Involutive factorisation systems & Dold-Kan correspondences

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¹joint with Christophe Cazanave and Ingo Waschkies $\rightarrow \langle B \rangle \langle B \rangle \langle B \rangle \langle B \rangle \langle B \rangle$



- 2 Simplicial objects
- Involutive factorisation systems
- 4 Dold-Kan correspondences





Theorem (Dold 1958, Kan 1958)

 $M: \underline{\operatorname{Ab}}^{\Delta^{\operatorname{op}}} \simeq \operatorname{Ch}(\mathbb{Z}): K$

Corollary

There is a simplicial abelian group K(A, n) such that $\pi_n(K(A, n)) = A$ and $\pi_i(K(A, n)) = 0$ for $i \neq n$.

Proof.

 $K : Ch(\mathbb{Z}) \to \underline{Ab}^{\Delta^{op}}$ takes homology into homotopy. K(A, n) is the image of the chain complex: $0 \leftarrow \cdots \leftarrow 0 \leftarrow \stackrel{n}{A} \leftarrow 0 \leftarrow \cdots$

Purpose of the talk

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Categorical structure of Δ inducing Dold-Kan correspondence.

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Definition (simplex category Δ)

 $\mathrm{Ob}\Delta = \{[n] = \{0, 1, \dots, n\}, n \ge 0\}, \mathrm{Mor}\Delta = \{\mathsf{monotone\ maps}\}$

Remark (epi-mono factorisation system)

The category Δ is generated by elementary

- face operators $\epsilon_i^n : [n-1] \rightarrow [n], 0 \le i \le n$, and
- degeneracy operators $\eta_i^n : [n+1] \rightarrow [n], 0 \le i \le n$.

Every simplicial operator $\phi : [m] \rightarrow [n]$ factors as



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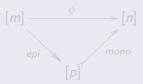
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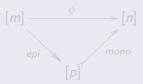
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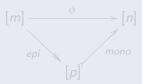
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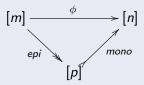
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Definition (geometric realisation, Milnor 1957)

 $\Delta \hookrightarrow \operatorname{Top} : [n] \mapsto \Delta_n$ yields by left Kan extension along Yoneda

 $-|_{\Delta} : \operatorname{Sets}^{\Delta^{\operatorname{op}}} \to \operatorname{Top}.$

Theorem (Quillen 1968)

Geometric realisation is left part of a Quillen equivalence.

Definition (simplicial homology, Eilenberg 1944)

 $\operatorname{Sets}^{\Delta^{\operatorname{op}}} \longrightarrow \underline{\operatorname{Ab}}^{\Delta^{\operatorname{op}}} \xrightarrow{N} \operatorname{Ch}(\mathbb{Z}) \longrightarrow \underline{\operatorname{Ab}}^{\mathbb{N}}$

 $X_{\bullet} \longmapsto \mathbb{Z}[X_{\bullet}] \longmapsto (N_{\bullet}(X), d_{\bullet}) \longmapsto H_{\bullet}(X)$

where $(N_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}[D_n(X)], d_n = \sum_k (-1)^k X(\epsilon_k^n))$

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Moore normalisation M admits a left adjoint K assigning to a chain complex $(C_{\bullet}, d_{\bullet})$ the simplicial abelian group

$$K(C_{\bullet}, d_{\bullet})_{n} = \bigoplus_{[n] \to [k]} C_{k} \text{ with } K(\phi) : \bigoplus_{[n] \to [k]} C_{k} \to \bigoplus_{[m] \to [j]} C_{j}$$
where $K(\phi)_{ab} = \begin{cases} d_{k} \text{ if } [m] \xrightarrow{\phi} [n] \\ a_{k} & b \\ [k-1] \xrightarrow{c_{k}} [k] \\ 0 \text{ otherwise} \end{cases}$

Remark

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A factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} is called *involutive* if there is a specified faithful, identity-on-objects functor $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$ sth.

11) $ee^* = 1$ (the split idempotent e^*e is called an \mathcal{E} -projector);

- 12) the morphisms f^*e form a subcategory of \mathcal{C} ;
- $(A \xrightarrow{m} B) \in \mathcal{M} \,\forall \phi \in \operatorname{Proj}_{\mathcal{E}}(A) \,\exists \psi \in \operatorname{Proj}_{\mathcal{E}}(B) : m\phi = \psi m;$
- (4) Proj_ε(A) is finite. Primitive ε-projectors can be linearly ordered such that if φ precedes ψ then ψφ is an ε-projector.

Remark (primitive *E*-projectors)

 $\operatorname{Proj}_{\mathcal{E}}(A) \cong \operatorname{Quot}_{\mathcal{E}}(A)$. Primitive \mathcal{E} -projectors are *covered* by 1_A .

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Each epi $e : [m] \rightarrow [n]$ has a *maximal* section $e^* : [n] \rightarrow [m]$.

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Dold-Kan correspondences

Definition (essential \mathcal{M} -maps)

An \mathcal{M} -map $m : A \to B$ is called *essential* if 1_B is the only \mathcal{E} -projector of B fixing m.

Remark (essential \mathcal{M} -maps of Δ)

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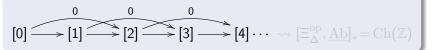
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Put $\Theta_1 = \Delta$ and for n > 1 : $\Theta_n = \Delta \wr \Theta_{n-1}$

Theorem (Makkai-Zawadowski 2003, B 2003)

 Θ_n embeds densely into nCat, i.e. there is a fully faithful functor

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A *Reedy category* C has a strict $(\mathcal{E}, \mathcal{M})$ -factorisation system, a grading deg : $ObC \to \mathbb{N}$ such that \mathcal{E} (resp. \mathcal{M})-maps lower (resp. increase) degree. C is *elegant* if \mathcal{E} has absolute pushouts.

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For any presheaf $X : \mathcal{C}^{\text{op}} \to \text{Sets}$, each $x \in X(c)$ equals $X(\phi)(y)$ for unique $\phi : c \to d$ in \mathcal{E} and "non-degenerate" $y \in X(d)$.

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- For each abelian group A there is an abelian group object BⁿA in nCat with one k-cell for 0 ≤ k < n;
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		1	2	3	4	5	6	7		9
$K(\mathbb{Z}/2\mathbb{Z},1)$	1	1	1	1	1	1	1	1	1	1
$K(\mathbb{Z}/2\mathbb{Z},2)$	1		1	1	2	3	5		13	21
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$$\underline{\operatorname{Ab}}^{\Theta^{\operatorname{op}}_n} \simeq [\Xi^{\operatorname{op}}_{\Theta_n}, \underline{\operatorname{Ab}}]_*$$

Remark (Θ_n -set model for Eilenberg-MacLane spaces)

- For each abelian group A there is an abelian group object BⁿA in nCat with one k-cell for 0 ≤ k < n;
- $|N_{\Theta_n}(B^nA)|$ is a cellular model for K(A, n)
- Its cellular chain complex is the "totalisation" of corresponding Ξ^{op}_{Θn}-complex.

	m		1	2	3	4	5	6	7		9
$K(\mathbb{Z}/2\mathbb{Z},1)$		1	1	1	1	1	1	1	1	1	1
$K(\mathbb{Z}/2\mathbb{Z},2)$		1		1	1	2	3	5		13	21
$K(\mathbb{Z}/2\mathbb{Z},3)$		1			1	1	2	4	7	13	24

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$K(\mathbb{Z}/2\mathbb{Z},2)$	1		1	1	2	3	5		13	21
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# cells in dim	0	1	2	3	4	5	6	7	8	9
$K(\mathbb{Z}/2\mathbb{Z},1)$	1	1	1	1	1	1	1	1	1	1
$K(\mathbb{Z}/2\mathbb{Z},2)$	1	0	1	1	2	3	5	8	13	21
$K(\mathbb{Z}/2\mathbb{Z},3)$	1	0	0	1	1	2	4	7	13	24