipode, a linear map $S\colon A \to A$ making the following diagram commute :

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, shift left, "1_A \otimes S"] \arrow[rr, shift right, "S \otimes 1_A"'] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"'] & & K \arrow[rr, "u"'] & & A
\end{tikzcd}$$

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A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"'] & & K \arrow[rr, "u"'] & & A
\end{tikzcd}
$$

$(A, m, u, \Delta, \epsilon, S)$ is cocommutative if the following triangle commutes :

$$
\begin{tikzcd}
& A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[rr, "tw", "\cong"'] & & A \otimes A
\end{tikzcd}
$$

In Sweedler's notations : $\Delta(a) = a_1 \otimes a_2 = a_2 \otimes a_1$, for any $a \in A$.

Example

Any group G gives the group-algebra

$$K[G] = \{\sum_g \alpha_g g \mid g \in G,\},$$

which becomes a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

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In the category $\mathrm{Hopf}_{K,coc}$ of cocommutative Hopf algebras there is the full subcategory

$$\mathrm{GrpHopf}_K \subset \mathrm{Hopf}_{K,coc}$$

of group Hopf algebras (= generated by grouplike elements).

Theorem (M. Gran, F. Sterck and J. Vercruysse, JPAA, 2019)

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The most difficult part is to prove that $\mathsf{Hopf}_{K,coc}$ is a regular category (this was explained by F. Sterck in her talk).

In particular, this result implies

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The category $\mathrm{Hopf}_{K,coc}^{comm}$ is abelian.

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Indeed :

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$A \in \mathrm{Hopf}_{K,coc}$ is abelian $\Leftrightarrow \Delta \colon A \to A \otimes A$ is a normal mono

$\Leftrightarrow$ A is commutative : $ab = ba$

$\Leftrightarrow$ $A \in \mathrm{Hopf}_{K,coc}^{comm}$

There is an adjunction

$$\mathsf{Hopf}_{K,coc}^{comm} = \mathsf{Ab}(\mathsf{Hopf}_{K,coc}) \underset{U}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Hopf}_{K,coc}$$

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$$\mathrm{Hopf}_{K,coc}^{comm} = \mathrm{Ab}(\mathrm{Hopf}_{K,coc}) \underset{U}{\overset{\mathrm{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Hopf}_{K,coc}$$

In general, if $\mathbb{C}$ is semi-abelian, $\mathrm{Ab}(\mathbb{C})$ is abelian

$$\mathrm{Ab}(\mathbb{C}) \underset{U}{\overset{\mathrm{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{C}$$

with unit of the adjunction

$$A \xrightarrow{\eta_A} \frac{A}{[A,A]}$$

Commutators

For general normal Hopf subalgebras M, N of $A \in \mathrm{Hopf}_{K,coc}$

$$M \rightarrowtail A \leftarrowtail N$$

one can compute the categorical commutator :

$$[M, N]_{Huq} = \langle \{ m_1 n_1 S(m_2) S(n_2) \mid m \in M, n \in N \} \rangle_A$$

(where $\Delta(m) = m_1 \otimes m_2$ and $\Delta(n) = n_1 \otimes n_2$).

In $\mathsf{Hopf}_{K,coc}$ the condition $[M, N]_{Huq} = 0$ is equivalent to the existence of a (unique) morphism $p\colon M \otimes N \to A$ making the diagram

$$\begin{tikzcd}
 & M \otimes N \arrow[dd, dotted, "p"] & \\
M \arrow[ur, "{(1_M,0)}"] \arrow[dr, hook] & & N \arrow[ul, "{(0,1_N)}"'] \arrow[dl, hook'] \\
 & A &
\end{tikzcd}$$

commute, where $p(m \otimes n) = mn$, for any $m \otimes n \in M \otimes N$.

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 & A &
\end{tikzcd}$$

commute, where $p(m \otimes n) = mn$, for any $m \otimes n \in M \otimes N$.

This allows one to apply methods of commutator theory to $\mathsf{Hopf}_{K,coc}$.

Outline

Split extensions

In a semi-abelian category $\mathbb{C}$ a split extension is a diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\kappa"] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd} \tag{1}$$

where $\kappa = \mathrm{Ker}\,(p)$ and $p \circ s = 1_B$.

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In a semi-abelian category $\mathbb{C}$ a split extension is a diagram

$$0 \longrightarrow X \xrightarrow{\kappa} A \underset{p}{\overset{s}{\leftrightarrows}} B \longrightarrow 0 \qquad (1)$$

where $\kappa = \mathrm{Ker}\,(p)$ and $p \circ s = 1_B$.

Example

In the category Grp of groups each split extension (1) is determined by a morphism

$$\chi \colon B \to \mathrm{Aut}(X)$$

where the action of B on X is given by

$$\chi(b)(x) = s(b)xs(b)^{-1}$$

for any $b \in B$ and $x \in X$.

Given any $X \in$ Grp there is a universal split extension

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i_1"] & X \rtimes \mathrm{Aut}(X) \arrow[r, shift right, "p_2"'] & \mathrm{Aut}(X) \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd}$$

(with kernel X) with the following universal property :

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\end{tikzcd}
$$

(with kernel X) with the following universal property :

for any other split extension, there is a unique morphism

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, dotted, "\exists ! \overline{\chi}"] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, dotted, "\exists ! \chi"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_1"'] & X \rtimes \mathrm{Aut(X)} \arrow[r, shift right, "p_2"'] & \mathrm{Aut(X)} \arrow[l, shift right, "i_2"'] \arrow[r] & 0.
\end{tikzcd}
$$

Given $X \in$ Grp, the group Aut(X) is the split extension classifier :

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, dotted, "\exists!\overline{\chi}"] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, dotted, "\exists!\chi"] & 0 \\
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\end{tikzcd}
$$

The category Grp has representable actions in the sense of
F. Borceux, G. Janelidze, G.M. Kelly, Comment. Math. Univ. Carolin. 2005.

The term “having representable actions” comes from the fact that

$$\mathrm{SplExt}(-, X)\colon \mathrm{Grp}^{op} \to \mathrm{Set}$$

is representable, with representing object Aut(X) :

$$\mathrm{SplExt}(-, X) \cong \hom(-, \mathrm{Aut(X)}).$$

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is representable, with representing object $\mathsf{Aut(X)}$:

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Split extensions in Grp correspond to actions :

$$\mathsf{Act}(-, X) \cong \mathsf{SplExt}(-, X) \cong \hom(-, \mathsf{Aut(X)})$$

Split extensions in the category of Lie algebras

Similarly, for any $L \in \mathrm{Lie}_K$ the Lie algebra $\mathrm{Der}(L)$ of derivations is a split extension classifier

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[rr, shift right, "p"'] \arrow[d, dotted, "\exists!\overline{\rho}"] & & B \arrow[ll, shift right, "s"'] \arrow[r] \arrow[d, dotted, "\exists!\rho"] & 0 \\
0 \arrow[r] & L \arrow[r, "i_1"'] & L \rtimes \mathrm{Der}(L) \arrow[rr, shift right, "p_2"'] & & \mathrm{Der}(L) \arrow[ll, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd}$$

where the Lie algebra action is

$$\rho(b)(l) = [s(b), l]$$

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Similarly, for any $L \in \mathrm{Lie}_K$ the Lie algebra $\mathrm{Der}(L)$ of derivations is a split extension classifier

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0 \arrow[r] & L \arrow[r, "i_1"'] & L \rtimes \mathrm{Der}(L) \arrow[r, shift right, "p_2"'] & \mathrm{Der}(L) \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd}
$$

where the Lie algebra action is

$$\rho(b)(l) = [s(b), l]$$

$$\mathrm{Act}(-, L) \cong \mathrm{SplExt}(-, L) \cong \hom(-, \mathrm{Der}(L))$$

In general, a semi-abelian category $\mathbb{C}$ has representable actions if any object $X \in \mathbb{C}$ has a split extension classifier, denoted by $[X]$, with

$$0 \longrightarrow X \xrightarrow{\kappa} \overline{X} \underset{p}{\overset{s}{\leftrightarrows}} [X] \longrightarrow 0$$

a universal split extension (with kernel X).

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Split extensions in cocommutative Hopf algebras

In $\mathrm{Hopf}_{K,coc}$ any split extension

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\kappa"] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

is canonically isomorphic to the semidirect product exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "i_1"'] & X \rtimes B \arrow[u, "\cong"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd}$$

Semidirect product

In the split exact sequence

$$0 \longrightarrow X \xrightarrow{i_1} X \rtimes B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B \longrightarrow 0 \tag{2}$$

the semidirect product $X \rtimes B$ is the vector space $X \otimes B$ equipped with the cocommutative Hopf algebra structure :

- $M_{X \rtimes B}(x \otimes b, x' \otimes b') = x(b_1 \cdot x') \otimes b_2 b'$
- $\Delta_{X \rtimes B} = (1_X \otimes tw \otimes 1_B)(\Delta_X \otimes \Delta_B)$
- $u_{X \rtimes B} = u_X \otimes u_B$ and $\epsilon_{X \rtimes B} = \epsilon_X \otimes \epsilon_B$
- $S(x \otimes b) = (S_B(b_1)) \cdot S_X(x) \otimes S_B(b_2)$

(here $b \cdot x$ denotes the action of b on x corresponding to

$$0 \longrightarrow X \xrightarrow{\kappa} A \underset{p}{\overset{s}{\leftrightarrows}} B \longrightarrow 0 \ )$$

When K is an algebraically closed field of characteristic 0 :

Theorem (Milnor-Moore, Ann. Math. 1965)

For any cocommutative Hopf K-algebra H there is a split extension

$$0 \longrightarrow \mathcal{U}(L_H) \xrightarrow{i_1} H \cong \mathcal{U}(L_H) \rtimes \mathrm{K[G_H]} \underset{p_2}{\overset{i_2}{\leftrightarrows}} \mathrm{K[G_H]} \longrightarrow 0$$

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- $\mathcal{U}(L_H)$ is the universal enveloping algebra of the Lie algebra

$$L_H = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

  of *primitive elements* of H;

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  of *primitive elements* of H;
- $\mathrm{K}[\mathrm{G_H}]$ is the group Hopf algebra generated by the *grouplike elements*

  $$G_H = \{x \in H \mid \Delta(x) = x \otimes x, \epsilon(x) = 1\}$$

  of H.

This result can be used to prove

Proposition (M.G., G. Kadjo and J. Vercruysse (APCS, 2016))

When K is an algebraically closed field with characteristic 0, the pair

$$(\mathsf{PrimHopf_K}, \mathsf{GrpHopf_K})$$

of full subcategories of $\mathsf{Hopf}_{K,coc}$ is a hereditary torsion theory.

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of full subcategories of $\mathsf{Hopf}_{K,coc}$ is a hereditary torsion theory.

Moreover, the category of groups is a localization of $\mathsf{Hopf}_{K,coc}$

$$\begin{tikzcd} \mathsf{Grp} \arrow[r, shift right, "{K[-]}"'] \arrow[r, phantom, "\perp"] & \mathsf{Hopf}_{K,coc} \arrow[l, shift right, "F"'] \end{tikzcd}$$

i.e. the reflector $F\colon \mathsf{Hopf}_{K,coc} \to \mathsf{Grp}$ preserves finite limits.

Split extension classifier in $\mathsf{Hopf}_{K,coc}$

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It is natural to look for an explicit description of the split extension classifier $[H]$ of any cocommutative Hopf algebra H.

The *"group Hopf algebra* part" of $[H]$ is

$$\mathrm{K}[\mathrm{Aut}_{Hopf}(H)]$$

where $Aut_{Hopf}(H)$ is the group of Hopf automorphisms of H.

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To define the *"primitive* part" of $[H]$ one needs the following

Definition

A Hopf derivation of a Hopf algebra $(H, m, u, \Delta, \epsilon, S)$ is a linear endomorphism $\psi \colon H \to H$ that is a *derivation*

$$\psi \circ m = m \circ (\psi \otimes id + id \otimes \psi)$$

and a *coderivation*

$$\Delta \circ \Psi = (\psi \otimes id + id \otimes \psi) \circ \Delta.$$

One writes $\mathrm{Der}_{Hopf}(H)$ for the Lie algebra of Hopf derivations, where

$$[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - \psi_2 \circ \psi_1, \qquad \forall \psi_1, \psi_2 \in \mathrm{Der}_{Hopf}(H).$$

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By applying the universal enveloping algebra functor
$\mathcal{U} \colon \mathrm{Lie}_K \to \mathrm{Hopf}_{K,coc}$ one gets the primitive Hopf algebra

$$\mathcal{U}(\mathrm{Der}_{Hopf}(H))$$

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One defines

$$[H] = \mathcal{U}(\mathsf{Der}_{Hopf}(H)) \rtimes_{\overline{\rho}} \mathsf{K}[\mathsf{Aut}_{Hopf}(H)]$$

where the action

$$\overline{\rho}\colon \mathsf{K}[\mathsf{Aut}_{Hopf}(H)] \otimes \mathcal{U}(\mathsf{Der}_{Hopf}(H)) \to \mathcal{U}(\mathsf{Der}_{Hopf}(H))$$

is determined by $\quad \overline{\rho}(\phi \otimes \psi) = \phi \circ \psi \circ \phi^{-1}$.

Theorem (M.G., G. Kadjo and J. Vercruysse, BBMS 2018)
Let K be an algebraically closed field of characteristic zero. Then

$$[H] = \mathcal{U}(\mathrm{Der}_{Hopf}(H)) \rtimes_{\overline{\rho}} \mathrm{K}[\mathrm{Aut}_{Hopf}(H)]$$

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There is a universal split extension

$$0 \longrightarrow H \longrightarrow H \rtimes_{\star} [H] \rightleftarrows [H] \longrightarrow 0$$

where the action $\star \colon [H] \otimes H \to H$ is defined by

$$(\phi \otimes \psi) \star h = \psi(\phi(h))$$

for any $\phi \otimes \psi \in [H] = \mathcal{U}(\mathrm{Der}_{Hopf}(H)) \rtimes_{\overline{\rho}} \mathrm{K}[\mathrm{Aut}_{Hopf}(H)]$, and $h \in H$.

Center

When a semi-abelian category $\mathbb{C}$ is action representable, the categorical center $Z(X)$ of an object X can be obtained as the kernel of the canonical arrow χ in

$$\begin{tikzcd}
 & & & 0 \arrow[d, dotted] & \\
 & & & Z(X) \arrow[d, dotted, "\ker(\chi)"] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & X \times X \arrow[r, shift right, "p_1"'] \arrow[d, dotted, "\overline{\chi}"] & X \arrow[l, shift right, "\Delta"'] \arrow[r] \arrow[d, dotted, "\chi"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_1"'] & X \rtimes [X] \arrow[r, shift right, "p_2"'] & {[X]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd}$$

(see A. Cigoli and S. Mantovani, JPAA, 2012).

Example

In the case of groups, this corresponds to the fact that the center $Z(G)$ of a group G is the kernel of the conjugation map χ in

$$\begin{tikzcd}
 & & & 0 \arrow[d, dotted] & \\
 & & & Z(G) \arrow[d, dotted, "\ker(\chi)"] & \\
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & G \times G \arrow[r, shift right] \arrow[d, dotted, "\overline{\chi}"] & G \arrow[l, shift right] \arrow[r] \arrow[d, dotted, "\chi"] & 0 \\
0 \arrow[r] & G \arrow[r] & G \rtimes Aut(G) \arrow[r, shift right] & Aut(G) \arrow[l, shift right] \arrow[r] & 0
\end{tikzcd}$$

where $\chi(g)(h) = ghg^{-1}$, for any $g, h \in G$.

Definition (N. Andruskiewitsch, Canad. J. Math. 1996)

Given a Hopf algebra A, the Hopf center $\mathrm{HZ}(A)$ is the largest Hopf subalgebra of A contained in the algebraic center $Z_{alg}(A)$ of A, where

$$Z_{alg}(A) = \{a \in A \mid ab = ba, \forall b \in A\}.$$

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Proposition (M.G., G. Kadjo and J. Vercruysse, 2018)

When A is cocommutative, the categorical center $\mathrm{Z}(A)$ of A coincides with the Hopf center $\mathrm{HZ}(A)$:

$$\mathrm{Z}(A) = \mathrm{HZ}(A) = \{a \in A \mid \Delta(a) \in A \otimes Z_{alg}(A)\}.$$

Final remarks

It is interesting to adopt the approach based on semi-abelian categories in the study of (cocommutative) Hopf algebras.

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The approach based on *Schreier split extensions* (due to Sobral, Martins-Ferreira, Montoli, Bourn) could be useful to study some exactness properties of $Hopf_K$.

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