# Involutive factorisation systems & Dold-Kan correspondences

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<sup>&</sup>lt;sup>1</sup>joint with Christophe Cazanave and Ingo Waschkies  $\rightarrow \langle B \rangle \langle B \rangle \langle B \rangle \langle B \rangle \langle B \rangle$ 



- 2 Simplicial objects
- Involutive factorisation systems
- 4 Dold-Kan correspondences





#### Theorem (Dold 1958, Kan 1958)

 $M: \underline{\operatorname{Ab}}^{\Delta^{\operatorname{op}}} \simeq \operatorname{Ch}(\mathbb{Z}): K$ 

#### Corollary

There is a simplicial abelian group K(A, n) such that  $\pi_n(K(A, n)) = A$  and  $\pi_i(K(A, n)) = 0$  for  $i \neq n$ .

#### Proof.

 $K : Ch(\mathbb{Z}) \to \underline{Ab}^{\Delta^{op}}$  takes homology into homotopy. K(A, n) is the image of the chain complex:  $0 \leftarrow \cdots \leftarrow 0 \leftarrow \stackrel{n}{A} \leftarrow 0 \leftarrow \cdots$ 

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Categorical structure of  $\Delta$  inducing Dold-Kan correspondence.

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# Definition (simplex category $\Delta$ )

 $\mathrm{Ob}\Delta = \{[n] = \{0, 1, \dots, n\}, n \ge 0\}, \mathrm{Mor}\Delta = \{\mathsf{monotone\ maps}\}$ 

# Remark (epi-mono factorisation system)

The category  $\Delta$  is generated by elementary

- face operators  $\epsilon_i^n : [n-1] \rightarrow [n], 0 \le i \le n$ , and
- degeneracy operators  $\eta_i^n : [n+1] \rightarrow [n], 0 \le i \le n$ .

Every simplicial operator  $\phi : [m] \rightarrow [n]$  factors as



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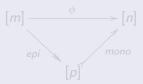
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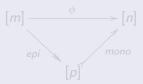
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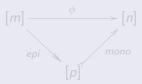
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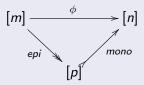
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 $\Delta \hookrightarrow \operatorname{Top} : [n] \mapsto \Delta_n$  yields by left Kan extension along Yoneda

 $-|_{\Delta} : \operatorname{Sets}^{\Delta^{\operatorname{op}}} \to \operatorname{Top}.$ 

Theorem (Quillen 1968)

Geometric realisation is left part of a Quillen equivalence.

Definition (simplicial homology, Eilenberg 1944)

 $\operatorname{Sets}^{\Delta^{\operatorname{op}}} \longrightarrow \underline{\operatorname{Ab}}^{\Delta^{\operatorname{op}}} \xrightarrow{N} \operatorname{Ch}(\mathbb{Z}) \longrightarrow \underline{\operatorname{Ab}}^{\mathbb{N}}$ 

 $X_{\bullet} \longmapsto \mathbb{Z}[X_{\bullet}] \longmapsto (N_{\bullet}(X), d_{\bullet}) \longmapsto H_{\bullet}(X)$ 

where  $(N_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}[D_n(X)], d_n = \sum_k (-1)^k X(\epsilon_k^n))$ 

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Moore normalisation M admits a left adjoint K assigning to a chain complex  $(C_{\bullet}, d_{\bullet})$  the simplicial abelian group

$$K(C_{\bullet}, d_{\bullet})_{n} = \bigoplus_{[n] \to [k]} C_{k} \text{ with } K(\phi) : \bigoplus_{[n] \to [k]} C_{k} \to \bigoplus_{[m] \to [j]} C_{j}$$
where  $K(\phi)_{ab} = \begin{cases} d_{k} \text{ if } [m] \xrightarrow{\phi} [n] \\ a_{k} & b \\ [k-1] \xrightarrow{c_{k}} [k] \\ 0 \text{ otherwise} \end{cases}$ 

# Remark

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A factorisation system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{C}$  is called *involutive* if there is a specified faithful, identity-on-objects functor  $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$  sth.

11)  $ee^* = 1$  (the split idempotent  $e^*e$  is called an  $\mathcal{E}$ -projector);

- 12) the morphisms  $f^*e$  form a subcategory of  $\mathcal{C}$ ;
- $(A \xrightarrow{m} B) \in \mathcal{M} \,\forall \phi \in \operatorname{Proj}_{\mathcal{E}}(A) \,\exists \psi \in \operatorname{Proj}_{\mathcal{E}}(B) : m\phi = \psi m;$
- (4) Proj<sub>ε</sub>(A) is finite. Primitive ε-projectors can be linearly ordered such that if φ precedes ψ then ψφ is an ε-projector.

# Remark (primitive *E*-projectors)

 $\operatorname{Proj}_{\mathcal{E}}(A) \cong \operatorname{Quot}_{\mathcal{E}}(A)$ . Primitive  $\mathcal{E}$ -projectors are *covered* by  $1_A$ .

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 $\operatorname{Proj}_{\mathcal{E}}(A) \cong \operatorname{Quot}_{\mathcal{E}}(A)$ . Primitive  $\mathcal{E}$ -projectors are *covered* by  $1_A$ .

# Remark (Involutive factorisation system for $\Delta$ )

Each epi  $e : [m] \rightarrow [n]$  has a *maximal* section  $e^* : [n] \rightarrow [m]$ .

A factorisation system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{C}$  is called *involutive* if there is a specified faithful, identity-on-objects functor  $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$  sth.

- (1)  $ee^* = 1$  (the split idempotent  $e^*e$  is called an  $\mathcal{E}$ -projector);
- (12) the morphisms  $f^*e$  form a subcategory of C;

$$(\mathsf{I3}) \ \forall (A \stackrel{m}{\to} B) \in \mathcal{M} \ \forall \phi \in \operatorname{Proj}_{\mathcal{E}}(A) \ \exists \psi \in \operatorname{Proj}_{\mathcal{E}}(B) : m\phi = \psi m;$$

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# Definition (essential $\mathcal{M}$ -maps)

An  $\mathcal{M}$ -map  $m : A \to B$  is called *essential* if  $1_B$  is the only  $\mathcal{E}$ -projector of B fixing m.

Remark (essential  $\mathcal{M}$ -maps of  $\Delta$ )

are precisely the "last" face operators  $\epsilon_n^n : [n-1] \rightarrow [n]$ .

#### \_emma (quotienting out inessential $\mathcal M$ -maps)

By axiom (13) the inessential  $\mathcal{M}$ -maps form an ideal  $\mathcal{M}_{iness}$  in  $\mathcal{M}$ . In particular, there is a *locally pointed* category  $\Xi_{\mathcal{C}} = \mathcal{M}/\mathcal{M}_{iness}$ .

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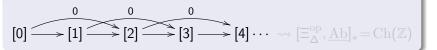
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For any small category  $\mathcal{A}$  the category  $\Delta \wr \mathcal{A}$  is defined by

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#### Definition (Joyal 1997, B 2007)

Put  $\Theta_1 = \Delta$  and for n > 1 :  $\Theta_n = \Delta \wr \Theta_{n-1}$ 

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For any presheaf  $X : \mathcal{C}^{\text{op}} \to \text{Sets}$ , each  $x \in X(c)$  equals  $X(\phi)(y)$  for unique  $\phi : c \to d$  in  $\mathcal{E}$  and "non-degenerate"  $y \in X(d)$ .

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If  $\mathcal{A}$  is an elegant Reedy category then so is  $\Delta \wr \mathcal{A}$ . In particular,  $\Theta_n$  is an elegant Reedy category.

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If  $\mathcal{A}$  is an elegant Reedy category then so is  $\Delta \wr \mathcal{A}$ . In particular,  $\Theta_n$  is an elegant Reedy category.

# Proposition (BCW 2019)

## Theorem (BCW 2019)

$$\underline{\operatorname{Ab}}^{\Theta_n^{\operatorname{op}}} \simeq [\Xi_{\Theta_n}^{\operatorname{op}}, \underline{\operatorname{Ab}}]_*$$

#### Remark ( $\Theta_n$ -set model for Eilenberg-MacLane spaces)

- For each abelian group A there is an abelian group object B<sup>n</sup>A in nCat with one k-cell for 0 ≤ k < n;</li>
- $|N_{\Theta_n}(B^n A)|$  is a cellular model for K(A, n)
- Its cellular chain complex is the "totalisation" of corresponding Ξ<sup>op</sup><sub>Θ<sub>a</sub></sub>-complex.

|                               |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
| $K(\mathbb{Z}/2\mathbb{Z},3)$ | 1 |   |   | 1 | 1 | 2 | 4 | 7 | 13 | 24 |

# Theorem (BCW 2019)

$$\underline{\mathrm{Ab}}^{\Theta^{\mathrm{op}}_n} \simeq [\Xi^{\mathrm{op}}_{\Theta_n}, \underline{\mathrm{Ab}}]_*$$

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|                               |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
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|                               |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
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- Its cellular chain complex is the "totalisation" of corresponding Ξ<sup>op</sup><sub>Θ<sub>n</sub></sub>-complex.

|                               |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
| $K(\mathbb{Z}/2\mathbb{Z},3)$ | 1 |   |   | 1 | 1 | 2 | 4 | 7 | 13 | 24 |

# Theorem (BCW 2019)

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- Its cellular chain complex is the "totalisation" of corresponding Ξ<sup>op</sup><sub>Θn</sub>-complex.

|                               | m |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ |   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ |   | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
| $K(\mathbb{Z}/2\mathbb{Z},3)$ |   | 1 |   |   | 1 | 1 | 2 | 4 | 7 | 13 | 24 |

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|                               |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 |   | 1 | 1 | 2 | 3 | 5 |   | 13 | 21 |
| $K(\mathbb{Z}/2\mathbb{Z},3)$ | 1 |   |   | 1 | 1 | 2 | 4 | 7 | 13 | 24 |

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| # cells in dim                | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8  | 9  |
|-------------------------------|---|---|---|---|---|---|---|---|----|----|
| $K(\mathbb{Z}/2\mathbb{Z},1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |
| $K(\mathbb{Z}/2\mathbb{Z},2)$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| $K(\mathbb{Z}/2\mathbb{Z},3)$ | 1 | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 24 |